*Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*
**33** (2017), 199–206
www.emis.de/journals
ISSN 1786-0091

# EXTENSIONS FOR A REGULAR FUNCTION WITH VALUES IN DUAL QUATERNIONS (CORRIGENDUM)

JI EUN KIM

ABSTRACT. This paper researches properties of a regular function defined on a domain in Clifford analysis. Also, the paper investigates properties of extensions for a regular function with values in $\mathbb{D}_{\mathbb{H}}$ and the corresponding Cauchy-Riemann system for a regular function of dual quaternionic variables.

## 1. Introduction

Dual quaternions generalize the notion of quaternions to an 8-tuple, and provide a convenient representation of rigid body transformations containing both rotations and translations in three-dimensional space. The mathematical structure of dual quaternions uses two quaternions that are combined using the algebra of dual numbers. A dual quaternion can be represented in the form $p+\varepsilon q$, where $p$, $q$ are ordinary quaternions and $\varepsilon$ is the dual unit with $\varepsilon^2=0$. The set of dual quaternions is a Clifford algebra which has the following form:

$$\mathbb{D}_{\mathbb{H}} := \{Z = p_1 + \varepsilon p_2 \mid p_1, p_2 \in \mathbb{H}\}$$

and it is isomorphic with $\mathbb{H}^2$ and $\mathbb{R}^8$, where $\mathbb{H}$ is the set of quaternions whose the basis elements are 1, $i$, $j$ and $k$. It has the product rule for $i$, $j$ and $k$ is written by

$$i^2 = j^2 = k^2 = ijk = -1,$$

and

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

For two quaternions $p = z_1 + z_2 j$ and $q = w_1 + w_2 j$, where $z_1 = x_0 + ix_1$, $z_2 = x_2 + ix_3$, $w_1 = y_0 + iy_1$ and $w_2 = y_2 + iy_3$, the rule of additions and

2010 *Mathematics Subject Classification.* 32A99, 32W50, 30G35, 11E88.

*Key words and phrases.* quaternion; dual number; regular function; Cauchy-Riemann system; Clifford analysis.

Corrigendum to the paper appeared in Vol. 33, No. 1, pp. 31-38 (2017) in this journal. The Editorial Board decided to publish the full version of the corrected paper.

multiplications are:

$$p+q=(z_1+w_1)+(z_2+w_2)j,$$
$$pq=(z_1w_1-z_2\overline{w_2})+(z_1w_2+z_2\overline{w_1})j.$$

From the above rules, we give the norm for a quaternion as follows:

$$|p|^2:=pp^*=z_1\overline{z_1}+z_2\overline{z_2},$$

where $p^*=\overline{z_1}+z_2j$ is the conjugation of $p$ with $z_r$ $(r=1,2)$ are the classical complex numbers. Hamilton [3] introduced quaternions in 1843, and by 1873 Clifford [1] obtained a broad generalization of these numbers that he called biquaternions, which is an example of what is now called a Clifford algebra [13]. Kotelnikov [12] developed dual vectors and dual quaternions. Study [15] realized that this associative algebra was the ideal for describing the group of motions of the three-dimensional space. Favaro [2] introduced spaces of entire functions of $\theta$-holomorphy type of bounded type and proved results involving these spaces. Saric [14] researched a real valued function $F$ defined at the end-points of an interval $[a,b]$ in $\mathbb{R}$ and showed that $\mathcal{KH}-\mathrm{vt}\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{f}=\mathrm{F(b)}-\mathrm{F(a)}$, where $\mathcal{KH}-\mathrm{vt}$ denotes the total value of the Kurzweil-Henstock integral. Kajiwara et al. [4] gave an inhomogeneous Cauchy Riemann system and applied the theory on a closed densely defined operator in a Hilbert space and brconvex domains. Kim et al. [5, 6, 8] obtained the regularity of functions on the reduced quaternion field and compared with hyperholomorphic functions and regular functions on the form of dual split quaternions in Clifford analysis. Also, Kim et al. [7, 9, 10] researched corresponding Cauchy-Riemann systems and properties of functions with values in special quaternions. We [11] investigated the differentiation and integration for regular functions of bicomplex numbers satisfying the commutative multiplicative rule.

