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STRONGLY P-INJECTIVE RINGS

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Abstract. A ring R is called right strongly P-injective (or right SP-injective for short) if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$, and for any $b \in R$, every R-homomorphism from $a^nR + bR$ to R extends to an endomorphism of R. We study the properties of right strongly P-injective rings, several conditions under which right strongly P-injective rings are quasi-Frobenius rings are given.

1. Introduction

Throughout this article, R is an associative ring with identity, and all modules are unitary. As usual, J and $S_l(resp., S_r)$ denote respectively the Jacobson radical and the left (resp., right) socle of R. The left (resp., right) annihilator of a subset X of R is denoted by $\mathbf{l}(X)$ (resp., $\mathbf{r}(X)$) . If M is an R-module, the notation $N \subseteq^{max} M$ means that N is a maximal submodule of M, and we write $N \subseteq^{\oplus} M$ if N is a direct summand of M for convenience.

Recall that a ring R is called *quasi-Frobenius*, if it is one-sided artinian (or one-sided noetherian), and one-sided self-injective. A ring R is called *right P-injective* [10] if, for any principal right ideal I of R, every R-homomorphism from I to R extends to an endomorphism of R. A ring R is called *right GP-injective* [3, 5, 9] if, for every $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and every R-homomorphism from a^nR to R extends to an endomorphism of R . GP-injective rings are also called YJ-injective rings in [19, 20, 21]. A ring R is called right mininjective [11] if for any minimal right ideal I of R, every R-homomorphism from I to R extends to an endomorphism of R. The following implications hold:

right self-injective $\Rightarrow$ right P-injective $\Rightarrow$ right GP-injective $\Rightarrow$ right mininjective.

P-injective rings, GP-injective rings and the relations of them with quasi-Frobenius rings have been studied by many authors. In this article, we introduce the concept of right

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strongly P-injective rings , some properties of them are studied, and several conditions under which strongly P-injective rings are quasi-Frobenius rings are given.

2. Strongly P-injective rings

We start with the following definition.

Definition 1. A ring R is called right strongly P-injective (or right SP-injective for short) if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$, and for any $b \in R$, every R-homomorphism from $a^nR + bR$ to R extends to an endomorphism of R.

Let M be a right R-module. We call a submodule K of M *extensive* if every homomorphism from K to M extends to an endomorphism of M.

Lemma 2. *Let M be a right R-module with $S = \mathrm{End}(\mathrm{M_R})$ and K, K' be two submodules of M.*

(1) If $K + K'$ is extensive then $\mathbf{l}_S(K \cap K') = \mathbf{l}_S(K) + \mathbf{l}_S(K')$.

(2) If $\mathbf{l}_S(K \cap K') = \mathbf{l}_S(K) + \mathbf{l}_S(K')$ and K, K' are extensive, then $K + K'$ is extensive.

Proof. (1). If $s \in \mathbf{l}_S(K \cap K')$, then $f : K + K' \to M$ is well defined by $f(k + k') = sk$, so $f = t\cdot$ for some $t \in S$ by hypothesis. Then $s - t \in \mathbf{l}_S(K)$ and $t \in \mathbf{l}_S(K')$, so $s = (s-t) + t \in \mathbf{l}_S(K) + \mathbf{l}_S(K')$. Hence $\mathbf{l}_S(K \cap K') \subseteq \mathbf{l}_S(K) + \mathbf{l}_S(K')$; the other inclusion always holds.

(2). Let $f : K + K' \to M$ be a right R-homomorphism. Then $f\mid_K = s\cdot$ and $f\mid_K = t\cdot$ for some $s, t \in S$ by hypothesis. Thus, $s-t \in \mathbf{l}_S(K\cap K') = \mathbf{l}_S(K)+\mathbf{l}_S(K')$, say $s-t = s'-t'$, where $s' \in \mathbf{l}_S(K)$ and $t' \in \mathbf{l}_S(K')$. Put $a = s-s' = t-t'$. Then $ak = (s-s')k = sk = f(k)$ and $ak' = (t - t')k' = tk' = f(k')$ for any $k \in K$ and $k' \in K'$. It follows that $f = a\cdot$, as required. □

Theorem 3. *The following statements are equivalent for a ring R:*

(1) R is a right SP-injective ring.

(2) R is a right P-injective ring and for any $0 \neq a \in R$ and any $b \in R$, there exists a positive integer n such that $a^n \neq 0$ and $\mathbf{l}(a^nR \cap bR) = \mathbf{l}(a^n) + \mathbf{l}(b)$.

