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A CLASS OF FINSLER METRICS WITH ISOTROPIC MEAN BERWALD CURVATURE

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Abstract. In this paper, we find a condition on (α, β)-metrics under which the notions of isotropic S-curvature, weakly isotropic S-curvature and isotropic mean Berwald curvature are equivalent.

1. Introduction

The S-curvature is introduced by Shen for a comparison theorem on Finsler manifolds [8]. Recent studies show that the S-curvature plays a very important role in Finsler geometry [11, 12]. A Finsler metric F is said to have isotropic S-curvature if $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on an n-dimensional manifold M.

Taking twice vertical covariant derivatives of the S-curvature gives rise the mean Berwald curvature. A Finsler metric F with vanishing mean Berwald curvature is called weakly Berwald metric. In [1], Bácsó and Yoshikawa studied some weakly Berwald metrics. Also, F is called to have isotropic mean Berwald curvature if $\mathbf{E} = \frac{n+1}{2} cF^{-1}\mathbf{h}$, for some scalar function c on M, where $\mathbf{h}$ is the angular metric. It is easy to see that every Finsler metric of isotropic S-curvature is of isotropic mean Berwald curvature. Now, is the equation $\mathbf{S} = (n+1)cF$ equivalent to the equation $\mathbf{E} = \frac{n+1}{2} cF^{-1}\mathbf{h}$?

Recently, Cheng and Shen proved that a Randers metric $F = \alpha + \beta$ is of isotropic S-curvature if and only if it is of isotropic mean Berwald curvature [2]. Then Xiang and Cheng extended this equivalency to the Finsler metric $F = \alpha^{-m}(\alpha+\beta)^{m+1}$ for every real constant m, including Randers metric [13]. In [7] Lee and Lee proved that this notions are equivalent for the Finsler metrics in the form $F = \alpha + \alpha^{-1}\beta^2$.

All of above metrics are special Finsler metrics so- called (α, β)-metrics. An (α, β)-metric is a scalar function on TM defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$ where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on a manifold M. A natural question arises:

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Is being of isotropic S-curvature equivalent to being of isotropic mean Berwald curvature for (α, β)-metrics?

In [6] Deng and Wang found the formula of the S-curvature of homogeneous (α, β)-metrics. Then Cheng and Shen classified (α, β)-metrics of isotropic S-curvature [3].

Let $F = \alpha\phi(s)$ be an (α, β)-metric on a manifold M of dimension n, where $s = \frac{\beta}{\alpha}$, $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. For an (α, β)-metric, put

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'}, \\ \Delta &= 1 + sQ + (b^2 - s^2)Q', \\ \Phi &= -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q'', \\ \Xi &= \frac{(b^2 Q + s)\Phi}{\Delta^2}. \end{aligned}$$

Using the same method as in [3], we give an affirmative answer to the above question for almost all (α, β)-metrics. More precisely, we prove the following.

Theorem 1.1. *Let $F = \alpha\phi(s)$ be an (α, β)-metric, where $s = \frac{\beta}{\alpha}$. Suppose that Ξ is not constant. Then F is of isotropic S-curvature if and only if it is of isotropic mean Berwald curvature.*

It is remarkable that if $\Xi = 0$, then F reduces to a Riemannian metric. But, in general, it is still an open problem if Theorem 1.1 is true when Ξ is a constant.

Example 1.2. The above mentioned (α, β)-metric correspond to $\phi = 1 + s$, $\phi = (1 + s)^{m+1}$ and $\phi = 1 + s^2$, respectively. Using a Maple program shows that for all these metrics Ξ is not constant.

2. Preliminaries

Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold M. There is a notion of distortion $\tau = \tau(x, y)$ on TM associated with a volume form $dV = \sigma(x)dx$, which is defined by

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

Then the S-curvature is defined by

$$\mathbf{S}(x, y) = \frac{d}{dt}\Big[\tau\Big(c(t), \dot{c}(t)\Big)\Big]\Big|_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$ [5, 10]. From the definition, we see that the S-curvature $\mathbf{S}(x, y)$ measures the rate of change in the distortion on (T_xM, F_x) in the direction $y \in T_xM$.