This paper gives some differential conditions and operators in $\mathbb{D}_{\mathbb{H}}$. From the definition of these operators, the paper researches properties of a function, called a regular function, defined on a domain in $\mathbb{D}_{\mathbb{H}}$. Also, the paper investigates some results of the extension for functions satisfying conditions of the differentiable cases on $\mathbb{D}_{\mathbb{H}}$ in Clifford analysis.

## 2. Preliminaries

We give a scalar part of $Z$ is

$$Sc(Z)=Sc(p_1)=x_0,$$

a dual part of $Z$ is

$$Du(Z)=p_2=y_0+iy_1+jy_2+ky_3$$

and a vector part of $Z$ is

$$Ve(Z)=Ve(p_1)+Ve(p_2)=i\lambda_1+j\lambda_2+k\lambda_3,$$

where $\lambda_r$ is a dual number $x_r+\varepsilon y_r$ with $x_r,y_r\in\mathbb{R}$ $(r=0,1,2,3)$.

For $Z = p_1+\varepsilon p_2$ and $W = q_1+\varepsilon q_2$, we have the following rules of an addition on $\mathbb{D}_{\mathbb{H}}$:

$$Z + W = (p_1 + q_1) + \varepsilon(p_2 + q_2)$$

and a multiplication on $\mathbb{D}_{\mathbb{H}}$:

$$ZW = p_1q_1 + \varepsilon(p_1q_2 + p_2q_1).$$

We give a complex conjugate element of $\mathbb{D}_{\mathbb{H}}$ as follows:

$$Z^* = p_1^* - \varepsilon p_1^* p_2 p_1^{-1}$$

and then, the module of $Z$, denoted by $M(Z)$, is described by

$$M(Z) = ZZ^* = p_1p_1^* = \sum_{r=0}^{3} x_r^2.$$

Hence, the inverse element of $\mathbb{D}_{\mathbb{H}}$ is

$$Z^{-1} = \frac{p_1^* - \varepsilon p_1^* p_2 p_1^{-1}}{M(Z)} = p_1^{-1} - \varepsilon p_1^{-1} p_2 p_1^{-1}.$$

*Remark* 2.1. Since $p_1p_1^* \in \mathbb{R}$, we have

$$p_1^* p_2 p_1^{-1} = p_1^* p_2 \frac{p_1^*}{p_1p_1^*} = \frac{p_1^*}{p_1p_1^*} p_2 p_1^* = p_1^{-1} p_2 p_1^*.$$

Hence, the module $M(Z)$ satisfies the following equations:

$$M(Z) = ZZ^* = Z^*Z.$$

## 3. Regular functions of dual quaternionic variables

We consider an open subset $\Omega$ of $\mathbb{D}_{\mathbb{H}}$ and a function $F : \Omega \to \mathbb{D}_{\mathbb{H}}$ of class $\mathcal{C}^1(\Omega, \mathbb{D}_{\mathbb{H}})$.

Now, we give the existence of a dual quaternionic derivative defined as the limit of a difference quotient using quaternionic derivative.

**Definition 3.1.** Let $\Omega$ be an open set of $\mathbb{D}_{\mathbb{H}}$. A function $F$ is dual quaternionic diffenrentiable on the left at $Z$ if the limit

$$\frac{dF}{dZ} := \lim_{\mathrm{h}\to 0} \mathrm{h}^{-1}\{F(Z + \mathrm{h}) - F(Z)\} \tag{3.1}$$

exists, where $\mathrm{h} = h_1 + \varepsilon h_2$ which is in the set of no zero divisors in dual quaternions and $\mathrm{h} \to 0$ means each component approaches to zero such that both $h_1 \to 0$ and $h_2 \to 0$.