Proof. (1) ⇒ (2). For any $0 \neq a \in R$, since R is right SP-injective, there exists a positive integer n such that $a^n \neq 0$ and every R-homomorphism from $a^nR + aR$ to R extends to an endomorphism of R, and so every R-homomorphism from aR to R extends to an endomorphism of R because $aR = a^nR + aR$. It shows that R is right P-injective. Moreover, by Lemma 2(1), we have $\mathbf{l}(a^nR \cap bR) = \mathbf{l}(a^n) + \mathbf{l}(b)$.

(2) ⇒ (1). It follows from Lemma 2(2). □

Recall that a ring R is called right *Kasch* if every simple right R-module embeds in R, equivalently if $\mathbf{l}(T) \neq 0$ for every maximal right ideal T of R; a ring R is called right *minfull* [11] if it is semiperfect, right mininjective and $Soc(eR) \neq 0$ for each local idempotent $e \in R$.

Theorem 4. *Let R be a right SP-injective, right Kasch ring. Then*

(1) R is left GP-injective, and hence right and left mininjective.

(2) $S_r = S_l \trianglelefteq {}_RR$.

(3) $J = \mathbf{r}(S_r) = \mathbf{rl}(J)$.

(4) $\mathbf{l}(J) \trianglelefteq {}_RR$.

(5) $J = Z_l = Z_r$.

(6) The map $\theta : T \mapsto \mathbf{l}(T)$ gives a bijection from the set of maximal right ideals of R to the set of minimal left ideals of R, whose inverse map is given by $K \mapsto \mathbf{r}(K)$.
Moreover, if R is semilocal, then

(7) R is left Kasch.

(8) $S_r = S_l \trianglelefteq R_R$.

(9) $\mathbf{r}(J) \trianglelefteq R_R$.

Proof. (1) Let $a \in R$. Then by the hypothesis, there exists a positive integer n such that $a^n \neq 0$ and for every $b \in R$, every R-homomorphism from $a^nR + bR$ to R extends to an endomorphism of R. We always have $a^nR \subseteq \mathbf{rl}(a^n)$. If $b \in \mathbf{rl}(a^n) - a^nR$, let $a^nR \subseteq T \subseteq^{max} (a^nR + bR)$. By the Kasch hypothesis, let $\sigma : (a^nR + bR)/T \to R$ be monic, and define $\gamma : a^nR + bR \to R$ by $\gamma(x) = \sigma(x + T)$. Then $\gamma = c\cdot$ for some $c \in R$. So $ca^n = \gamma(a^n) = 0$. This gives $cb = 0$ because $b \in \mathbf{rl}(a^n)$. But $cb = \sigma(b+T) \neq 0$ because $b \notin T$, this is a contradiction. Hence $\mathbf{rl}(a^n) = a^nR$, it shows that R is left GP-injective by [20, Lemma 3].

(2)-(5) follows from [3, Theorem 2.3].

(6). It follows from (1) and [11, Lemma 1.1, Theorem 2.3(2)].

(7) Since R is semilocal, then by (1), it is a semilocal, right and left mininjective right Kasch ring. By [14, Lemma 5.49], R is a left Kasch ring.

(8). Since R is left GP-injective by (1) and left Kasch by (7), we have that $S_r = S_l \trianglelefteq R_R$ by [3, Theorem 2.3(2)].

(9). Since R is left GP-injective and left Kasch, by [3, Theorem 2.3(4)], $\mathbf{r}(J) \trianglelefteq R_R$. □

By Theorem 3, we see that right SP-injective rings are right P-injective, our next example shows that right P-injective rings need not be right SP-injective.

Example 5. Let K be a field and L be a proper subfield of K such that $\rho : K \to L$ is an isomorphism, and let $K[\rho; x]$ be the ring of twisted left polynomials over K where

$xk = \rho(k)x$ for all $k \in K$. Set $R = K[\rho; x]/(x^2)$. Then R is right P-injective, but R is not right SP-injective.

Proof. By Rutter [16, Example 1], R is a right P-injective , left artinian ring but R is not quasi-Frobenius. Hence R is right minfull. By [11, Theorem 3.7(1)], R is right Kasch . If R is right SP-injective, then by Theorem 4(1), R is left and right mininjective and left artinian. It follows that R is quasi-Frobenius by [11, Corollary 4.8], a contradiction. □

Corollary 6. *Let R be a left perfect right SP-injective ring. Then*

(1) R is right and left Kasch.

