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INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR FUNCTIONS WHOSE DERIVATIVES ARE STRONGLY α-PREINVEX

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Abstract. In the paper, the authors introduce a new notion "strongly α-preinvex function", establish an integral identity for the newly introduced function, and find some Hermite–Hadamard type integral inequalities for a function that the power of the absolute value of its first derivative is strongly α-preinvex.

1. Introduction

Let us recall some definitions of various convex functions.

Definition 1.1. A function $f\colon I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.2 ([1, 2, 14])**.** A set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta\colon S \times S \to \mathbb{R}^n$ if for every $x, y \in S$ and $t \in [0,1]$

$$y + t\eta(x,y) \in S. \tag{1.2}$$

Definition 1.3 ([8])**.** Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta\colon S \times S \to \mathbb{R}^n$. For every $x, y \in S$, the η-path P_{xv} joining the points x and $v = x + \eta(y,x)$ is defined by

$$P_{xv} = \{z \mid z = x + t\eta(y,x), t \in [0,1]\}. \tag{1.3}$$

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Definition 1.4 ([2]). Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta\colon S \times S \to \mathbb{R}^n$. A function $f\colon S \to \mathbb{R}$ is said to be preinvex with respect to η, if for every $x, y \in S$ and $t \in [0,1]$,

$$f(y + t\eta(x,y)) \le tf(x) + (1-t)f(y). \tag{1.4}$$

Definition 1.5 ([10]). For $f\colon [a,b] \to \mathbb{R}$, if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) - c\lambda(1-\lambda)(x-y)^2 \tag{1.5}$$

is valid for all $x, y \in [a,b]$ and $\lambda \in [0,1]$,$c > 0$, then we say that $f(x)$ is a strongly convex function on $[a,b]$.

Let us reformulate some inequalities of Hermite–Hadamard type for the above mentioned convex functions.

Theorem 1.1 ([5, Theorem 2.2]). *Let $f\colon I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{8}\bigl(|f'(a)| + |f'(b)|\bigr). \tag{1.6}$$

Theorem 1.2 ([2, Theorem 2.1]). *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta\colon A \times A \to \mathbb{R}$ and let $f\colon A \to \mathbb{R}$ be a differentiable function. If $|f'|$ is preinvex on A, then for every $a, b \in A$ with $\theta(a,b) \ne 0$ we have*

$$\left|\frac{f(b)+f(b+\theta(a,b))}{2} - \frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}\,x\right| \le \frac{|\theta(a,b)|}{8}\bigl[|f'(a)| + |f'(b)|\bigr]. \tag{1.7}$$

For more information on Hermite–Hadamard type inequalities for various convex functions, please refer to recently published articles [3, 4, 6, 7, 11, 12, 13] and closely related references therein.

In this article, we will introduce a new notion "α-preinvex function", establish an integral identity for such a kind of functions, and find some Hermite–Hadamard type integral inequalities for a function that the power of the absolute value of its first derivative is α-preinvex.

2. A new definition and two lemmas

The so-called "strongly α-preinvex function" may be introduced as follows.

Definition 2.1. Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta\colon S \times S \to \mathbb{R}^n$. A function $f\colon S \to \mathbb{R}$ is said to be strongly α-preinvex with respect to η for $\alpha \in (0,1]$ and $c > 0$,if for every $x, y \in S$ and $t \in [0,1]$,

$$f(y + t\eta(x,y)) \le t^\alpha f(x) + (1-t^\alpha)f(y) - ct(1-t)(x-y)^2. \tag{2.1}$$

For establishing our new integral inequalities of Hermite–Hadamard type for strongly α-preinvex functions, we need the following integral identity.