*Remark* 3.2. The above limit has the form

$$\begin{aligned}
\frac{dF}{dZ} &= \lim_{\mathrm{h}\to 0} \mathrm{h}^{-1}\{F(Z+\mathrm{h}) - F(Z)\} \\
&= \lim_{\mathrm{h}\to 0}(h_1^{-1} - \varepsilon h_1^{-1}h_2h_1^{-1})\{(f_1(p_1+h_1,p_2+h_2) - f_1(p_1,p_2)) \\
&\quad +\varepsilon(f_2(p_1+h_1,p_2+h_2) - f_2(p_1,p_2))\} \\
&= \lim_{\mathrm{h}\to 0}\{h_1^{-1}(f_1(p_1+h_1,p_2+h_2) - f_1(p_1,p_2)) \\
&\quad +\varepsilon\{h_1^{-1}(f_2(p_1+h_1,p_2+h_2) - f_2(p_1,p_2)) \\
&\quad -h_1^{-1}h_2h_1^{-1}(f_1(p_1+h_1,p_2+h_2) - f_1(p_1,p_2))\}.
\end{aligned}$$

If a function $F$ is dual quaternionic diffenrentiable on the left at $Z$, then for any pathes, the limit $\frac{dF}{dZ}$ exists. Hence, for $h_1 \to 0$ and $h_2 = 0$, we have

$$\begin{aligned}
\frac{dF}{dZ} &= \lim_{\mathrm{h}\to 0}\{h_1^{-1}(f_1(p_1+h_1,p_2) - f_1(p_1,p_2)) \\
&\quad +\varepsilon\{h_1^{-1}(f_2(p_1+h_1,p_2) - f_2(p_1,p_2))\}.
\end{aligned}$$

So, we obtain

$$\frac{dF}{dZ} = \frac{\partial f_1}{\partial p_1} + \varepsilon\frac{\partial f_2}{\partial p_1}. \tag{3.2}$$

Also, for $h_1 = h_2$ and $h_1 \to 0$, we have

$$\begin{aligned}
\frac{dF}{dZ} &= \lim_{\mathrm{h}\to 0}\{h_1^{-1}(f_1(p_1+h_1,p_2+h_1) - f_1(p_1,p_2)) \\
&\quad +\varepsilon\{h_1^{-1}(f_2(p_1+h_1,p_2+h_1) - f_2(p_1,p_2)) \\
&\quad -h_1^{-1}(f_1(p_1+h_1,p_2+h_1) - f_1(p_1,p_2))\}.
\end{aligned}$$

and then, we obtain

$$\frac{dF}{dZ} = \frac{\partial f_1}{\partial p_1} + \frac{\partial f_1}{\partial p_2} + \varepsilon\{\frac{\partial f_2}{\partial p_1} + \frac{\partial f_2}{\partial p_2} - \frac{\partial f_1}{\partial p_1} - \frac{\partial f_1}{\partial p_2}\}, \tag{3.3}$$

where $\frac{\partial f_r}{\partial p_t}$ $(r, t = 1, 2)$ are quaternion-differentiable on the left, cited by [16]. Therefore, from the relation between equations (3.2) and (3.3), we have the following corresponding Cauchy-Riemann system:

$$0 = \frac{\partial f_1}{\partial p_2} + \varepsilon\{\frac{\partial f_2}{\partial p_2} - \frac{\partial f_1}{\partial p_1}\},$$

that is,

$$\begin{cases} \dfrac{\partial f_1}{\partial p_2} = 0, \\ \dfrac{\partial f_2}{\partial p_2} = \dfrac{\partial f_1}{\partial p_1}. \end{cases} \tag{3.4}$$