(2) R is two-sided mininjective and $S_r = S_l$ is essential both as a right and a left ideal.

Proof. (1). Since R is left perfect, it is semilocal and right semiartinian by [14, Theorem B.32], and so every nonzero right R-module has an essential socle by [14, Theorem B.31]. In particular, $S_r \trianglelefteq R_R$ and $Soc(eR) \neq 0$ for every local $e^2 = e \in R$. Therefore, R is right minfull, which implies that R is right and left Kasch by [11, Theorem 3.7(1)].

(2). It follows from (1) and Theorem 4(1)(2)(8). □

Theorem 7. *Let R be a right SP-injective right Kasch ring. Then the following conditions are equivalent:*

(1) R is semilocal.

(2) R is left finitely cogenerated.

(3) R is left finite dimensional.

(4) R is right finitely cogenerated and left Kasch.

(5) R is right finite dimensional.

(6) S_r is a finitely generated left ideal.

Proof. (1) ⇒ (2). By [3, Theorem 2.8], every right GP-injective right Kasch semilocal ring is left finitely cogenerated.

(2) ⇒ (3), and (4) ⇒ (5) are obvious.

(3) ⇒ (1). By [6, Corollary 3.2], every right Kasch left finite dimensional ring is semilocal.

(1) ⇒ (4). By Theorem 2.4(7), R is left Kasch. And so, by [3, Theorem 2.8], R is right finitely cogenerated.

(5) ⇒ (1). By [10, Theorem 3.3(2)], every right P-injective right finite dimensional ring is semilocal.

(2) ⇒ (6). It follows from Theorem 4(2).

(6) ⇒ (1). Since R is a right SP-injective right Kasch ring, by Theorem 2.4(2), $S_r = S_l$. By (6), S_r is a finitely generated left ideal, so $S_r = S_l = Ra_1 + Ra_2 + \cdots + Ra_n$, where Ra_i

is a simple left ideal, $i = 1, 2, \cdots, n$. Then, by Theorem 4(3), $J = \mathbf{r}(S_r) = \cap_{i=1}^n \mathbf{r}(a_i)$. Note that each $\mathbf{r}(a_i) = \mathbf{r}(Ra_i)$ is a maximal right ideal by Theorem 4(6), so R is semilocal. $\square$

Recall that a module M is called *C2* [14] if every submodule that is isomorphic to a direct summand of M is itself a direct summand of M ; a module M is called *C3* [14] if N and K are both direct summands of M and $N \cap K = 0$, then $N \oplus K$ is also a direct summand of M ; a module M is called a *min-CS module* [11] if every simple submodule of M is essential in a direct summand of M; a ring R is called a *left (right) min-CS* ring [11] if ${}_RR$ (R_R) is a min-CS module; a ring R is called a *left (right) C2* ring [10] if ${}_RR$ (R_R) is a C2 module.

Lemma 8. *Let M_R be a finitely cogenerated, min-CS , C2 module with $S = \mathrm{End(M_R)}$. Then S is semiperfect.*

Proof. Since M_R is finitely cogenerated, $\mathrm{Soc}(M_R)$ is finitely cogenerated and $\mathrm{Soc}(M_R) \trianglelefteq M_R$. Let $\mathrm{Soc}(M_R) = K_1 \oplus K_2 \oplus \cdots \oplus K_n$, where each K_i is a simple submodule of M_R, $i = 1, 2, \cdots, n$. Since M_R is min-CS, there exists idempotents $e_i \in S, i = 1, 2, \cdots, n$ such that $K_i \trianglelefteq e_iM$, $i = 1, 2, \cdots, n$. This implies that the sum $N = \sum_{i=1}^n e_iM$ is also direct.

Note that M_R is C2 and so it is C3, we have that $N = \oplus_{i=1}^n e_iM$ is a direct summand of M_R. Since $\mathrm{Soc}(M_R) \subseteq N \subseteq M$ and $\mathrm{Soc}(M_R) \trianglelefteq M_R$, $N \trianglelefteq M_R$, and hence $N = M$, i.e., $M = \oplus_{i=1}^n e_iM$. Let $0 \neq A_i$ be a submodule of e_iM. Since $K_i \trianglelefteq e_iM$, $K_i \cap A_i \neq 0$, and so $K_i \cap A_i = K_i$ because K_i is simple. It shows that each e_iM is uniform.