Let $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ denote the spray of F and $dV_{BH} = \sigma(x)dx$ be the Busemann-Hausdorff volume form on M, where the spray coefficients G^i are defined by

$$G^i(x,y) := \frac{1}{4} g^{il}(x,y) \left\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^l}(x,y) y^k - \frac{\partial [F^2]}{\partial x^l}(x,y) \right\}, \quad y \in T_xM.$$

Then the S-curvature is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}(\ln \sigma).$$

The mean Berwald curvature $\mathbf{E} = E_{ij} dx^i \otimes dx^j$ is given by

$$E_{ij} = \frac{1}{2} \frac{\partial^2 S}{\partial y^i \partial y^j}.$$

Definition 2.1. Let (M, F) be an n-dimensional Finsler manifold. Then

(a) F is of isotropic S-curvature if $\mathbf{S} = (n+1)cF$,
(b) F is of weak isotropic S-curvature if $\mathbf{S} = (n+1)cF + \eta$,
(c) F is of isotropic mean Berwald curvature if $\mathbf{E} = \frac{n+1}{2} c F^{-1} \mathbf{h}$,

where $c = c(x)$ is a scalar function on M, $\eta = \eta_i(x) y^i$ is a 1-form on M and $\mathbf{h}$ is the angular metric [9].

Consider the (α, β)-metric $F = \alpha \phi \left(\frac{\beta}{\alpha} \right)$ where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on a manifold M. For an (α, β)-metric, put

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_j := b^i r_{ij}, \ s_j := b^i s_{ij}, \ r_{i0} := r_{ij} y^j, \quad s_{i0} := s_{ij} y^j, \ r_0 := r_j y^j, \ s_0 := s_j y^j.$$

Let $\bar{G}^i$ denote the spray coefficients of α. We have the following formula for the spray coefficients G^i of F [5]:

$$G^i = \bar{G}^i + \alpha Q s^i{}_0 + \Theta \Big\{ -2Q\alpha s_0 + r_{00} \Big\} \frac{y^i}{\alpha} + \Psi \Big\{ -2Q\alpha s_0 + r_{00} \Big\} b^i,$$

where $s^i{}_j := a^{ih} s_{hj}, s^i{}_0 := s^i{}_j y^j$ and $r_{00} := r_{ij} y^i y^j$. In [3], Cheng-Shen found the S-curvature as follows

$$\mathbf{S} = \left\{ 2\Psi - \frac{f'(b)}{bf(b)} \right\} (r_0 + s_0) - \alpha^{-1} \frac{\Phi}{2\Delta^2} (r_{00} - 2\alpha Q s_0), \tag{1}$$

where

$$Q = \frac{\phi'}{\phi - s\phi'}, \quad \Delta = 1 + sQ + (b^2 - s^2) Q', \quad \Psi = \frac{Q'}{2\Delta}$$
$$\Phi = -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ) Q''.$$

Recently, Cheng and Shen characterized (α, β)-metrics with isotropic S-curvature and proved the following.

Lemma 2.2 ([3]). *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on an n-manifold. Then, F is of isotropic S-curvature $\mathbf{S} = (n+1)cF$, if and only if one of the following holds*

(i) *β satisfies*

$$r_{ij} = \varepsilon\Big\{b^2 a_{ij} - b_i b_j\Big\}, \quad s_j = 0, \tag{2}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{3}$$

where k is a constant. In this case, $c = k\epsilon$.

(ii) *β satisfies*

$$r_{ij} = 0, \quad s_j = 0. \tag{4}$$

In this case, $c = 0$.

It is remarkable that Cheng, Wang and Wang proved that the condition $\Phi = 0$ characterizes the Riemannian metrics among (α, β)-metrics [4]. Hence, in the continue, we suppose that $\Phi \neq 0$.

3. Proof of Theorem 1.1

First, we find the formula of mean Berwald curvature of (α, β)-metrics. After a long and tedious computation, we obtain the following.

Proposition 3.1. *Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be an (α, β)-metric. Put $\Omega := \frac{\Phi}{2\Delta^2}$. Then the mean Berwald curvature of F is given by the following*

$$\begin{aligned} E_{ij} = {} & C_1 b_i b_j + C_2(b_i y_j + b_j y_i) + C_3 y_i y_j + C_4 a_{ij} + C_5(r_{i0}b_j + r_{j0}b_i) \\ & + C_6(r_{i0}y_j + r_{j0}y_i) + C_7 r_{ij} + C_8(s_i b_j + s_j b_i) + C_9(s_i y_j + s_j y_i) \\ & + C_{10}(r_i b_j + r_j b_i) + C_{11}(r_i y_j + r_j y_i), \end{aligned} \tag{5}$$