We give a differential, denoted by $dF_Z$, of $F$ at $Z \in \mathbb{D}_{\mathbb{H}}$ such that

$$dF_Z := \frac{\partial f_1}{\partial p_1} dp_1 + \frac{\partial f_1}{\partial p_2} dp_2 + \varepsilon(\frac{\partial f_2}{\partial p_1} dp_1 + \frac{\partial f_2}{\partial p_2} dp_2),$$

where

$$\frac{\partial f_t}{\partial p_1} dp_1 := \sum_{r=0}^{3} \frac{\partial f_t}{\partial x_r} dx_r \quad \text{and} \quad \frac{\partial f_t}{\partial p_2} dp_2 := \sum_{r=0}^{3} \frac{\partial f_t}{\partial y_r} dy_r \qquad (t = 1, 2).$$

**Theorem 3.3.** *Let $\Omega$ be an open subset of $\mathbb{D}_{\mathbb{H}}$. If a function $F$ is dual quaternionic diffenrentiable for the left at $Z$, defined on $\Omega$, then $F$ has the form*

$$F(Z) = A + Z\beta$$

*for $A \in \mathbb{D}_{\mathbb{H}}$ and $\beta \in \mathbb{H}$.*

*Proof.* From the definition 3.1, if $F$ is dual quaternionic diffenrentiable, then the function $F$ satisfies

$$dF_Z(\mathrm{h}) = \mathrm{h}\frac{dF}{dZ},$$

that is,

$$dF_Z = dZ \, \frac{dF}{dZ}.$$

By the existence of the limit (3.1), the limit gives the equations (3.4). Hence, by using relations between the two equations in (3.4) and citing [16], we have

$$\frac{df_1}{dp_1} = \frac{df_1}{dx_0} = -i\frac{df_1}{dx_1} = -j\frac{df_1}{dx_2} = -k\frac{df_1}{dx_3}$$

and

$$\frac{df_2}{dp_2} = \frac{df_2}{dy_0} = -i\frac{df_2}{dy_1} = -j\frac{df_2}{dy_2} = -k\frac{df_2}{dy_3}.$$

Hence, we have

$$f_1 = \alpha + p_1\beta \quad \text{and} \quad f_2 = \gamma + p_2\beta \quad (\alpha, \beta, \gamma \in \mathbb{H}).$$

Therefore, we obtain

$$F(Z) = f_1 + \varepsilon f_2 = A + Z\beta,$$

where $A = \alpha + \varepsilon\gamma \in \mathbb{D}_{\mathbb{H}}$ and $\beta \in \mathbb{H}$ on $\Omega$. □

We give differential operators in $\mathbb{D}_{\mathbb{H}}$.

$$\widetilde{D} := D^*_{p_2} - \varepsilon D^*_{p_1},$$

where

$$D^*_{p_1} = \frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}}, \; D^*_{p_2} = \frac{\partial}{\partial \overline{w_1}} + j\frac{\partial}{\partial \overline{w_2}}$$

with $\partial/\partial\overline{z_r}$ and $\partial/\partial\overline{w_r}$ $(r = 1, 2)$ are usual differential operators in complex analysis. Then the equation (3.4) is equal to the equation $\widetilde{D}F = 0$ in $\mathbb{D}_{\mathbb{H}}$.

**Definition 3.4.** The function $F$ is said to be (L-)regular in $\Omega$ of $\mathbb{D}_{\mathbb{H}}$ if the function $F$ has continuously differentiable components $f_1$ and $f_2$ and satisfies the equation

$$\widetilde{D}F = 0. \tag{3.5}$$

*Example.* Let a function $F(Z) = Z$. Then the function $F$ satisfies

$$\widetilde{D}Z = D^*_{p_2}f_1 + \varepsilon(-D^*_{p_1}f_1 + D^*_{p_2}f_2) = 0.$$

Hence, the function $F(Z) = Z$ is regular in $\Omega$. Also, for a function $F(Z) = Z^n$ $(n \geq 1)$, it satisfies $\widetilde{D}Z^n = 0$. Thus, the function $Z^n$ is regular in $\Omega$. Since $F(Z) = Z^*$ and $F(Z) = Z^{-1}$ don't satisfy the equation (3.5), so both functions are not regular in $\Omega$.

