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MAXIMAL ZERO DIVISOR AND SEMI-ZERO DIVISOR GRAPHS OF A RING

ASAAD ABOZEID JUND AND NECHIRVAN BADAL IBRAHIM

Abstract. This paper introduces two new zero divisor graph concepts for commutative rings with two maximal ideals, called the maximal zero divisor graph and the semi-zero divisor graph. These graphs are constructed based on specific conditions for vertices and edges. The paper also calculates various properties of these graphs, such as diameter, nullity, Wiener index, Wiener polynomial, and energy for the maximal zero divisor graph.

1. Introduction

Let G be a simple undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A graph G is said to be a singular graph provided that its adjacency matrix $A(G)$ is a singular matrix. The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_p$ of $A(G)$ are eigenvalues of the graph G, which form the spectrum of G. The algebraic multiplicity of the number zero in the spectrum of a graph G is called its nullity (degree of singularity) denoted by $\eta(G)$ and studied by Gutman in [5]. The concept of zero divisor graphs $\Gamma(R)$ was introduced by I. Beck in [4] but his motive was in coloring of graphs. In [2], Anderson and Livingston studied the diameter of a graph $\Gamma(R)$ of a commutative ring R with identity. In [8, 11], the Wiener index of a graph was the first reported topological index based on graph distances, this index is defined as the sum of all distances between vertices of the graph. The energy of a graph is the sum of absolute value of eigenvalues of the adjacency matrix, which initiated by Gutman in [6]. The energy and Wiener index of zero divisor graphs was studied in [1]. The zero divisor graph of Z_{p^nq} is studied in [9, 10]. For undefined terms in graph theory, we refer to [3]. Two vertices u and v in a graph G are said to be coneighbor vertices if and only if $N_G(u) = N_G(v)$.

Lemma 1. [7] *(Coneighbor Lemma)*
If u and v be two coneighbor vertices in a graph G, then

$$\eta(G) = \eta(G - u) + 1 = \eta(G - v) + 1.$$

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Corollary 2 ([5], p.234). *(End Vertex Corollary)*
If G is a bipartite graph with an end vertex, and H be an induced subgraph of G obtained by deleting this end vertex together with the vertex adjacent to it, then $\eta(G) = \eta(H)$.

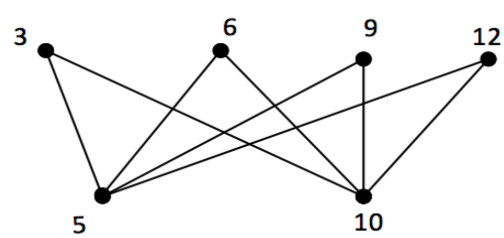
2. Maximal zero divisor graphs of a ring

In this section, we define a new concept of zero divisor graph of a commutative ring and determine the diameter, energy, nullity, Wiener index and Wiener polynomial of this type of graphs.

Definition 3. Let R be a commutative ring with identity which has only two maximal ideals M_1 and M_2 and let $Z^*(R)$ be the set of all nonzero zero divisor elements of R, we associate a simple graph $\Gamma(M(R))$ to R with vertices set $M(R) = Z^*(R) \cap (M_1 \cup M_2)$ and two distinct vertices $a \in M_1$ and $b \in M_2$ are adjacent if and only if $a * b = 0$ in R, this graph called the maximal zero divisor graph.

Clearly, where R is finite, $\Gamma(M(R))$ is a subgraph of $\Gamma(R)$. Let $R = Z_{pq}$ and $p < q$ be prime numbers. The commutative ring Z_{pq} has exactly two maximal ideals M_1 of order q and M_2 of order p.

Example 4. Let $Z_{3*5} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ has maximal ideals $M_1 = 3Z_5$ and $M_2 = 5Z_3$, the maximal zero divisor graph of Z_{3*5} is



The maximal zero divisor graph of Z_{pq}, where $p < q$ are prime numbers has the following properties:

(1) The vertex set of maximal zero divisor graph of Z_{pq} is
$V(\Gamma(M(Z_{pq}))) = \{v_{pi} : i = 1, 2, \cdots, q-1\} \cup \{v_{qj} : j = 1, \cdots, p-1\}$.
(2) The maximal zero divisor graph of Z_{pq} is isomorphic to the complete bipartite graph $K_{p-1,q-1}$.
(3) The order of maximal zero divisor graph of Z_{pq} is $(p + q - 2)$, while the size is $(p - 1)(q - 1)$.