where

$$\begin{aligned} C_1 := {} & \frac{1}{2\alpha^3\Delta^2}\Big\{\Phi\alpha Q'' s_0 + 2\alpha\Delta^2\Psi'' r_0 - \Delta^2\Omega'' r_0 + 2\Delta^2\alpha\Omega'' Q s_0 \\ & + 4\Delta^2\alpha\Omega' Q' s_0 + 2\alpha\Delta^2\Psi'' s_0\Big\}, \\ C_2 := {} & \frac{-1}{2\alpha^4\Delta^2}\Big\{2\alpha\Delta^2\Psi'' s_0 - 2\Omega'\Delta^2 r_0 + 2\Omega'\Delta^2\alpha Q s_0 - \Delta^2\Omega'' s r_0 \\ & + 2\Delta^2\alpha\Omega'' s Q s_0 + 4\Delta^2\alpha\Omega' Q' s_0 s + 2\alpha\Delta^2\Psi' r_0 + 2\alpha\Delta^2\Psi'' s r_0 \\ & + 2\alpha\Delta^2\Psi'' s s_0 + \Phi\alpha Q' s_0 + \Phi\alpha Q'' s_0 s\Big\}, \\ C_3 := {} & \frac{1}{4\alpha^5\Delta^2}\Big\{4\Delta^2 s^2\Omega''\alpha Q s_0 - 2\Delta^2 s^2\Omega'' r_0 + 12\alpha\Delta^2\Psi' s r_0 + 12\alpha\Delta^2\Psi' s s_0 \\ & + 4\alpha\Delta^2\Psi'' s^2 r_0 + 4\alpha\Delta^2\Psi'' s^2 s_0 + 8\Delta^2 s^2\Omega'\alpha Q' s_0 + 2\Phi\alpha Q'' s_0 s^2 \\ & - 10\Omega'\Delta^2 s r_0 + 12\Omega'\Delta^2 s\alpha Q s_0 + 6\Phi\alpha Q' s_0 s - 3\Phi r_0\Big\}, \end{aligned}$$

$$C_4 := \frac{-1}{4\alpha^3\Delta^2}\Big\{4\alpha\Delta^2\Psi' s s_0 - \Phi r_0 - 2\Omega'\Delta^2 s r_0 + 4\Omega'\Delta^2 s\alpha Q s_0 + 4\alpha\Delta^2\Psi' s r_0 + 2\Phi\alpha Q' s_0 s\Big\},$$

$$C_5 := \frac{-\Omega'}{\alpha^2}, \quad C_6 := \frac{2\Delta^2 s\Omega' + \Phi}{2\alpha^3\Delta^2}, \quad C_7 := \frac{-\Phi}{2\alpha\Delta^2},$$

$$C_8 := \frac{1}{2\alpha\Delta^2}\{2\Omega'\Delta^2 Q + 2\Delta^2\Psi' + \Phi Q'\},$$

$$C_9 := \frac{-s}{\alpha}C_8, \quad C_{10} := \frac{\Psi'}{\alpha}, \quad C_{11} := \frac{-s}{\alpha}C_{10}.$$

The formula of mean Berwald curvature of Randers metrics and Kropina metrics computed from Proposition 3.1 coincides with the one computed in [1].

It is easy to see that F is of isotropic mean Berwald curvature if and only if F is of weak isotropic S-curvature. Hence, we consider an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ with weak isotropic S-curvature, $\mathbf{S} = (n+1)cF + \eta$, where $\eta = \eta_i(x)y^i$ is a 1-form on underlying manifold M. Using the same method used in [3], one can obtain that the condition that F is of weak isotropic S-curvature $\mathbf{S} = (n+1)cF + \eta$ is equivalent to the following equation

$$\alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0) - 2\Psi(r_0 + s_0) = -(n+1)cF + \widetilde{\theta}, \tag{6}$$

where

$$\widetilde{\theta} := -\frac{f'(b)}{bf(b)}(r_0 + s_0) - \eta. \tag{7}$$

To simplify the equation (6), we choose special coordinates $\psi\colon (s, u^A) \to (y^i)$ as follows

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad y^A = u^A, \tag{8}$$

where

$$\bar{\alpha} = \sqrt{\sum_{A=2}^{n}(u^A)^2}.$$

Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}. \tag{9}$$

Fix an arbitrary point x. Take a local coordinate system at x as in (8). We have

$$r_1 = br_{11}, \quad r_A = br_{1A},$$

$$s_1 = 0, \quad s_A = bs_{1A}.$$

Let

$$\bar{r}_{10} := \sum_{A=2}^{n} r_{1A} y^A, \quad \bar{s}_{10} := \sum_{A=2}^{n} s_{1A} y^A, \quad \bar{r}_{00} := \sum_{A,B=2}^{n} r_{AB} y^A y^B,$$

$$\bar{r}_0 := \sum_{A=2}^{n} r_A y^A, \quad \bar{s}_0 := \sum_{A=2}^{n} s_A y^A.$$