Since M_R is finite dimensional and C2, by [12, Proposition 3.7(1)], every monomorphism in $\mathrm{End(M_R)}$ is epic. Therefore, by [14, Lemma 4.26], S is semiperfect. $\square$

Theorem 9. *Let R be a right SP-injective ring. Then the following conditions are equivalent:*

(1) R is semiperfect and right Kasch.

(2) R is semiperfect and $S_r \trianglelefteq R_R$.

(3) R is semiperfect and $S_r \trianglelefteq {}_RR$.

(4) R is semiperfect and $Soc(eR) \neq 0$ for every local idempotent e of R.

(5) R is left min-CS and right Kasch.

(6) R is right min-CS and right finitely cogenerated.

(7) R is semilocal, right Kasch and right min-CS.

(8) R is right min-CS, left mininjective and left Kasch.

Proof. $(1) \Rightarrow (2)$. By Theorem 4(8) as R is semilocal.

$(1) \Rightarrow (3)$. By Theorem 4(2).

$(2) \Rightarrow (4)$, and $(1), (6) \Rightarrow (7)$ are obvious .

(3) $\Rightarrow$ (1). Since $S_r \trianglelefteq {}_RR$, $S_r \cap Re \neq 0$ for every local idempotent $e \in R$. Let $0 \neq a \in S_r \cap Re$, then $a = ae \in S_r e$. Thus $S_r e \neq 0$, and then R is right *Kasch* by [11, Proposition 3.3(2)].

(4) $\Rightarrow$ (1). By hypothesis, R is right minfull, and so R is right Kasch by [11, Theorem 3.7(1)].

(1), (3) $\Rightarrow$ (5). Firstly, by (1), R is right Kasch. Secondly, since R is semiperfect and $S_r \trianglelefteq {}_RR$, by [14, Lemma 4.2(1)], $\mathbf{lr}(L)$ is essential in a summand of ${}_RR$ for each left ideal L of R. Let Ra be a minimal left ideal. Then we have that $\mathbf{lr}(Ra)$ is essential in a summand of ${}_RR$. Since R is left mininjective by Theorem 4(1), aR is a minimal right ideal by [11, Theorem 1.14(1)]. So, observing that R is right minimal injective, according to [11, Lemma 1.1], we have $\mathbf{lr}(Ra) = Ra$. Thus, Ra is essential in a summand of ${}_RR$, that is, R is left min-CS.

(5) $\Rightarrow$ (1). Since R is right Kasch, right mininjective, by [14, Theorem 2.31], for every maximal right ideal M of R, $\mathbf{l}(M)$ is a minimal left ideal, which implies that $\mathbf{l}(M)$ is essential in a summand of ${}_RR$ because R is left min-CS. Therefore, R is semiperfect by [14, Lemma 4.1].

(1) $\Rightarrow$ (6). Assume (1). Then since R is right SP-injective, semiperfect and right Kasch, by Theorem 7(4), R is right finitely cogenerated. And by Theorem 4(8), $S_l \trianglelefteq R_R$. So, by [14, Lemma 4.2(1)], $\mathbf{rl}(T)$ is essential in a summand of R_R for each right ideal T of R. Let aR be a minimal right ideal. Then we have that $\mathbf{rl}(aR)$ is essential in a summand of R_R . Since R is right mininjective by Theorem 4(1), Ra is a minimal left ideal by [11, Theorem 1.14(1)]. So, observing that R is left minimal injective, according to [11, Lemma 1.1], we have $\mathbf{rl}(aR) = aR$. Thus, aR is essential in a summand of R_R , that is, R is right min-CS.

(6) $\Rightarrow$ (2). Assume (6). Then it is easy to see that $S_r \trianglelefteq R_R$. Since R is right SP-injective, by Theorem 3, R is right P-injective, and so it is right C2 by [10, Theorem 1.2(1)]. Thus , by hypothesis, R is right min-CS right C2 and right finitely cogenerated. Hence, by Lemma 8, R is semiperfect.

(7) $\Rightarrow$ (8). Since R is right SP-injective and right Kasch, by Theorem 4(1), it is left mininjective. But R is also semilocal, by Theorem 4(7), it is left Kasch.

