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ALMOST BALCOBALANCING NUMBERS II

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Abstract. In [24], we defined almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first and second type and determined the general terms of them in terms of balancing and Lucas-balancing numbers. In this work, we derive some new algebraic relations on them.

1. Introduction

A positive integer n is called a balancing number ([1]) if the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1}$$

holds for some positive integer r which is called balancer corresponding to n. If n is a balancing number with balancer r, then from (1)

$$r=\frac{-2n-1+\sqrt{8n^2+1}}{2}. \tag{2}$$

Though the definition of balancing numbers suggests that no balancing number should be less than 2. But based on (2), the authors of [1] noted that $8(0)^2+1=1$ and $8(1)^2+1=3^2$ are perfect squares. So they accepted 0 and 1 to be balancing numbers.

Panda and Ray ([10]) defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1+2+\cdots+n=(n+1)+(n+2)+\cdots+(n+r) \tag{3}$$

holds for some positive integer r which is called cobalancer corresponding to n. If n is a cobalancing number with cobalancer r, then from (3)

$$r=\frac{-2n-1+\sqrt{8n^2+8n+1}}{2}. \tag{4}$$

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From (4), the authors of [10] noted that $8(0)^2 + 8(0) + 1 = 1$ is a perfect square. So they accepted 0 to be a cobalancing number, just like Behera and Panda accepted 0 and 1 to be balancing numbers.

Let B_n denote the balancing number and let b_n denote the cobalancing number. Then from (2), B_n is a balancing number if and only if $8B_n^2 + 1$ is a perfect square and from (4), b_n is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. Thus

$$C_n = \sqrt{8B_n^2 + 1} \quad \text{and} \quad c_n = \sqrt{8b_n^2 + 8b_n + 1}$$

are integers which are called Lucas-balancing number and Lucas-cobalancing number, respectively (see also [8, 9, 14]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [6], Liptai proved that there is no Fibonacci balancing number except 1 and in [7] he proved that there is no Lucas-balancing number. In [17], Szalay considered the same problem and obtained some nice results by a different method. In [4], Kovács, Liptai, Olajos extended the concept of balancing numbers to the (a, b)-balancing numbers defined as follows: Let $a > 0$ and $b \geq 0$ be coprime integers. If

$$(a + b) + \cdots + (a(n - 1) + b) = (a(n + 1) + b) + \cdots + (a(n + r) + b)$$

for some positive integers n and r, then $an+b$ is an (a, b)-balancing number. The sequence of (a, b)-balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$. In [5], Liptai, Luca, Pintér and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let $y, k, l \in \mathbb{Z}^+$ with $y \geq 4$. A positive integer x with $x \leq y - 2$ is called a (k, l)-power numerical center for y if

$$1^k + \cdots + (x - 1)^k = (x + 1)^l + \cdots + (y - 1)^l.$$

They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for (k, l)-power numerical centers. For positive integers k, x, let $\Pi_k(x) = x(x + 1) \ldots (x + k - 1)$. Then it was proved in [4] that the equation $B_m = \Pi_k(x)$ for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2, 3, 4\}$ all solutions were determined. In [26] Tengely, considered the case $k = 5$ and proved that this Diophantine equation has no solution for $m \geq 0$ and $x \in \mathbb{Z}$. In [12], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers. In [15], Ray considered the sums of balancing and Lucas-balancing numbers by matrix methods. In [2], Dash, Ota, Dash considered the t-balancing numbers for an integer $t \geq 1$. They called that a positive integer n is a t-balancing number if the Diophantine equation

$$1 + 2 + \cdots + n - 1 = (n + 1 + t) + (n + 2 + t) + \cdots + (n + r + t)$$

holds for some positive integer r which is called t-balancer. A positive integer n is called a t-cobalancing number if the Diophantine equation

$$1 + 2 + \cdots + n = (n + 1 + t) + (n + 2 + t) + \cdots + (n + r + t)$$

holds for some positive integer r which is called t-cobalancer. In [21], Tekcan and Aydın determined the general terms of t-balancing and Lucas t-balancing numbers, and in [20], Tekcan and Erdem determined the general terms of t-cobalancing and Lucas t-cobalancing numbers in terms of balancing and Lucas-balancing numbers. In [11], Panda and Panda defined that a positive integer n is called an almost balancing number if the Diophantine equation