Put

$$\widetilde{\theta} = t_i y^i - \eta_i y^i.$$

Then t_i are given by

$$t_1 = -\frac{f'(b)}{f(b)} r_{11}, \quad t_A = -\frac{f'(b)}{f(b)} (r_{1A} + s_{1A}). \tag{10}$$

From (8), we have

$$r_0 = \frac{s b r_{11}}{\sqrt{b^2 - s^2}} \bar{\alpha} + b \bar{r}_{10}, \quad s_0 = \bar{s}_0 = b \bar{s}_{10}, \tag{11}$$

and

$$r_{00} = \frac{s^2 \bar{\alpha}^2}{b^2 - s^2} r_{11} + 2 \frac{s \bar{\alpha}}{\sqrt{b^2 - s^2}} \bar{r}_{10} + \bar{r}_{00}, \tag{12}$$

$$\widetilde{\theta} = t_1 \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha} - \frac{f'(b)}{f(b)} \bar{r}_{10} - \frac{f'(b)}{f(b)} \bar{s}_{10} - \eta. \tag{13}$$

Substituting (11), (12) and (13) into (6) and by using (9), we find that (6) is equivalent to the following equations:

$$\frac{\Phi}{2\Delta^2}(b^2 - s^2)\bar{r}_{00} = -\left\{ s \left(\frac{s\Phi}{2\Delta^2} - 2\Psi b^2 \right) r_{11} + (n+1) c b^2 \phi - s b t_1 \right\} \bar{\alpha}^2, \tag{14}$$

$$\left(\frac{s\Phi}{\Delta^2} - 2\Psi b^2 \right) (r_{1A} + s_{1A}) - (b^2 Q + s) \frac{\Phi}{\Delta^2} s_{1A} + b \eta_A - b t_A = 0. \tag{15}$$

$$\eta_1 = 0. \tag{16}$$

Let

$$\Upsilon := \left[\frac{s\Phi}{\Delta^2} - 2\Psi b^2 \right]'.$$

We see that $\Upsilon = 0$ if and only if

$$\frac{s\Phi}{\Delta^2} - 2\Psi b^2 = b^2 \mu,$$

where $\mu = \mu(x)$ is independent of s.

Let us suppose that $\Xi = \frac{(b^2 Q + s)\Phi}{\Delta^2}$ is not constant. Now we shall divide the proof into two cases:

(i) $\Upsilon = 0$ and (ii) $\Upsilon \neq 0$.

3.1. $\Upsilon = 0$**.** First, note that $\Upsilon = 0$ implies that

$$\frac{s\Phi}{\Delta^2} - 2\Psi b^2 = b^2\mu, \tag{17}$$

where $\mu = \mu(x)$ is a function on M independent of s. First, we prove the following.

Lemma 3.2. *Let (M, F) be an n-dimensional Finsler manifold. Suppose that $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric and $\Upsilon = 0$. If F has weak isotropic S-curvature, $\mathbf{S} = (n+1)cF + \eta$, then β satisfies*

$$r_{ij} = ka_{ij} - \varepsilon b_i b_j + \frac{1}{b^2}(r_i b_j + r_j b_i), \tag{18}$$

where $k = k(x)$, $\varepsilon = \varepsilon(x)$, and $\phi = \phi(s)$ satisfies the following ODE:

$$(k - \epsilon s^2)\frac{\Phi}{2\Delta^2} = \Big\{\nu + (k - \varepsilon b^2)\mu\Big\}s - (n+1)c\phi, \tag{19}$$

where $\nu = \nu(x)$. If $s_0 \neq 0$, then ϕ satisfies the following additional ODE:

$$\frac{\Phi}{\Delta^2}(Qb^2 + s) = b^2(\mu + \lambda), \tag{20}$$

where $\lambda = \lambda(x)$.