**Theorem 3.5.** *Let $\Omega$ be an open subset of $\mathbb{D}_{\mathbb{H}}$. If a function $F$ satisfies*

$$dZ \wedge dF = 0,$$

*where $dZ = dp_1 + \varepsilon dp_2$ with $dp_1 = dx_0 + idx_1 + jdx_2 + kdx_3$ and $dp_2 = dy_0 + idy_1 + jdy_2 + kdy_3$, then $F$ is regular in $\Omega$.*

*Proof.* Since $F$ satisfies

$$\begin{aligned} 0 = dZ \wedge dF &= (dp_1 + \varepsilon dp_2) \wedge \{D^*_{p_1}f_1dp_1 + D^*_{p_2}f_1dp_2 \\ &\quad + \varepsilon(D^*_{p_1}f_2dp_1 + D^*_{p_2}f_2dp_2)\} \\ &= D^*_{p_2}f_1dp_1 \wedge dp_2 + \varepsilon(D^*_{p_2}f_2dp_1 \wedge dp_2 + D^*_{p_1}f_1dp_2 \wedge dp_1), \end{aligned}$$

we have the system

$$\begin{cases} D^*_{p_2}f_1 = 0, \\ D^*_{p_2}f_2 - D^*_{p_1}f_1 = 0. \end{cases}$$

Therefore, the function $F$ is regular in $\Omega$. $\square$

Consider a function $F = \varphi_1 + \varphi_2 j + \varepsilon(\psi_1 + \psi_2 j)$, where $\varphi_r$ and $\psi_r$ are continuously differential functions with values in $\mathbb{C}$ and the equation $\overline{\partial}G = F$, where

$$\overline{\partial} = \frac{\partial}{\partial\overline{z_1}}d\overline{z_1} + \frac{\partial}{\partial\overline{z_2}}d\overline{z_2} + \frac{\partial}{\partial\overline{w_1}}d\overline{w_1} + \frac{\partial}{\partial\overline{w_2}}d\overline{w_2}.$$

**Theorem 3.6.** *Let $\Omega$ be an open set in $\mathbb{D}_{\mathbb{H}}$. Suppose two functions $F$ and $G$ satisfy the following equation*

$$\overline{\partial} G = F.$$

*Then $F$ is regular if and only if $G$ is regular in $\mathbb{D}_{\mathbb{H}}$.*

*Proof.* Suppose a function $F$ is regular in $\mathbb{D}_{\mathbb{H}}$. Since the equation $\overline{\partial} G = F$ is represented by

$$\begin{aligned}\overline{\partial} G = & \frac{\partial g_1}{\partial \overline{z_1}} d\overline{z_1} + \frac{\partial g_1}{\partial \overline{z_2}} d\overline{z_2} + \frac{\partial g_1}{\partial \overline{w_1}} d\overline{w_1} + \frac{\partial g_1}{\partial \overline{w_2}} d\overline{w_2} \\ & + \varepsilon \Big( \frac{\partial g_2}{\partial \overline{z_1}} d\overline{z_1} + \frac{\partial g_2}{\partial \overline{z_2}} d\overline{z_2} + \frac{\partial g_2}{\partial \overline{w_1}} d\overline{w_1} + \frac{\partial g_2}{\partial \overline{w_2}} d\overline{w_2} \Big),\end{aligned}$$

we have the following equations:

$$\frac{\partial g_1}{\partial \overline{w_1}} = \frac{\partial g_1}{\partial \overline{w_2}} = 0, \ \frac{\partial g_1}{\partial \overline{z_1}} = \frac{\partial g_2}{\partial \overline{w_1}}, \ \frac{\partial g_1}{\partial \overline{z_2}} = \frac{\partial g_2}{\partial \overline{w_2}},$$

by multiplying each the following terms

$$\frac{\partial}{\partial \overline{z_r}} \ \text{and} \ \frac{\partial}{\partial \overline{w_r}} \quad (r = 1, 2).$$