Lemma 2.1. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta\colon A\times A \to \mathbb{R}$ and let $a,b \in A$ with $\theta(a,b) \neq 0$. If $f\colon A \to \mathbb{R}$ is a differentiable function and f' is integrable on the θ-path P_{bc}, $c = b+\theta(a,b)$, then*

$$\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}\,x - f(b+\theta(a,b)) = \frac{\theta(a,b)}{4}\int_0^1\left[(1+t)f'\left(b+\frac{1+t}{2}\theta(a,b)\right)+tf'\left(b+\frac{t}{2}\theta(a,b)\right)\right]\mathrm{d}\,t.$$

Proof. Since $a,b\in A$ and A is an invex set with respect to θ, for every $t\in[0,1]$, we have $b+t\theta(a,b)\in A$. Integrating by part gives

$$\begin{aligned}
&\int_0^1\left[(1+t)f'\left(b+\frac{1+t}{2}\theta(a,b)\right)+tf'\left(b+\frac{t}{2}\theta(a,b)\right)\right]\mathrm{d}\,t\\
&=\frac{2}{\theta(a,b)}\left[-(1+t)f\left(b+\frac{1+t}{2}\theta(a,b)\right)\Big|_0^1+\int_0^1 f\left(b+\frac{1+t}{2}\theta(a,b)\right)\mathrm{d}\,t\right.\\
&\qquad\left.-tf\left(b+\frac{t}{2}\theta(a,b)\right)\Big|_0^1+\int_0^1 f\left(b+\frac{t}{2}\theta(a,b)\right)\mathrm{d}\,t\right]\\
&=\frac{2}{\theta(a,b)}\left[-2f(b+\theta(a,b))+f\left(\frac{2b+\theta(a,b)}{2}\right)\right.\\
&\qquad\left.+\frac{2}{\theta(a,b)}\int_{\frac{2b+\theta(a,b)}{2}}^{b+\theta(a,b)} f(x)\,\mathrm{d}\,x - f\left(\frac{2b+\theta(a,b)}{2}\right)+\frac{2}{\theta(a,b)}\int_b^{[2b+\theta(a,b)]/2} f(x)\,\mathrm{d}\,x\right]\\
&=\frac{4}{\theta(a,b)}\left[\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}\,x - f(b+\theta(a,b))\right].
\end{aligned}$$

The proof of Lemma 2.1 is completed. □

By the same way, we obtain another lemma.

Lemma 2.2. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta\colon A\times A \to \mathbb{R}$ and let $a,b \in A$ with $\theta(a,b) \neq 0$. If $f\colon A \to \mathbb{R}$ is a differentiable function and f' is integrable on the θ-path P_{bc}, $c = b+\theta(a,b)$, then*

$$f(b)-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}\,x = \frac{\theta(a,b)}{4}\int_0^1\left[(1-t)f'\left(b+\frac{1+t}{2}\theta(a,b)\right)+(2-t)f'\left(b+\frac{t}{2}\theta(a,b)\right)\right]\mathrm{d}\,t.$$

3. Some new integral inequalities of Hermite–Hadamard type

We are now in a position to establish some Hermite–Hadamard type integral inequalities for a function that the power of the absolute value of its first derivative is strongly α-preinvex.

Theorem 3.1. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta\colon A\times A \to \mathbb{R}$, $a,b \in A$ with $\theta(a,b) \neq 0$. Suppose $f\colon A \to \mathbb{R}$ be a differentiable function and f' is integrable on the θ-path P_{bc}, $c = b+\theta(a,b)$, some $\alpha \in (0,1]$. If $|f'|^q$ is strongly α-preinvex on A for $q \ge 1$, then*

$$\begin{aligned}
(3.1)\quad &\left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}\,x - f(b+\theta(a,b))\right| \\
&\le \frac{|\theta(a,b)|}{8}\left(\frac{1}{(\alpha+2)2^{\alpha-1}}\right)^{1/q}\Big\{3\big[3(2^{\alpha+2}-1)|f'(a)|^q \\
&\quad + (9\times 2^{\alpha-1}(\alpha+2) - 3(2^{\alpha+2}-1))|f'(b)|^q - 11\times 2^{\alpha-4}(\alpha+2)c(a-b)^2\big]^{1/q} \\
&\quad + \big[3|f'(a)|^q + 3(2^{\alpha-1}(\alpha+2)-1)|f'(b)|^q - 5\times 2^{\alpha-4}(\alpha+2)c(a-b)^2\big]^{1/q}\Big\}.
\end{aligned}$$