(8) $\Rightarrow$ (1). Let $L \subseteq^{max} {}_RR$. we show that $\mathbf{r}(L)$ is essential in a summand of R_R. Since R is left Kasch, $\mathbf{r}(L) \neq 0$. Let $La = 0$, where $0 \neq a \in R$. Then $L = \mathbf{l}(a)$, and so Ra is minimal. Note that R is left mininjective, we have that $\mathbf{r}(L) = \mathbf{rl}(a) = aR$. Moreover, the left mininjectivity of R implies that aR is a minimal right ideal by [11, Theorem 1.14(1)]. But R is right min-CS, aR is essential in a summand of R_R. This shows that $\mathbf{r}(L)$ is essential in a summand of R_R. Hence R is semiperfect by [14, Lemma 4.1]. Finally, R is right Kasch by [14, Lemma 5.49]. □

3. Applications to quasi-Frobenius rings

In this section, we will give some new characterizations of quasi-Frobenius rings in terms of strongly P-injective rings.

Lemma 10. *Let R be a left Kasch right SP-injective ring. If every closed right ideal of R is cyclic , then R is semiperfect.*

Proof. We show that every maximal left ideal of R has a supplement in R and apply [14, Theorem B.28]. Let M be any maximal left ideal of R. Since R is a left Kasch ring, by [14, Proposition 1.44(4)], there exists $0 \neq a \in R$ such that $M = \mathbf{l}(a)$. Let C be a closed right ideal which is maximal with respect to $\mathbf{rl}(a) \cap C = 0$. Then by hypothesis, C is cyclic. Since R is right SP-injective, by Theorem 3, there exists a positive integer n such that $a^n \neq 0$ and $\mathbf{l}(a^nR \cap C) = \mathbf{l}(a^n) + \mathbf{l}(C)$. Observing that $a^nR \subseteq \mathbf{rl}(a^n) = \mathbf{rl}(a)$, we have $M + \mathbf{l}(C) = \mathbf{l}(a^n) + \mathbf{l}(C) = \mathbf{l}(a^nR \cap C) = \mathbf{l}(0) = R$. Now we claim that $\mathbf{l}(C)$ is a supplement for M. To see this, let $M + X = R$, where $X \subseteq \mathbf{l}(C)$ is a left ideal. Then $C \subseteq \mathbf{r}(X)$. Take $x \in X - M$, then $M + Rx = R$ and $C \subseteq \mathbf{r}(x)$. So, since $\mathbf{rl}(a) \cap \mathbf{r}(x) = \mathbf{r}(\mathbf{l}(a) + Rx) = 0$, the maximality of C implies that $C = \mathbf{r}(x)$. Hence $\mathbf{l}(C) = \mathbf{lr}(x) = Rx$ because R is right P-injective, and so $\mathbf{l}(C) = X$. It shows that $\mathbf{l}(C)$ is a supplement for M. Therefore, by [14, Theorem B.28], R is semiperfect. □

Recall that a ring R is said to be *left (right) CS* if every left (right) ideal of R is essential in a summand of ${}_RR$ (R_R); a ring R is said to be *left (right) CF* if every cyclic left (right) R-module can be embedded in a free module; a ring R is said to be a *right Goldie ring* if it has ACC on right annihilator and R_R is finite dimensional; a ring R is said to be a *right min-PF ring* if R is a semiperfect, right mininjective ring in which $S_r \trianglelefteq R_R$ and $\mathbf{lr}(K) = K$ for every simple left ideal $K \subseteq Re$, where $e^2 = e$ is local. These concepts can be found in [14].

Theorem 11. *The following statements are equivalent for a ring R:*

(1) R is a quasi-Frobenius ring.

(2) R is a right artinian right SP-injective ring.

(3) R is a right noetherian right SP-injective ring.

(4) R is right SP-injective with the ascending chain condition on annihilator right ideals.

(5) R is a left artinian right SP-injective ring.

(6) R is a right SP-injective semilocal ring with ACC on essential right ideals.

(7) R is a right SP-injective semilocal ring such that R/S_r is right Goldie.

(8) R is a left CF left CS right SP-injective ring.

(9) R is a left CF, right Kasch right SP-injective ring.

(10) R is a left noetherian right SP-injective, left Kasch ring, and every closed right ideal of R is cyclic.

(11) R is a right SP-injective right CS ring with ACC on essential right ideals.

Proof. (1) ⇒ (2) − (11) and (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5). Since R is right P-injective with the ascending chain condition on annihilator right ideals, by [14, Proposition 5.15], it is left artinian.