Proposition 5. *If $p < q$ are prime numbers, then $diam(\Gamma(M(Z_{pq})) \leq 2$.*

Proof. A diameter is the maximum distance between two vertices in a graph, the maximal zero divisor graph of Z_{pq} is a bipartite graph, let u be the vertex in M_1 and v be any vertex in M_2, then the shortest path between u and v is of length one, so $diam(u, v) = 1$, if both vertices are in M_1 (or in M_2), the shortest path between them is two, then $diam(\Gamma(M(Z_{pq})) \leq 2$. □

Theorem 6. *An energy of the maximal zero divisor graph of Z_{pq} is given by*

$$E(\Gamma(M(Z_{pq})) = 2\sqrt{(pq - p - q + 1)}.$$

Proof. Let $V(\Gamma(M(Z_{pq}))) = A \cup B$ be a partition with $|A| = p - 1$ and $|B| = q - 1$. Then the adjacency matrix of $\Gamma(M(Z_{pq}))$ is of the form:

$$A(\Gamma(M(Z_{pq})) = \begin{pmatrix} 0 & B_{p-1,q-1} \\ A_{p-1,q-1} & 0 \end{pmatrix}$$

where $J = A_{p-1,q-1} = B_{p-1,q-1}$ stand for $p-1$-by-$q-1$ matrix of $1s$, then the characteristic polynomial of $\Gamma(M(Z_{pq}))$ is given by

$$\varphi(\Gamma(M(Z_{pq}))) = |\mu I_{p-1,q-1} - A(\Gamma(M(Z_{pq})))| = \begin{vmatrix} \mu & -J_{p-1,q-1} \\ -J_{p-1,q-1} & \mu \end{vmatrix}$$

$$\varphi(\Gamma(M(Z_{pq}))) = \mu^{p+q-4}(\mu^2 - (p - 1(q - 1)).$$

Hence, the spectra of $\Gamma(M(Z_{pq})))$ is

$$S_p(\Gamma(M(Z_{pq}))) = \begin{pmatrix} \sqrt{pq - p - q + 1} & 0 & -\sqrt{pq - p - q + 1} \\ 1 & p + q - 4 & 1 \end{pmatrix}.$$

Thus, $E(\Gamma(M(Z_{pq}))) = 2\sqrt{pq - p - q + 1}$. □

Theorem 7. *The nullity of maximal zero divisor graph of Z_{pq} is $p + q - 4$.*

Proof. By previous theorem, the result is clear. □

The complement of a graph G, (G^c) is a graph that $|V(G)| = |V(G^c)|$ and u and v are adjacent in G^c if and only if they are not adjacent in G.

Proposition 8. *If $p < q$ are prime numbers then*

$$\eta(\Gamma(M(Z_{pq}))^c) = \begin{cases} 1 & if \quad p = 2 \\ 0 & if \quad p \geq 3 \end{cases}$$

Proof. If $p = 2$ and $q > 2$, then the complement of a maximal zero divisor graph of Z_{2q} is $K_1 \cup K_{q-1}$, then $\eta(\Gamma(M(Z_{2q}))^c) = \eta(K_1) + \eta(K_{q-1}) = 1 + 0 = 1$. If $p \geq 3$ and $q > 3$, then the complement of a maximal zero divisor graph of Z_{pq} is $K_{p-1} \cup K_{q-1}$, then $\eta(\Gamma(M(Z_{pq}))^c) = \eta(K_{p-1}) + \eta(K_{q-1}) = 0$. □

Proposition 9. *If $p < q$ are prime numbers. Then the Wiener index of the maximal zero divisor graphs of Z_{pq} is given by $W(\Gamma(M(Z_{pq}))) = (p + q)^2 - 4(p + q) - pq + 5$.*

Proof. The maximal zero divisor graph of Z_{pq} is isomorphic to the complete bipartite graph $K_{p-1,q-1}$, put $m = p-1$ and $n = q-1$ in $[1, Theorem 1.3]$, the result has been approved. □

Proposition 10. *The Weiner polynomial of* $\Gamma(M(Z_{pq}))$ *is given by*

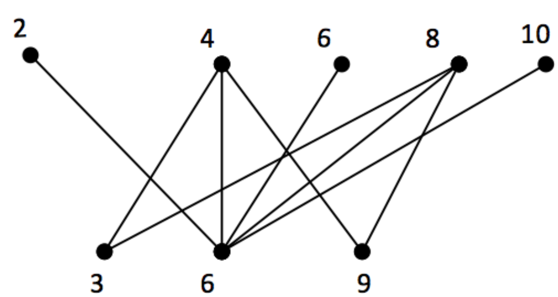
$$W(\Gamma(M(Z_{pq}, x)) = (p-1)(q-1)x + [\binom{p-1}{2} + \binom{q-1}{2}]x^2.$$