Proof. Since $\Phi \neq 0$ and $\bar{r}_{00}$ and $\bar{\alpha}$ are independent of s, it follows from (14) and (15) that in a special coordinate system (s, y^a) at a point x, the following relations hold

$$r_{AB} = k\delta_{AB}, \tag{21}$$

$$s\left(\frac{s\Phi}{2\Delta^2} - 2\Psi b^2\right)r_{11} + (n+1)cb^2\phi + k\frac{\Phi}{2\Delta^2}(b^2 - s^2) = bst_1, \tag{22}$$

$$\left(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\right)(r_{1A} + s_{1A}) - (b^2Q + s)\frac{\Phi}{\Delta^2}s_{1A} - bt_A = -b\eta_A, \tag{23}$$

where $k = k(x)$ is independent of s. Let

$$r_{11} = -(k - \epsilon b^2).$$

Then (18) holds. By (17), we have

$$\frac{s\Phi}{2\Delta^2} - 2\Psi b^2 = b^2\mu - \frac{s\Phi}{2\Delta^2}.$$

Then (22) and (23) become

$$b(k - \varepsilon s^2)\frac{\Phi}{2\Delta^2} = st_1 + sb\mu(k - b^2\varepsilon) - (n+1)cb\phi. \tag{24}$$

$$b^2\mu(r_{1A} + s_{1A}) - \frac{\Phi}{\Delta^2}(Qb^2 + s)s_{1A} - bt_A = -b\eta_A. \tag{25}$$

Letting $t_1 = b\nu$ in (24) we get (19). Now, suppose that $s_0 \neq 0$. Rewrite (25) as

$$\Big\{b^2\mu - \frac{\Phi}{\Delta^2}(Qb^2+s)\Big\}\, s_{1A} = bt_A - b\eta_A - b^2\mu r_{1A}.$$

We can see that there is a function $\lambda = \lambda(x)$ on M such that

$$\mu b^2 - \frac{\Phi}{\Delta^2}(Qb^2+s) = -b^2\lambda.$$

This gives (20).

□

Lemma 3.3 ([3]). *Let $F = \alpha\phi(\beta/\alpha)$ be an (α,β)-metric. Assume that*

$$\phi \neq k_1\sqrt{1+k_2s^2} + k_3s$$

for any constants $k_1 > 0, k_2$ and k_3. If $\Upsilon = 0$, then $b = constant$.

An (α,β)-metric is called Randers-type if $\phi = k_1\sqrt{1+k_2s^2} + k_3s$ for any constants $k_1 > 0, k_2$ and k_3. Now, we consider the equivalency of the notions weak isotropic S-curvature and isotropic S-curvature for a non-Randers type (α,β)-metric.

Lemma 3.4. *Let $F = \alpha\phi(\beta/\alpha)$ be a non-Randers type (α,β)-metric. Suppose that Ξ is not constant and $\Upsilon = 0$. Then F is of weak isotropic S-curvature if and only if F is of isotropic S-curvature.*

Proof. It is sufficient to prove that if F is of weak isotropic S-curvature, then F is of isotropic S-curvature. By $db = (r_0 + s_0)/b$ and Lemma 3.3, we have

$$r_0 + s_0 = 0.$$

Then by the formula of S-curvature of an (α,β)-curvature, we get

$$\mathbf{S} = -\alpha^{-1}\frac{\Phi}{2\Delta^2}\Big\{r_{00} - 2\alpha Q s_0\Big\}.$$

By Lemma 3.2,

$$r_{00} = (k - \varepsilon s^2)\alpha^2 + \frac{2s}{b^2}r_0\alpha.$$

Then

$$\mathbf{S} = -(k-\varepsilon s^2)\frac{\Phi}{2\Delta^2}\alpha + \frac{\Phi}{b^2\Delta^2}(b^2Q+s)s_0.$$

By (19), we have

$$\mathbf{S} = -s\Big\{\nu + (k-\varepsilon b^2)\mu\Big\}\alpha + \frac{\Phi}{b^2\Delta^2}(b^2Q+s)s_0 + (n+1)c\phi\alpha. \tag{26}$$

Since $\mathbf{S} = (n+1)cF + \eta$, then by (26) we obtain the following

$$-s\Big\{\nu + (k-\varepsilon b^2)\mu\Big\}\alpha + \frac{\Phi}{b^2\Delta^2}(b^2Q+s)s_0 = \eta. \tag{27}$$