Proof. Since $b+t\theta(a,b) \in A$ for every $t \in [0,1]$, by Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}\,x - f(b+\theta(a,b))\right| \\
&\le \frac{|\theta(a,b)|}{4}\int_0^1\left[(1+t)\left|f'\left(b+\frac{1+t}{2}\theta(a,b)\right)\right| + t\left|f'\left(b+\frac{t}{2}\theta(a,b)\right)\right|\right]\mathrm{d}\,t \\
&\le \frac{|\theta(a,b)|}{4}\left\{\left(\int_0^1 (1+t)\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 (1+t)\left|f'\left(b+\frac{1+t}{2}\theta(a,b)\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right. \\
&\quad \left. + \left(\int_0^1 t\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 t\left|f'\left(b+\frac{t}{2}\theta(a,b)\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right\} \\
&\le \frac{|\theta(a,b)|}{4}\left\{\left(\frac{3}{2}\right)^{1-1/q}\left[\int_0^1 (1+t)\left(\left(\frac{1+t}{2}\right)^\alpha |f'(a)|^q\right.\right.\right. \\
&\quad \left.\left. + \left(1-\left(\frac{1+t}{2}\right)^\alpha\right)|f'(b)|^q - \frac{1-t^2}{4}c(a-b)^2\right)\mathrm{d}\,t\right]^{1/q} \\
&\quad + \left(\frac{1}{2}\right)^{1-1/q}\left[\int_0^1 t\left(\left(\frac{t}{2}\right)^\alpha |f'(a)|^q\right.\right. \\
&\quad \left.\left.\left. + \left(1-\left(\frac{t}{2}\right)^\alpha\right)|f'(b)|^q - \frac{2t-t^2}{4}c(a-b)^2\right)\mathrm{d}\,t\right]^{1/q}\right\} \\
&= \frac{|\theta(a,b)|}{8}\left(\frac{1}{(\alpha+2)2^{\alpha-1}}\right)^{1/q}\Big\{3^{1-\frac{1}{q}}\big[(2^{\alpha+2}-1)|f'(a)|^q \\
&\quad + (3\times 2^{\alpha-1}(\alpha+2) - (2^{\alpha+2}-1))|f'(b)|^q - 11\times 3^{-1}\times 2^{\alpha-4}(\alpha+2)c(a-b)^2\big]^{1/q} \\
&\quad + \big[|f'(a)|^q + (2^{\alpha-1}(\alpha+2)-1)|f'(b)|^q - 5\times 3^{-1}\times 2^{\alpha-4}(\alpha+2)c(a-b)^2\big]^{1/q}\Big\}.
\end{aligned}$$

The proof of Theorem 3.1 is completed. □

Corollary 1. *Under the conditions of Theorem 3.1, if* $\alpha = q = 1$*, we have*

$$\left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}\,x - f(b+\theta(a,b))\right| \le \frac{|\theta(a,b)|}{12}[4|f'(a)|^q + 2|f'(b)|^q - c(a-b)^2].$$

Theorem 3.2. *Let* $A \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\theta\colon A\times A \to \mathbb{R}$*,* $a,b \in A$ *with* $\theta(a,b) \neq 0$*. Suppose* $f\colon A \to \mathbb{R}$ *be a differentiable function and* f' *is integrable on the* θ*-path* P_{bc}*,* $c = b+\theta(a,b)$*, some* $\alpha \in (0,1]$*. If* $|f'|^q$ *is strongly* α*-preinvex on* A *for* $q > 1$*, then*

$$(3.2)\quad \left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}\,x - f(b+\theta(a,b))\right|$$
$$\le \frac{|\theta(a,b)|}{4}\left(\frac{q-1}{(2q-1)(\alpha+1)2^\alpha}\right)^{1-1/q}\left(2^{\frac{2q-1}{q-1}}-1\right)^{1-1/q}$$
$$\times\left\{\left[(2^\alpha-1)|f'(a)|^q + (2^\alpha\alpha+1)|f'(b)|^q - 3^{-1}\times 2^{\alpha-1}(\alpha+1)c(a-b)^2\right]^{1/q}\right.$$
$$\left.+\left[|f'(a)|^q + (2^\alpha(\alpha+1)-1)|f'(b)|^q - 3^{-1}\times 2^{\alpha-1}(\alpha+1)c(a-b)^2\right]^{1/q}\right\}.$$