Proof. Put $m = p-1$ and $n = q-1$ in $[1, Theorem 1.2]$, to see the result. □

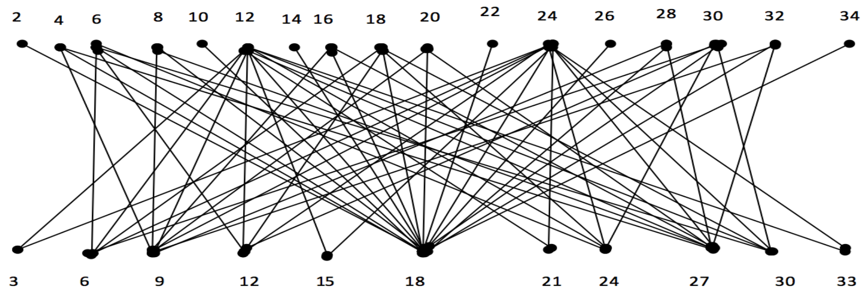
Let $R = Z_{p^{k_1}q^{k_2}}$, if $k_1 = 2$ and $k_2 = 1$ and $p < q$ are prime numbers, then the maximal zero divisor graph of Z_{p^2q} have the following properties:

(1) The maximal zero divisor graphs of Z_{p^2q} consists of two maximal ideals $M_1 = \{p_i : i = 1, 2, \cdots, pq-1\} - \{0\}$ and $M_2 = \{q_j : j = 1, 2, \cdots, p+q\} - \{0\}$.
(2) In the set $M_1 = \{p_i : i = 1, 2, \cdots, pq-1\} - \{0\}$ there are $q-1$ coneighbor vertices of degree p^2-1 adjacent to all vertices in the set $M_2 = \{q_j : j = 1, 2, \cdots, p+q\} - \{0\}$ and there are $q(p-1)$ coneighbor vertices of degree $p-1$.
(3) In the set $M_2 = \{q_j : j = 1, 2, \cdots, p+q\} - \{0\}$ there are $p-1$ coneighbor vertices of degree $pq-1$ adjacent to all vertices in the set $M_1 = \{p_i : i = 1, 2, \cdots, pq-1\} - \{0\}$ and there are $p(q-1)$ coneighbor vertices of degree $q-1$.
(4) The order of maximal zero divisor graphs of Z_{p^2q} is $p^2(q-1)$ and the size of maximal zero divisor graphs of Z_{p^2q} is q^2.
(5) Let $R = Z_{p^2q}$ where $p < q$ are prime numbers, then the order of M_1 is pq and the order of M_2 is p^2.

Example 11. (1) Let $R = Z_{2^23}$, the maximal ideals are $M_1 = 2Z_6$ and $M_2 = 3Z_4$. The maximal zero divisor graph of Z_{2^23} is



(2) Let $R = Z_{2^23^2}$, the maximal ideals are $M_1 = 2Z_{18}$ and $M_2 = 3Z_4$. The maximal zero divisor graph of $Z_{2^23^2}$ is



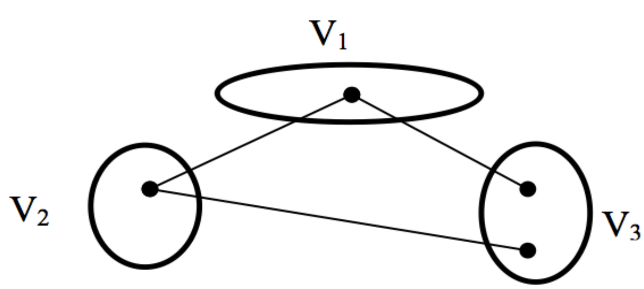
In $\Gamma(M(Z_{2^2*3^2}))$ there are 6 coneighbor vertices of degree one, 6 coneighbor vertices of degree 3, 3 coneighbor vertices of degree 5, 2 coneighbor vertices of degree 11, 4 coneighbor vertices of degree 2, 4 coneighbor vertices of degree 5, 2 coneighbor vertices of degree 8 and one coneighbor vertex of degree 17. Then by Coneighbor Lemma: $\eta(\Gamma(M(Z_{2^2*3^2})) = 5+5+2+1+3+3+1 = 20$.

Theorem 12. *The maximal zero divisor graph of Z_{p^2q} is bipartite graph.*

Proof. The vertex set of maximal zero divisor graph of Z_{p^2q} can be partitioned into two sets
$V_1 = M_1 - \{0\}$ and $V_2 = M_2 - \{0\}$. The vertices in each vertex set are non-adjacent but the vertices between the sets are adjacent. Clearly the maximal zero divisor graph of Z_{p^2q} is bipartite graph. □

Proposition 13. *If $p < q$ are prime numbers, then $diam(\Gamma(M(Z_{p^2q}))) \leq 3$.*

Proof. Up to the maximal zero divisor graph as shown, the sets
$V_1 = \{p-1$ coneighbor vertices of degree $pq-1$ adjacent to all vertices in set
$M_1 = \{pi : i = 1, 2, \cdots, pq-1\} - \{0\}\}$, $V_2 = \{q-1$ coneighbor vertices of degree
p^2-1 adjacent to all vertices in the set $M_2 = \{qj : j = 1, 2, \cdots, p+q\} - \{0\}\}$ and
$V_3 = \{q(p-1)$ and $p(p-1)$ coneighbor vertices of degree $p-1$ and $q-1\}$, let u be the vertex in V_1 and v be any vertex in V_2, then the shortest path between u and v is of length one, so $diam(u,v) = 1$, if u is a vertex in V_2 and v be a vertex in V_3, then the shortest path between u and v is one or two, so $diam(u,v) \leq 2$, if u be a vertex in V_1 and v be a vertex in V_3, then the shortest path between u and v is one or three. Thus $diam(u,v) \leq 3$.