(5) ⇒ (1). Since R is a left artinian right SP-injective ring, it is a semiperfect, right mininjective ring with essential right socle, and so it is right minfull. By [11, Theorem 3.7(1)], R is right Kasch. Then, by Theorem 4(1), it is left and right mininjective. Therefore R is a quasi-Frobenius ring by [11, Corollary 4.8].

(6) ⇒ (7). Since R has ACC on essential right ideals, by [8, Lemma 3], R/S_r is right noetherian and hence R/S_r is right Goldie.

(7) ⇒ (1). Since R/S_r has ACC on right annihilators, by [14, Lemma 4.20(2)], Z_r is nilpotent. Since R is right P-injective, by [10, Theorem 2.1], $J = Z_r$. So, J is nilpotent, and hence it is semiprimary as it is semilocal. Thus, R is semiperfect, right mininjective and with essential right socle, it is right minfull. By [14, Theorem 3.12(1)], R is right Kasch, and so, by Theorem 4(1), it is a two-sided min-PF ring and R/S_r is right Goldie, by [14, Theorem 3.38], it is a quasi-Frobenius ring.

(8) ⇒ (5). By [7, Corollary 3.10], every left CF left CS ring is left artinian.

(9) ⇒ (5). By [7, Corollary 2. 6], every left CF right Kasch ring is left artinian.

(10) ⇒ (5) . By Lemma 10, R is semiperfect, and hence it is semilocal. Since R is left noetherian and right P-injective, J is nilpotent by [14, Lemma 8.6], so R is semiprimary, and thus it is left artinian.

(11) ⇒ (2). Since R is right SP-injective, it is right P-injective. So, by [14, Theorem 1.2(1)], it is right C2. Thus, R is right continuous and satisfies ACC on essential right ideals, by [8, Theorem(*ii*)], it is right artinian. □

Corollary 12. *The following statements are equivalent for a ring R:*

(1) R is a quasi-Frobenius ring.

(2) R is a right artinian right 2-injective ring.

(3) R is a right noetherian right 2-injective ring.

(4) [16, Corollary 3] *R is right 2-injective with the ascending chain condition on annihilator right ideals.*

(5) [23, Corollary 3(2)] *R is a left artinian right 2-injective ring.*

(6) R is a right 2-injective semilocal ring with ACC on essential right ideals.

(7) R is a right 2-injective semilocal ring such that R/S_r *is right Goldie.*

(8) [4, Corollary 2.15(4)] *R is a left CF left CS right 2-injective ring.*

(9) [4, Corollary 2.15(5)] *R is a left CF, right Kasch right 2-injective ring.*

(10) R is a left noetherian right 2-injective, left Kasch ring, and every closed right ideal of R is cyclic.

(11) R is a right 2-injective right CS ring with ACC on essential right ideals.

Recall that a ring R is called left *GC2* [18] if every left ideal that is isomorphic to $_RR$ is itself a direct summand of R. The following theorem improves the results in [23, Corollary 2.3]

Theorem 13. *The following statements are equivalent for a ring R:*

(1) R is a quasi-Frobenius ring.

(2) R is left noetherian right SP-injective and right Kasch.

(3) R is left noetherian right SP-injective and left $C2$.

(4) R is left noetherian right SP-injective and left $GC2$.

(5) R is a left noetherian right SP-injective semilocal ring.

(6) R is left noetherian right SP-injective and the ascending chain $\mathbf{r}(a_1) \subseteq \mathbf{r}(a_2a_1) \subseteq \mathbf{r}(a_3a_2a_1) \subseteq \cdots$ terminates for every sequence $\{a_1, a_2, \cdots\} \subseteq R$.

(7) R is left noetherian right SP-injective and right finite dimensional.

Proof. $(1) \Rightarrow (2), (6), (7)$ is obvious. Since right Kasch ring is left C2, and left C2 ring is left GC2, we have $(2) \Rightarrow (3) \Rightarrow (4)$.

(4)$\Rightarrow$ (5). Since left noetherian ring is left finite dimensional, and left finite dimensional left GC_2 ring is semilocal [22, Lemma 1.1], so (5) follows from (4).

(5)$\Rightarrow$ (1). Since R is left noetherian right P-injective, By [14, Lemma 8.6(1)], J is nilpotent. Thus R is left noetherian and semiprimary by hypothesis, and so it is left aritinian. By Theorem 11(5), R is quasi-Frobenius.