Thus, $G$ satisfies the equation (3.4). Conversely, if a function $G$ is regular, then we have the system which is equal to the equation (3.4), by multiplying

$$\frac{\partial}{\partial \overline{z_r}} \ \text{and} \ \frac{\partial}{\partial \overline{w_r}} \quad (r = 1, 2),$$

respectively. Therefore, we obtain the result. □

**Acknowledgement.** This work was supported by the Dongguk University Research Fund of 2017.

## References


[1] *W. K. Clifford*: Preliminary sketch of bi-quaternions. Proc. London Math. Soc. Vol. **4** (1873), 381-395.

[2] *V. V. Favaro, A. M. Jatoba*: Holomorphy types and spaces of entire functions of bounded type on Banach spaces. Czech. Math. J. **59(4)** (2009), 909–927.

[3] *W. R. Hamilton*: On quaternions, or on a new system of imaginaries in algebra. Philos. Mag. **18** (2000).

[4] *J. Kajiwara, X. D. Li, K. H. Shon*: Function spaces in complex and Clifford analysis. Inhomogeneous Cauchy Riemann system of quaternion and Clifford analysis in ellipsoid, The 14 international conference on Finite or Infinite Dimensional Complex Analysis and its Applications, Hue, Vietnam, Hue University. **14** (2006), 127–155.

[5] *J. E. Kim, S. J. Lim, K. H. Shon*: Regular functions with values in ternary number system on the complex Clifford analysis. Abst. Appl. Anal. **2013** Artical ID 136120 (2013), 7 pages.

[6] *J. E. Kim, S. J. Lim, K. H. Shon*: Regularity of functions on the reduced quaternion field in Clifford analysis. Abst. Appl. Anal. **2014** Artical ID 654798 (2014), 8 pages.
[7] *J. E. Kim, K. H. Shon*: The Regularity of functions on Dual split quaternions in Clifford analysis. Abst. Appl. Anal. **2014** Artical ID 369430 (2014), 8 pages.
[8] *J. E. Kim, K. H. Shon*: Polar Coordinate Expression of Hyperholomorphic Functions on Split Quaternions in Clifford Analysis. Adv. Appl. Clifford Alg. **25(4)** (2015), 915–924.
[9] *J. E. Kim, K. H. Shon*: Coset of hypercomplex numbers in Clifford analysis. Bull. Korean Math. Soc. **52(5)** (2015), 1721–1728.
[10] *J. E. Kim, K. H. Shon*: Inverse Mapping Theory on Split Quaternions in Clifford Analysis. To appear in Filomat (2016).
[11] *J. E. Kim, K. H. Shon*: Properties of regular functions with values in bicomplex numbers. To appear Bull. Korean Math. Soc. (2016).
[12] *A. P. Kotelnikov*: Screw calculus and some applications to geometry and mechanics. Annal. Imp. Univ. Kazan (1895).
[13] *J. M. McCarthy*: An Introduction to Theoretical Kinematics. MIT Press (1990), 62–65.
[14] *B. Saric*: Cauchy's residue theorem for a class of real valued functions. Czech. Math. J. **60(4)** (2010), 1043–1048.
[15] *E. Study*: Geometrie der Dynamen, Teubner, Leipzig, (1901).
[16] *A. Sudbery*: Quaternionic analysis. In Math. Proc. Cambridge Phil. Soc. **85(02)** (1979), 199–225.

*Received March 21, 2016.*

Department of Mathematics,
Dongguk University,
Gyeongju-si 38066, Republic of Korea,

*E-mail address*: jeunkim@pusan.ac.kr