□

Theorem 14. *The nullity of maximal zero divisor graph of Z_{p^2q} is $p^2 + pq - 6$.*

Proof. In $\Gamma(M(Z_{p^2q}))$, we have $q-1$ coneighbor vertices of degree p^2-1 in set M_1, $q(p-1)$ coneighbor vertices of degree $qp-1$ in the set M_1, $q(p-1)$ coneighbor vertices of degree $p-1$ in set M_2 and $p(p-1)$ coneighbor vertices of degree $q-1$ in set M_2, then by using Coneighbor Lemma:
$\eta(\Gamma(M(Z_{p^2q})) = q - 2 + p - 2 + qp - q - 1 + p2 - p - 1 + \eta(P_4)$ and $\eta(P_4) = 0$ see [[7], Lemma 1.3.9]. Thus $\eta(M(Z_{p^2q})) = p^2 + pq - 6$. □

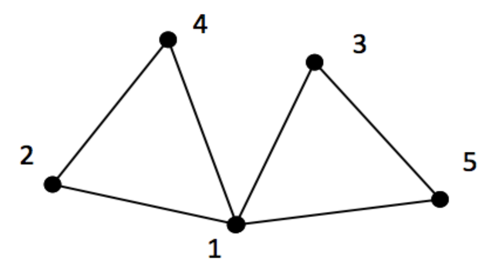
Question 15. What is the diameter, nullity, Weiner index and Weiner polynomial of $\Gamma(M(Z_{p^{k_1}q^{k_2}}))$, if $k_1 = k_2 = 2$ and $p < q$?

3. Semi-zero divisor graph of a ring

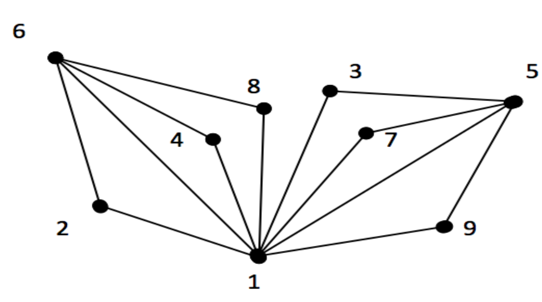
Definition 16. Let R be a commutative ring with identity, let $a, b \in R$. A simple graph $\Gamma(S(R))$ associated to R with vertices set $S(R) \subseteq Z^*(R)$, where two distinct vertices a and b in $S(R)$ are adjacent if and only if $a * b = a$ or $a * b = b$. This graph is called the semi-zero divisor graph of R.

Let $R = Z_{pq}$ where $p < q$ and p and q are prime numbers.

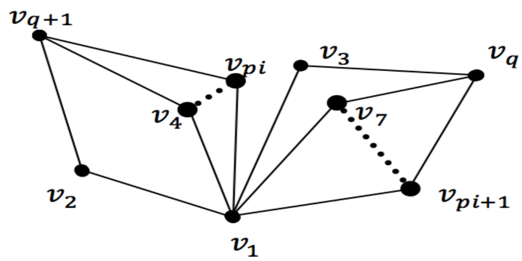
Example 17. (1) If $p = 2$ and $q = 3$, we have $Z_6 = \{0, 1, 2, 3, 4, 5\}$, the semi-zero divisor graph of Z_6 is



(2) For $p = 2$ and $q = 5$, we have $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, the semi-zero divisor graph of Z_{10} is



The vertex v_1 is adjacent to every vertex in the set $S(Z_{2q}))$, the vertex v_q adjacent to every vertex of set five, the vertex v_{q+1} adjacent to every vertex of set four, and the set of vertices of set three and four are non-adjacent vertices. For $q > 3$, the semi-zero divisor graph will be:

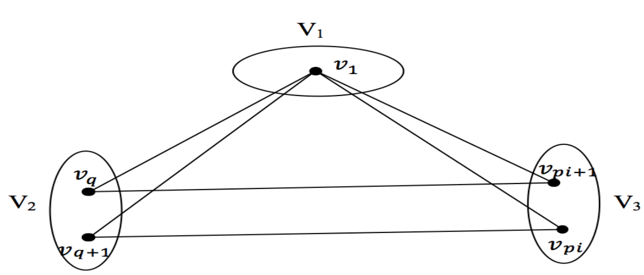


The semi-zero divisor graph of Z_{2q} has the following properties:

(1) The vertices in the set $\{v_{pi}, i = 1, 2, \cdots, q-1\}$ are $q-2$ coneighbor vertices, the vertices in the set $\{v_{pi+1}, i = 1, 2, \cdots, q-1\}$ are $q-2$ coneighbor vertices.
(2) The vertex v_1 adjacent to all vertices in the set $S(Z_{2q}) - \{0\}$.
(3) The vertex v_q adjacent to all vertices in the set $\{v_{pi}, i = 1, 2, \cdots, q-1\}$.
(4) The vertex v_{q+1} adjacent to all vertices in the set $\{v_{pi+1}, i = 1, 2, \cdots, q-1\}$.
(5) The order of the semi-zero divisor graph of Z_{2q} is $pq-1$ and the size of the semi-zero divisor graph of Z_{2q} is $2(q+2)$ where $q > 3$.

Theorem 18. *The semi-zero divisor graph of Z_{2q} is tripartite graph.*

Proof. The vertex set of $\Gamma(S(Z_{2q}))$ is $\{v_1, v_2, \cdots, v_{2q-1}\}$ can be divided in to three sets as shown in following figure, the vertex set $V_1 = \{v_1\}$, the vertex set is $V_2 = \{v_q, v_{q+1}\}$, the vertex set
$V_3 = \{q-2$ coneighbor vertices adjacent to v_1 and $v_{q+1}, q-2$ coneighbor vertices adjacent to v_1 and $v_q\}$, then $\Gamma(S(Z_{2q}))$ is tripartite graph.



□

Theorem 19. $diam(\Gamma(S(Z_{2q}))) \leq 2$.

Proof. Up to the semi-zero divisor graph as shown in previous figure, let $u = v_1$ be the vertex in V_1 and v be any vertex in any other vertex sets V_2 and V_3, then the shortest path between u and v is of length one, then $diam(u, v) = 1$. For $u = v_q$ and $v = v_{q+1}$ are the vertex in V_2 , then the shortest path between u and v is two, therefore $diam(u, v) = 2$. If $u = v_q(v_{q+1})$ in V_2 and $v = v_{pi+1}$ or v_{pi} are vertices in V_2, respectively, then the shortest path is of length one or two. Hence $diam(u, v) = 2$, then $diam(\Gamma(S(Z_{2q}))) \leq 2$. □

Theorem 20. *The nullity of semi-zero divisor graph Z_{2q} is $2q - 6$, $q > 3$.*

Proof. In the $\Gamma(S(Z_{2q}))$, we have two set of $q-2$ coneighbor vertices, by using Coneighbor Lemma: $\eta(\Gamma(S(Z_{2q}))) = q - 3 + q - 3 + \eta(\Gamma(S(Z_6))$ and $\eta(\Gamma(S(Z_6)) = 0$.
Hence, $\Gamma(S(Z_{2q})) = q - 3 + q - 3 = 2q - 6$. □

Proposition 21. *The Wiener index of $\Gamma(S(Z_{2q}))$ is given by $W(\Gamma(S(Z_{2q})) = 4q^2-10q+8$.*

Proof. Graph $\Gamma(S(Z_{2q}))$ has a center vertex with degree $2q - 2$, i.e. a center vertex has distance one with $2q - 2$ vertices. There are two vertices that each one is adjacent with $q - 2$ vertices except center one. Moreover, there are $(q - 2)(3q - 5)$ and $(q - 2)(q - 3)$ different paths of length two. □

Proposition 22. *The Wiener polynomial of $\Gamma(S(Z_{2q}))$ is given by*

$$W(\Gamma(S(Z_{2q}, x)) = (4q - 6)x + (4q^2 - 14q + 14)x^2.$$

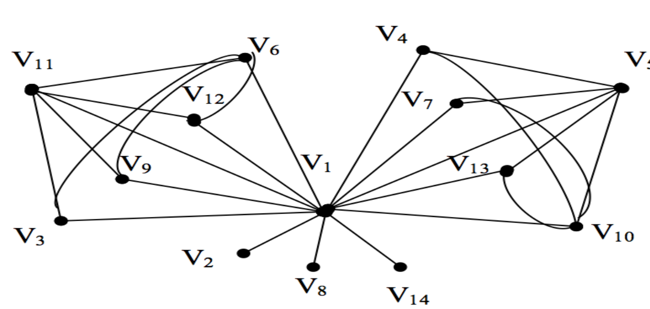
Proof. The distance between vertex v_1 and all other vertices is $2q - 2$ of distance one, the distance between the vertices v_q and v_{q+1} and all other $2(q - 2)$ vertices are of distance one. The distance between the $(q - 2)(3q - 5) + (q - 2)(q - 3)$ vertices of degree two is of distance two. Then

$$\begin{aligned} W(\Gamma(S(Z_{2q}, x)) &= [2q - 2 + 2(q - 2)]x + [(q - 2)(3q - 5) + (q - 2)(q - 3)]x^2 \\ &= (4q - 6)x + (4q^2 - 14q + 14)x^2. \end{aligned}$$