(6)$\Rightarrow$ (5). Since R is right P-injective and the ascending chain $\mathbf{r}(a_1) \subseteq \mathbf{r}(a_2a_1) \subseteq \mathbf{r}(a_3a_2a_1) \subseteq \cdots$ terminates for every sequence $\{a_1, a_2, \cdots\} \subseteq R$, by [3, Theorem 3.4], R is right perfect, and so it is semilocal .

(7)$\Rightarrow$ (3). Since R is right P-injective, by [10, Theorem 1.2(1)], R is right C2. □

Recall that a ring R is called *right coherent* if every finitely generated right ideal of R is finitely presented. We call a ring R *right min-coherent* if every minimal right ideal of R is finitely presented. We recall also that a ring R is called right AGP-injective [15] if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and Ra^n is a direct summand of $\mathbf{lr}(a^n)$.

Theorem 14. *Let R be a right SP-injective ring. Then the following statements are equivalent:*

(1) R is a quasi-Frobenius ring.

(2) R is a right artinian ring.

(3) R is left perfect and every cyclic right R-module is finite dimensional.

(4) R is left perfect, right min-coherent.

(5) R is right Kasch with left annihilators ACC.

(6) R is left GP-injective with left annihilators ACC.

(7) R is left AGP-injective with left annihilators ACC.

(8) R is semiprimary with left annihilators ACC.

(9) R is left perfect with left annihilators ACC.

Proof. (1) ⇒ (2) − (8) ; (6) ⇒ (7); and (8) ⇒ (9) are clear.

(2) ⇒ (1). By Theorem 11(2).

(3) ⇒ (2). Let I be any right ideal of R. Then R/I is finite dimensional by hypothesis, so $Soc(R/I)$ is finite dimensional and then finitely cogenerated. Since R is left perfect, it is right semiartinian, and so $Soc(R/I) \trianglelefteq R/I$. It follows that R/I is finitely cogenerated. Therefore, R is right artinian .

(4) ⇒ (2). Suppose (4) holds. Then R is left perfect and right mininjective and right min-coherent, so R is right Artinian by [13, Theorem 10].

(5) ⇒ (2). Since R is right Kasch, it is left GP-injective by Theorem 4(1). But R has ACC on left annihilators , it is right artinian by [3, Theorem 3.7(1)] .

(7) ⇒ (8). By [22, Corollary 1.6].

(9) ⇒ (1). Since R is left perfect, right SP-injective, by Corollary 6(2), it is left and right mininjective and $S_l \trianglelefteq {}_R R$, and thus R is quasi-Frobenius by [17, Theorem 2.5] because R has left annihilators ACC. □

Recall that a ring R is called left pseudo-coherent [2] if the left annihilator of every finite subsets of R is finitely generated. The following theorem improve the results of [24, Theorem 2.8].

Theorem 15. *The following statements are equivalent for a ring R:*

(1) R is a quasi-Frobenius ring.

(2) R is a right SP-injective left perfect, left pseudo-coherent ring.

(3) R is a right SP-injective , semiprimary, left pseudo-coherent ring.

(4) R is a right SP-injective , right perfect, left pseudo-coherent ring.

(5) R is a right SP-injective left perfect, right pseudo-coherent ring.

Proof. (1)⇒ (2)-(5). It is clear.

(2)⇒ (3). Since R is left perfect and right SP-injective, by Corollary 6(1), R is left and right Kasch. Since R is left Kasch, we have $J = \mathbf{lr}(J)$ by [24, Lemma 2.6]. Since R is right Kasch and right SP-injective, by Theorem 2.4(1), we have that it is left mininjective. Note that R is semilocal, by [1, Proposition 15.17] and [14, Theorem 5.52], $\mathbf{r}(J) = S_l$ is a finitely generated right ideal. But R is left pseudo-coherent, J is a finitely generated left ideal, and so J is nilpotent by [14, Lemma 5.64] since J is left T-nilpotent. Thus, R is semiprimary.

(3)⇒ (4) is clear.

(4)⇒ (1). Since R is right perfect, R has DCC on finitely generated left ideals. Note that R is left pseudo-coherent, every left annihilator of a finite subset of R is a finitely generated left ideal. So R has DCC on left annihilators of finite subsets of R. By [24, Lemma 2.7], every left annihilator of a subset of R is a left annihilator of a finite subset of R, and thus every left annihilator in R is a finitely generated left ideal. It follows that R has DCC on left annihilators and so R has ACC on right annihilators. By Theorem 11(4), R is a quasi-Frobenius ring.