Letting $y^i = \delta b^i$ for a sufficiently small $\delta > 0$ yields

$$-\delta\Big\{\nu + (k - \varepsilon b^2)\mu\Big\}b^2 = \delta\eta_i b^i.$$

It is easy to see that in the special coordinate $\eta_i b^i = 0$, hence in general $\eta_i b^i = 0$. We conclude that

$$\nu + (k - \varepsilon b^2)\mu = 0. \tag{28}$$

Then (27) reduces to

$$\frac{\Xi}{b^2}s_0 = \eta. \tag{29}$$

If $s_0 \neq 0$, then from the last equation, we obtain that Ξ is constant, which is excluded here. Hence, we have $s_0 = 0$. Thus by (29), we conclude that $\eta = 0$ and F has isotropic S-curvature $\mathbf{S} = (n+1)cF$. □

3.2. $\Upsilon \neq 0$**.** Here, we consider the case when $\phi = \phi(s)$ satisfies

$$\Upsilon \neq 0 \tag{30}$$

We need the following two lemmas. The proofs mainly follow the proof of Lemma 6.1 and Lemma 6.2 in [3], respectively. Thus we omit the proofs.

Lemma 3.5. *Let $F = \alpha\phi(s), s = \beta/\alpha$, be an (α,β)-metric on an n-dimensional manifold. Assume that $\Upsilon \neq 0$. Suppose that F has weak isotropic S-curvature, $\mathbf{S} = (n+1)cF + \eta$. Then*

$$r_{ij} = ka_{ij} - \varepsilon b_i b_j - \lambda(s_i b_j + s_j b_i), \tag{31}$$

where $\lambda = \lambda(x), k = k(x)$ and $\varepsilon = \varepsilon(x)$ are scalar functions of x and

$$-2s(k - \varepsilon b^2)\Psi + (k - \varepsilon s^2)\frac{\Phi}{2\Delta^2} + (n+1)c\phi - s\nu = 0, \tag{32}$$

where

$$\nu := -\frac{f'(b)}{bf(b)}(k - \varepsilon b^2). \tag{33}$$

If in addition $s_0 \neq 0$, i.e., $s_{A_o} \neq 0$ for some A_o, then

$$-2\Psi - \frac{Q\Phi}{\Delta^2} - \lambda\left(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\right) = \delta, \tag{34}$$

where

$$\delta := -\frac{f'(b)}{bf(b)}(1 - \lambda b^2) - \frac{\eta_{A_o}}{s_{A_o}}. \tag{35}$$

Lemma 3.6. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric. Suppose that $\phi = \phi(s)$ satisfies (30) and $\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$ for any constants $k_1 > 0$, k_2 and k_3. If F has weak isotropic S-curvature, then*

$$r_j + s_j = 0.$$

Proposition 3.7. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric. Suppose that $\phi = \phi(s)$ satisfies (30) and $\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$ for any constants $k_1 > 0$, k_2 and k_3. Suppose that Ξ is not constant. If F is of weak isotropic S-curvature, $\mathbf{S} = (n + 1)cF + \eta$, then*

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \tag{36}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function on M and $\phi = \phi(s)$ satisfies

$$\varepsilon(b^2 - s^2)\frac{\Phi}{2\Delta^2} = -(n + 1)c\phi. \tag{37}$$

Proof. Contracting (31) with b^i yields

$$r_j + s_j = (k - \varepsilon b^2)b_j + (1 - \lambda b^2)s_j. \tag{38}$$

By Lemma 3.6, $r_j + s_j = 0$. It follows from (38) that

$$(1 - \lambda b^2)s_j + (k - \epsilon b^2)b_j = 0. \tag{39}$$

Contracting (39) with b^j yields

$$(k - \epsilon b^2)b^2 = 0.$$

We get

$$k = \epsilon b^2.$$

Then (31) is reduced to

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j) - \lambda(b_i s_j + b_j s_i).$$

By (33),

$$\nu = 0.$$

Then (32) is reduced to (37).

We claim that $s_0 = 0$. Suppose that $s_0 \neq 0$. By (39), we conclude that

$$\lambda = \frac{1}{b^2}.$$

By (35),

$$\delta = -\frac{\eta_{A_o}}{s_{A_o}}.$$

It follows from (34) that

$$\frac{(b^2 Q + s)\Phi}{\Delta^2} = \frac{b\eta_{A_o}}{s_{A_o}},$$

which implies that Ξ is constant. This is impossible by the assumption on non-constancy of Ξ. Therefore, $s_j = 0$. This completes the proof. □

By Proposition 3.7 and Lemma 2.2, we have the following.

Corollary 3.8. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type (α, β)-metric. Suppose that $\Upsilon \neq 0$ and Ξ is not constant. Then F is of weak isotropic S-curvature, if and only if it is of isotropic S-curvature.*

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