Proof. Since $b + t\theta(a,b) \in A$ for every $t \in [0,1]$, by Lemma 2.1, Hölder's inequality, we have

$$(3.3)\left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}\,x - f(b+\theta(a,b))\right|$$
$$\le \frac{|\theta(a,b)|}{4}\int_0^1\left[(1+t)\left|f'\left(b+\frac{1+t}{2}\theta(a,b)\right)\right| + t\left|f'\left(b+\frac{t}{2}\theta(a,b)\right)\right|\right]\mathrm{d}\,t$$
$$\le \frac{|\theta(a,b)|}{4}\left\{\left(\int_0^1 (1+t)^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1\left|f'\left(b+\frac{1+t}{2}\theta(a,b)\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right.$$
$$\left.+\left(\int_0^1 t^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1\left|f'\left(b+\frac{t}{2}\theta(a,b)\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right\}$$
$$\le \frac{|\theta(a,b)|}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left\{\left(2^{\frac{2q-1}{q-1}}-1\right)^{1-1/q}\left[\int_0^1\left(\left(\frac{1+t}{2}\right)^\alpha|f'(a)|^q\right.\right.\right.$$
$$\left.\left.+\left(1-\left(\frac{1+t}{2}\right)^\alpha\right)|f'(b)|^q - \frac{1-t^2}{4}c(a-b)^2\right)\mathrm{d}\,t\right]^{1/q}$$
$$\left.+\left[\int_0^1 t\left(\left(\frac{t}{2}\right)^\alpha|f'(a)|^q + \left(1-\left(\frac{t}{2}\right)^\alpha\right)|f'(b)|^q - \frac{2t-t^2}{4}c(a-b)^2\right)\mathrm{d}\,t\right]^{1/q}\right\}$$
$$= \frac{|\theta(a,b)|}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{(\alpha+1)2^\alpha}\right)^{1-1/q}\left\{\left(2^{\frac{2q-1}{q-1}}-1\right)^{1-1/q}\right.$$

$$\times\left[(2^\alpha-1)|f'(a)|^q+(2^\alpha\alpha+1)|f'(b)|^q-3^{-1}\times 2^{\alpha-1}(\alpha+1)c(a-b)^2\right]^{1/q}$$
$$+\left[|f'(a)|^q+(2^\alpha(\alpha+1)-1)|f'(b)|^q-3^{-1}\times 2^{\alpha-1}(\alpha+1)c(a-b)^2\right]^{1/q}\bigg\}.$$

The proof of Theorem 3.2 is complete. □

Corollary 2. *Under the conditions of Theorem 3.2, if $\alpha = 1$, we have*

$$\left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}\,x - f(b+\theta(a,b))\right|$$
$$\le \frac{|\theta(a,b)|}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{4}\right)^{1-1/q}\left\{\left(2^{\frac{2q-1}{q-1}}-1\right)^{1-1/q}\right.$$
$$\times\left[|f'(a)|^q+3|f'(b)|^q-3^{-1}\times 2c(a-b)^2\right]^{1/q}$$
$$\left.+\left[|f'(a)|^q+3|f'(b)|^q-3^{-1}\times 2c(a-b)^2\right]^{1/q}\right\}.$$

Theorem 3.3. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta\colon A\times A\to\mathbb{R}$, $a,b\in A$ with $\theta(a,b)\ne 0$. Suppose $f\colon A\to\mathbb{R}$ be a differentiable function and f' is integrable on the θ-path P_{bc}, $c=b+\theta(a,b)$, some $\alpha\in(0,1]$. If $|f'|^q$ is strongly α-preinvex on A for $q\ge 1$, then*

$$\left|f(b)-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}\,x\right| \tag{3.4}$$
$$\le \frac{|\theta(a,b)|}{8}\left(\frac{1}{2^{\alpha-1}(\alpha+1)(\alpha+2)}\right)^{1/q}\Big\{[(2^{\alpha+3}-2^{\alpha+2}-\alpha-3)|f'(a)|^q$$
$$+[2^{\alpha-1}(\alpha+1)(\alpha+2)-(2^{\alpha+3}-2^{\alpha+2}-\alpha-3)]|f'(b)|^q$$
$$-5\times 2^{\alpha-4}\times 3^{-1}(\alpha+1)(\alpha+2)c(a-b)^2]^{1/q}+3^{1-1/q}[(\alpha+3)|f'(a)|^q$$
$$+(3\times 2^{\alpha-1}(\alpha+1)(\alpha+2)-\alpha-3)|f'(b)|^q$$
$$-11\times 3^{-1}\times 2^{\alpha-4}(\alpha+1)(\alpha+2)c(a-b)^2]^{1/q}\Big\}.$$