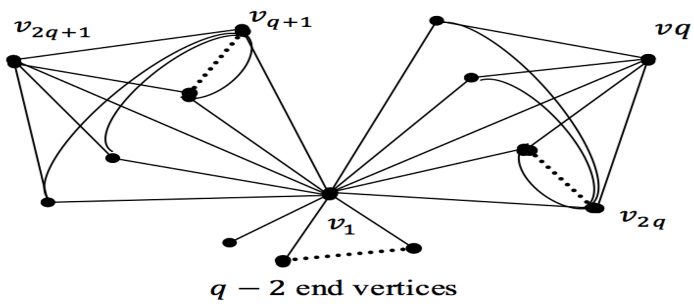
□

Let $R = Z_{pq}$ and $p = 3 < q$ are prime numbers.

Example 23. Let $R = Z_{3*5}$, then $\Gamma(S(Z_{2q}))$ is



The semi-zero divisor graph of Z_{3q} is indicated in the following figure:



The semi-zero divisor graph of Z_{3q} has the following properties:

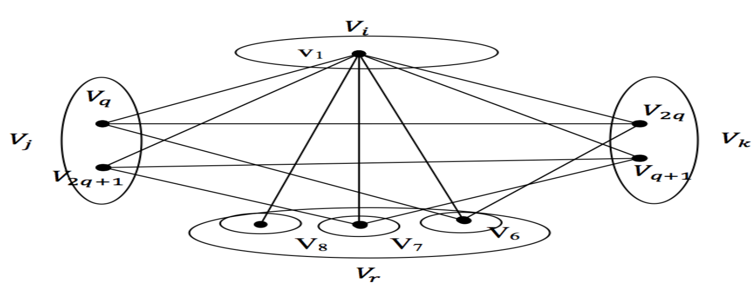
(1) The vertex set of $V(\Gamma(S(Z_{3q}))) = \{v_1, v_2, \cdots, v_{pq-1}\}$. The order of the semi-zero divisor graph of Z_{3q} is $pq-1$ and the size of the semi-zero divisor graph of Z_{3q} is $6q-8$.

(2) $V(\Gamma(S(Z_{3q})))$ can be partitioned into 8 sets, the vertex set $V_1 = \{v_1\}$, the vertex set $V_2 = \{v_q\}$, the vertex set $V_3 = \{v_{2q}\}$, the vertex set $V_4 = \{v_{2q+1}\}$, the vertex set $V_5 = \{v_{q+1}\}$, the vertex set $V_6 = \{v_{3i+1}, i = 1, 2, \cdots, q-1\} - \{v2q\}$ are $q-2$ coneighbor and non-adjacent vertices, the vertex set is $V_7 = \{v_{3i}, i = 1, 2, \cdots, q-1\} - \{v_{q+1}\}$ are $q-2$ coneighbor and non-adjacent vertices, the vertex set $V_8 = \{q-2$ end vertices $\}$.

(3) The vertex V_1 adjacent to all vertices in $\Gamma(S(Z_{3q}))$, the vertex in V_2 adjacent to the vertices in V_3 and V_6 and the vertex in V_3 adjacent to the vertices in V_6. The vertex in V_4 adjacent to the vertices in V_5 and V_7 and the vertex in V_5 adjacent to the vertices in V_7.

Theorem 24. *The semi-zero divisor graph of Z_{3q} is 4-partite graph.*

Proof. The semi-zero divisor graph of Z_{3q} can be partitioned into four sets, the vertex set $V_i = \{v_1\}$, the vertex set $V_j = \{v_q, v_{2q+1}\}$, $V_k = \{v_{2q}, v_{q+1}\}$ and the vertex set $V_r = \{V_6, V_7, V_8\}$, the vertices in each set are non-adjacent but the vertices between the sets are adjacent. Therefore, the semi-zero divisor graph of Z_{3q} is 4-partite graph.