(5)⇒ (1). Since R is left perfect and right SP-injective, we have that R is two-sided Kasch and two-sided mininjective by Corollary 6. Since R is right Kasch, we have $J = \mathbf{rl}(J)$ by [24, Lemma 2.6]. Since R is semilocal and right mininjective, by [1, Proposition 15.17] and [14, Theorem 5.52], $\mathbf{l}(J) = S_r$ is a finitely generated left ideal. But R is right pseudo-coherent, J is a finitely generated right ideal, and then J/J^2 is a finitely generated right R-module. Now, by Osofsky's Lemma [14, Lemma 6.50], we have that R is right artinian, and therefore R is a quasi-Frobenius ring by Theorem 11(2). □

References

[1] F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, Second edition, Springer-verlag, New Yock, 1992.

[2] J. E. Björk, *Rings satisfying certain chain conditions*, J. Reine Angew. Math. **245**(1970), 63-73.

[3] J. L. Chen and N. Q. Ding, *On general principally injective rings*, Comm. Algebra **27**(5) (1999), 2097-2116.

[4] J. L. Chen, N. Q. Ding and M. F. Yousif, *On noetherian rings with essential socle*, J. Aust. Math. Soc. **76**(1) (2004), 39-49.

[5] J. L. Chen, Y. Q. Zhou and Z. M. Zhu, *GP-injective rings need not be P-injective*, Comm. Algebra **35**(7) (2005), 2395-2402.

[6] C. Faith and P. Menal, *A counter example to a conjecture of Johns rings*, Proc. Amer. Math. Soc. **116**(1) (1992), 21–26.

[7] J. L. Gómez and P. A. Guil Asensio, *Torsionless modules and rings with finite essential socle*, Lecture Notes in Pure and Appl. Math. **201**v(1998), 261-278.

[8] S. K. Jain, S. R. Lopez-Permouth and S. T. Rizvi, *Continuous rings with ACC on essentials are Artinian*, Proc. AMS. **108**(3) (1990), 583-586.

[9] S. B. Nam, N. K. Kim and J. Y. Kim, *On simple GP-injective modules*, Comm. Algebra **23**(14) (1995), 5437-5444.

[10] W. K. Nicholson and M. F. Yousif, *Principally injective rings*, J. Algebra **174**(1) (1995), 77-93.

[11] W. K. Nicholson and M. F. Yousif, *Mininjective rings*, J. Algebra **187**(2) (1997), 548-578.

[12] W. K. Nicholson and M. F. Yousif, *Weakly continuous and C2-rings* , Comm. Algebra **29**(6) (2001), 2429-2446.
[13] W. K. Nicholson, J. K. Park and M. F. Yousif, *On simple-injective rings* , Algebra Colloq. **9**(3) (2002), 259-264.
[14] W. K. Nicholson and M. F. Yousif, *Quasi-Frobenius Rings*, Cambridge Tracts in Math. Cambridge University Press, Cambridge, (2003)
[15] S. S. Page and Y. Q. Zhou, *Generalizations of principally injective rings* , J. Algebra **206**(2) (1998), 706-721.
[16] E. A. Rutter, *Rings with the principle extension property* , Comm. Algebra **3**(3) (1975), 203-212.
[17] L. Shen and J. L. Chen, *New characterizations of quasi-Frobenius rings* , Comm. Algebra **34**(6) (2006), 2157-2165.
[18] M. F. Yousif and Y. Q. Zhou, *Rings for which certain elements have the principal extension property,* Algebra Colloq. **10**(4) (2003), 501-512.
[19] R. Yue Chi Ming, *On regular rings and self-injective rings II* , Glasnik Mat. **18**(38) (1983), 221-229.
[20] R. Yue Chi Ming, *On regular rings and artinian rings II*, Riv.Math.Univ.Parma **11**(4) (1985), 101-109 .
[21] R. Yue Chi Ming, *On YJ-injectivity and annihilators* , Georgian Math. J. **12**(3) (2005), 573-581.
[22] Y. Q. Zhou, *Rings in which certain right ideals are direct summands of annihilators*, J. Aust. Math. Soc. **73**(3) (2002), 335-346.
[23] Z. M. Zhu, *Some results on MP-injectivity and MGP-injectivity of rings and modules* , Ukrainian Mathematical Journal **63**(10) (2012), 1623-1632.
[24] Z. M. Zhu, *Some results on quasi-Frobenius rings*, Comment. Math. Univ. Carolin **58**(2) (2017), 147-151.

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