Proof. Since $b+t\theta(a,b)\in A$ for every $t\in[0,1]$, by Lemma 2.2, Hölder's inequality, we have

$$\left|f(b)-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}\,x\right|$$
$$\le \frac{|\theta(a,b)|}{4}\int_0^1\left[(1-t)\left|f'\left(b+\frac{1+t}{2}\theta(a,b)\right)\right|+(2-t)\left|f'\left(b+\frac{t}{2}\theta(a,b)\right)\right|\right]\mathrm{d}\,t$$
$$\le \frac{|\theta(a,b)|}{4}\left\{\left(\int_0^1(1-t)\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1(1-t)\left|f'\left(b+\frac{1+t}{2}\theta(a,b)\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right.$$
$$\left.+\left(\int_0^1(2-t)\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1(2-t)\left|f'\left(b+\frac{t}{2}\theta(a,b)\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right\}$$

$$\le \frac{|\theta(a,b)|}{4}\bigg\{\Big(\frac{1}{2}\Big)^{1-1/q}\bigg[\int_0^1(1-t)\bigg(\Big(\frac{1+t}{2}\Big)^{\alpha}|f'(a)|^q+\bigg(1-\Big(\frac{1+t}{2}\Big)^{\alpha}\bigg)|f'(b)|^q$$
$$-\frac{1-t^2}{4}c(a-b)^2\bigg)\,\mathrm{d}\,t\bigg]^{1/q}+\Big(\frac{3}{2}\Big)^{1-1/q}\bigg[\int_0^1(2-t)\bigg(\Big(\frac{t}{2}\Big)^{\alpha}|f'(a)|^q$$
$$+\bigg(1-\Big(\frac{t}{2}\Big)^{\alpha}\bigg)|f'(b)|^q-\frac{2t-t^2}{4}c(a-b)^2\bigg)\,\mathrm{d}\,t\bigg]^{1/q}\bigg\}$$
$$=\frac{|\theta(a,b)|}{8}\bigg(\frac{1}{2^{\alpha-1}(\alpha+1)(\alpha+2)}\bigg)^{1/q}\Big\{[(2^{\alpha+3}-2^{\alpha+2}-\alpha-3)|f'(a)|^q$$
$$+[2^{\alpha-1}(\alpha+1)(\alpha+2)-(2^{\alpha+3}-2^{\alpha+2}-\alpha-3)]|f'(b)|^q$$
$$-5\times2^{\alpha-4}\times3^{-1}(\alpha+1)(\alpha+2)c(a-b)^2]^{1/q}+3^{1-1/q}[(\alpha+3)|f'(a)|^q$$
$$+(3\times2^{\alpha-1}(\alpha+1)(\alpha+2)-\alpha-3)|f'(b)|^q$$
$$-11\times3^{-1}\times2^{\alpha-4}(\alpha+1)(\alpha+2)c(a-b)^2]^{1/q}\Big\}.$$

The proof of Theorem 3.3 is complete. □

Corollary 3. *Under the conditions of Theorem 3.3, if $\alpha=q=1$, we have*

$$\bigg|f(b)-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}\,x\bigg|\le\frac{|\theta(a,b)|}{24}\big[4|f'(a)|^q+7|f'(b)|^q-2c(a-b)^2\big].$$

Theorem 3.4. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\theta\colon A\times A\to\mathbb{R}$, $a,b\in A$ with $\theta(a,b)\neq0$. Suppose $f\colon A\to\mathbb{R}$ be a differentiable function and f' is integrable on the θ-path P_{bc}, $c=b+\theta(a,b)$,some $\alpha\in(0,1]$. If $|f'|^q$ is strongly α-preinvex on A for $q>1$, then*

$$\text{(3.5)}\quad\bigg|f(b)-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}\,x\bigg|$$
$$\le\frac{|\theta(a,b)|}{8}\bigg(\frac{1}{2^{\alpha-1}(\alpha+1)}\bigg)^{1/q}\Big\{[(2^{\alpha+1}-1)|f'(a)|^q+(2^{\alpha}(\alpha+1)$$
$$+3^{1-1/q}[|f'(a)|^q-(2^{\alpha+1}-1))|f'(b)|^q-3^{-1}\times2^{\alpha-1}(\alpha+1)c(a-b)^2]^{1/q}$$
$$+(2^{\alpha}(\alpha+1)-1)|f'(b)|^q-3^{-1}\times2^{\alpha-1}(\alpha+1)c(a-b)^2]^{1/q}\Big\}.$$