□

Theorem 25. *$diam(\Gamma(S(Z_{3q}))) \leq 2$.*

Proof. It is clear from previous figure. $\square$

Theorem 26. *The nullity of $\Gamma(S(Z_{3q}))$ is $3q-9$.*

Proof. By using Coneighbor Lemma and End Vertex Corollary: $\eta(\Gamma(S(Z_{3q})) = q-3+q-3+q-3+2\eta(K_3)$, in [[7], Lemma 1.3.9] $\eta(K_3) = 0$. Hence, $\eta(\Gamma(S(Z_{3q})) = 3q-9$. $\square$

Proposition 27. *The Wiener index of $\Gamma(S(Z_{3q}))$ is given by*

$$W(\Gamma(S(Z_{3q}))) = \sum_{i=0}^{q-3} 2(pq-p-i) + \sum_{i=0}^{q-p-1} 2(q-p-i) + \sum_{i=0,j=1}^{q-3,q-2} (3q-i-j) + 11q - 8.$$

Proof. The semi-zero divisor graph of Z_{3q} has eight sets, the Wiener index of the sets V_8, V_1, V_2, V_3, V_4 and V_5 are $\sum_{i=0}^{q-3} 2(pq-p-i)+q-2$, $2q$, $3q-1$, $3q-2$, $q-1$ and $q-2$. The Wiener index of the vertices adjacent to vertex sets V_2 and V_3 is $\sum_{i=0,j=1}^{q-3,q-2}(3q-i-j)$ and the Wiener index of the vertices adjacent to the vertices in set V_4 and V_5 is $\sum_{i=0}^{q-p-1} 2(q-p-i)$. Hence the proof is complete. $\square$

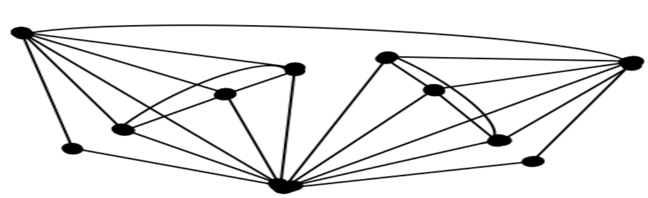
Proposition 28. *The Wiener polynomial of $\Gamma(S(Z_{3q})$ is given by*

$$W(\Gamma(S(Z_{3q},x))) = (7q-6)x + [(\sum_{i=0}^{q-p-2} (q-p-i) + 2q) + 2\sum_{i=0}^{q-p-1} (q-p-i) + 2q]x^2.$$

Proof. The distance between a vertex in set V_1 with vertices in set V_8 is $q-2$ of distance one and the vertex in set V_1 with all vertices in all other sets has distance $2q$ of distance one. The vertex in set V_2 with the vertices in set $_6$ is of distance $q-1$ of distance one and the vertex in set V_4 with the vertices in set V_7 is of distance $q-1$ of distance one. The vertex in set V_3 with the vertices in set V_6 is of distance $q-2$ of distance one and the vertex in set V_5 with the vertices in set V7 is q-2 of distance one. The vertices in set V_8 with vertices in sets V_2, V_3, V_4, V_5, V_6 and V_7 have $\sum_{i=0}^{q-p-2}((q-p-i)+2q)$ of distance two. The vertex V_3 with vertices in sets V_4, V_5 and V_7 have $2q$ of distance two, the vertices in set V_6 with itself have $\sum_{i=0}^{q-p-1}(p-q-i)+2q$ of distance two and the vertices in set V_7 with itself have $\sum_{i=0}^{q-p-i}(p-q-i)+2q$, then the proof is complete. $\square$

Let $R = Z_{p^{k_1}q^{k_2}}$, where $k_1 = 2$, $k_2 = 1$ and $p = 2 < q$ are prime numbers.

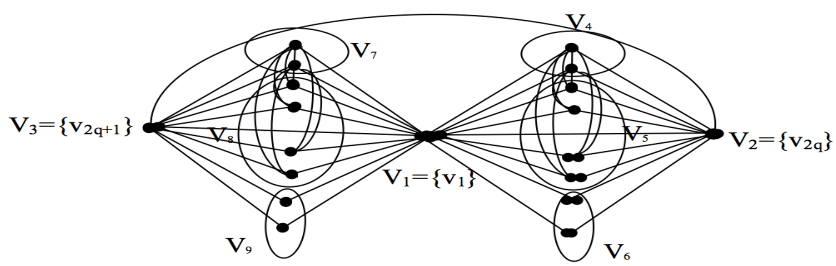
Example 29. Let $R = Z_{2^2 3}$, then $\Gamma(S(Z_{2^2 3})$ is



The nullity of $\Gamma(S(Z_{2^23})$ is one.