Proof. Since $b+t\theta(a,b)\in A$ for every $t\in[0,1]$, by Lemma 2.2, Hölder's inequality, we have

$$\bigg|f(b)-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}\,x\bigg|$$
$$\le\frac{|\theta(a,b)|}{4}\int_0^1\bigg[(1-t)\bigg|f'\bigg(b+\frac{1+t}{2}\theta(a,b)\bigg)\bigg|+(2-t)\bigg|f'\bigg(b+\frac{t}{2}\theta(a,b)\bigg)\bigg|\bigg]\,\mathrm{d}\,t$$
$$\le\frac{|\theta(a,b)|}{4}\bigg\{\bigg(\int_0^1(1-t)^{q/(q-1)}\,\mathrm{d}\,t\bigg)^{1-1/q}\bigg[\int_0^1\bigg|f'\bigg(b+\frac{1+t}{2}\theta(a,b)\bigg)\bigg|^q\,\mathrm{d}\,t\bigg]^{1/q}$$

$$+\left(\int_0^1(2-t)^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1\left|f'\left(b+\frac{t}{2}\theta(a,b)\right)\right|^q\mathrm{d}\,t\right]^{1/q}\Bigg\}$$
$$\le\frac{|\theta(a,b)|}{4}\left\{\left(\frac{1}{2}\right)^{1-1/q}\left[\int_0^1\left(\left(\frac{1+t}{2}\right)^{\alpha}|f'(a)|^q+\left(1-\left(\frac{1+t}{2}\right)^{\alpha}\right)|f'(b)|^q\right.\right.\right.$$
$$\left.\left.-\frac{1-t^2}{4}c(a-b)^2\right)\mathrm{d}\,t\right]^{1/q}+\left(\frac{3}{2}\right)^{1-1/q}\left[\int_0^1\left(\left(\frac{t}{2}\right)^{\alpha}|f'(a)|^q\right.\right.$$
$$\left.\left.\left.+\left(1-\left(\frac{t}{2}\right)^{\alpha}\right)|f'(b)|^q-\frac{2t-t^2}{4}c(a-b)^2\right)\mathrm{d}\,t\right]^{1/q}\right\}$$
$$=\frac{|\theta(a,b)|}{8}\left(\frac{1}{2^{\alpha-1}(\alpha+1)}\right)^{1/q}\Big\{[(2^{\alpha+1}-1)|f'(a)|^q+(2^{\alpha}(\alpha+1)$$
$$-(2^{\alpha+1}-1))|f'(b)|^q-3^{-1}\times2^{\alpha-1}(\alpha+1)c(a-b)^2]^{1/q}+3^{1-1/q}[|f'(a)|^q$$
$$+(2^{\alpha}(\alpha+1)-1)|f'(b)|^q-3^{-1}\times2^{\alpha-1}(\alpha+1)c(a-b)^2]^{1/q}\Big\}.$$

The proof of Theorem 3.4 is complete. □

Corollary 4. *Under the conditions of Theorem 3.4, if* $\alpha=1$*, we have*

$$\left|f(b)-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}\,x\right|$$
$$\le\frac{|\theta(a,b)|}{8}\left(\frac{1}{2}\right)^{1/q}\Big\{[3|f'(a)|^q+3|f'(b)|^q-3^{-1}\times2c(a-b)^2]^{1/q}$$
$$+3^{1-1/q}[|f'(a)|^q+3|f'(b)|^q-3^{-1}\times2c(a-b)^2]^{1/q}\Big\}.$$

Remark. On 8 April 2014, Professor S. S. Dragomir, Australia pointed out that there are errors appeared in the paper [9] as follows. Definition 2.1 from [9] has no meaning if i stands for the imaginary unit. So, no inequalities like in the hypothesis of Theorem 3.1 or eq. (3.1) and others can be stated for general φ.

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