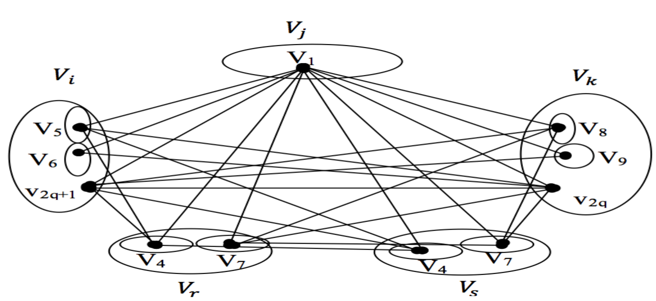
The semi-zero divisor graph of Z_{2^2q} has the following properties:

(1) $|V(\Gamma(S(Z_{2^2q})))| = p^{2q} - 1$ and its size is $(p+q)^2$.
(2) The vertices set of $\Gamma(S(Z_{2^2q}))$ can be partitioned into nine sets,
$V_1 = \{v_1\}$, $V_2 = \{v_{2q}$ of degree $2q\}$, $V_3 = \{v_{2q+1}$ of degree $2q\}$,
$V_4 = \{2$ adjacent vertices of degree $q+1\}$,
$V_5 = \{q-2$ coneighbor and non-adjacent vertices of degree $4\}$,
$V_6 = \{q-2$ coneighbor and non-adjacent vertices of degree $2\}$,
$V_7 = \{2$ adjacent vertices of degree $q+1\}$,
$V_8 = \{q-2$ coneighbor and non-adjacent vertices of degree $4\}$ and
$V_9 = \{q-2$ coneighbor and non-adjacent vertices of degree $2\}$.



Theorem 30. *The semi-zero divisor graph of Z_{2^2q} is 5-partite graph.*

Proof. The vertices in the semi-zero divisor graph of Z_{2^2q} can be partitioned into five partite sets V_i, V_j, V_k, V_r and V_s as shown in the following figure such that the vertices in each vertex set are non-adjacent but there is a connected vertex by an edge between the given sets, hence the graph is 5-partite graph.



□

Theorem 31. $diam(\Gamma(S(Z_{2^2q}))) \leq 2$.

Proof. Up to the inverse graph as shown in previous figure, the distance of the different vertices in the different partite sets V_i,V_j, V_k, V_r and V_s is one or two. Hence, $diam(u, v) \leq 2$, for all u and v in the semi-zero divisor graph of Z_{2^2q}. □

Theorem 32. *The nullity of* $\Gamma(S(Z_{2^2q}))$ *is* $4(q-2)-3$.

Proof. In the semi-zero divisor graph of Z_{2^2q}, we have two sets of $q-2$ coneighbor vertices of degree 2 and two sets of $q-2$ coneighbor vertices of degree 4. Then by using Coneighbor Lemma, once can obtain:

$$\eta(\Gamma(S(Z_{2^2q}))) = 2(q-2) - 2 + 2(q-2) - 2 + \eta(Z_{2^23}) = 4(q-2) - 4 + 1.$$

Hence, $\eta(\Gamma(S(Z_{2^2q}))) = 4(q-2)-3$. □

Proposition 33. *The Wiener index of* $\Gamma(S(Z_{2^2q}))$ *is given by:*

$$W(\Gamma(S(Z_{2^2q}))) = 44q - 12 + 2(\sum_{i=0}^{q-3}(3q-i) + \sum_{i=0}^{q-3}(2q-i)) + \sum_{i=1}^{q-2}(3q-i) + 2\sum_{i=3}^{q-1}(2q-i)].$$

Proof. The distance between vertex v_{2q} and all vertices is $2q$ of distance one and $2(2q-2)$ of distance two. The distance between vertex v_1 and all vertices is $4q-2$ of distance one. The distance between vertex v_{2q+1} and all vertices is $2(q-1)$ of distance one and $2(2q-2)$ of distance two. The distance between $2(q-2)$ vertices of degree 2 is $2\sum_{i=0}^{q-3}(3q-i) + \sum_{i=1}^{q-2}(3q-i)$, the distance between $2(q-2)$ vertices of degree 4 is $2(q-2)+2(q-2)$ of distance one and is $2\sum_{i=0}^{q-3}(2q-i) + 2\sum_{i=3}^{q-1}(2q-i)$ of distance two. The distance between 2 vertices of degree $2q$ is 2 of distance one and $4(2q-2)$ of distance two. □

Proposition 34. *The Weiner polynomial of* $\Gamma(S(Z_{p^2q}))$ *is given by*

$$W(\Gamma(S(Z_{p^2q}, x))) = (12q-10)x + [16(q-1) + 2(\sum_{i=0}^{q-3}(5q-2i)) + \sum_{i=1}^{q-2}(3q-i) + 2\sum_{i=3}^{q-1}(q-i)]x^2.$$

Proof. The distance between all vertices of distance one is $12q-10$ and the distance between all vertices of distance two is

$$16(q-1) + 2(\sum_{i=0}^{q-3}(3q-i) + \sum_{i=0}^{q-3}(2q-i)) + \sum_{i=1}^{q-2}(3q-i) + 2\sum_{i=3}^{q-1}(q-i).$$

□

Question 35. What is the diameter, nullity, Weiner index and Weiner polynomial of $\Gamma(S(Z_{p^{k_1}q^{k_2}}))$, if $k_1 = k_2 = 2$ and $p < q$ are prime numbers?

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