$$|[(n+1)+(n+2)+\cdots+(n+r)]-[1+2+\cdots+(n-1)]|=1 \tag{5}$$

holds for some positive integer r which is called the almost balancer. From (5), they have two cases: If $[(n+1)+(n+2)+\cdots+(n+r)]-[1+2+\cdots+(n-1)]=1$, then n is called an almost balancing number of first type and r is called an almost balancer of first type and if $[(n+1)+(n+2)+\cdots+(n+r)]-[1+2+\cdots+(n-1)]=-1$, then n is called an almost balancing number of second type and r is called an almost balancer of second type. In [13], Panda defined that a positive integer n is called an almost cobalancing number if the Diophantine equation

$$|[(n+1)+(n+2)+\cdots+(n+r)]-(1+2+\cdots+n)|=1 \tag{6}$$

holds for some positive integer r which is called an almost cobalancer. From (6), he has two cases: If $[(n+1)+(n+2)+\cdots+(n+r)]-(1+2+\cdots+n)=1$, then n is called an almost cobalancing number of first type and r is called an almost cobalancer of first type and if $[(n+1)+(n+2)+\cdots+(n+r)]-(1+2+\cdots+n)=-1$, then n is called an almost cobalancing number of second type and r is called an almost cobalancer of second type. In [25], Tekcan and Erdem determined the general terms of all almost balancing numbers and almost cobalancing numbers in terms of balancing and Lucas-balancing numbers. In [18], Tekcan considered the sums and spectral norms of all almost balancing numbers and in [19], Tekcan derived some results on almost balancing numbers, triangular numbers and square triangular numbers. In [22] and [23], we defined balcobalancing numbers, Lucas-balcobalancing numbers and balcobalancers and determined the general terms of them.

In [24], we defined that a positive integer n is called an almost balcobalancing number if the Diophantine equation

$$\left|\begin{array}{c}[1+2+\cdots+(n-1)+1+2+\cdots+(n-1)+n]\\ -2[(n+1)+(n+2)+\cdots+(n+r)]\end{array}\right|=1 \tag{7}$$

holds for some positive integer r which is called almost balcobalancer. From (7), we have two cases: If $[1+2+\cdots+(n-1)+1+2+\cdots+(n-1)+n]-2[(n+1)+(n+2)+\cdots+(n+r)]=1$, then n is called an almost balcobalancing number of first type, r is called an almost balcobalancer of first type and in this case

$$r=\frac{-2n-1+\sqrt{8n^2+4n-3}}{2}. \tag{8}$$

If $[1+2+\cdots+(n-1)+1+2+\cdots+(n-1)+n]-2[(n+1)+(n+2)+\cdots+(n+r)]=-1$, then n is called an almost balcobalancing number of second type, r is called an almost balcobalancer of second type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 4n + 5}}{2}. \tag{9}$$

Let B_n^{bc*} denote the almost balcobalancing number of first type and let B_n^{bc**} denote the almost balcobalancing number of second type. Then from (8), B_n^{bc*} is an almost balcobalancing number of first type if and only if $8(B_n^{bc*})^2 + 4B_n^{bc*} - 3$ is a perfect square. So

$$C_n^{bc*} = \sqrt{8(B_n^{bc*})^2 + 4B_n^{bc*} - 3} \tag{10}$$

is an integer which is called almost Lucas-balcobalancing number of first type, and from (9), B_n^{bc**} is an almost balcobalancing number of second type if and only if $8(B_n^{bc**})^2 + 4B_n^{bc**} + 5$ is a perfect square. So

$$C_n^{bc**} = \sqrt{8(B_n^{bc**})^2 + 4B_n^{bc**} + 5} \tag{11}$$

is an integer which is called almost Lucas-balcobalancing number of second type. We denote the almost balcobalancer of first type by R_n^{bc*} and denote the almost balcobalancer of second type by R_n^{bc**}. We proved in [24, Theorem 2.2] that the general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first type are

$$\begin{aligned} B_{2n-1}^{bc*} &= \frac{-4B_{2n-1} + 3C_{2n-1} - 1}{4} \\ B_{2n}^{bc*} &= \frac{4B_{2n-1} + 3C_{2n-1} - 1}{4} \\ C_{2n-1}^{bc*} &= 6B_{2n-1} - C_{2n-1} \\ C_{2n}^{bc*} &= 6B_{2n-1} + C_{2n-1} \\ R_{2n-1}^{bc*} &= \frac{16B_{2n-1} - 5C_{2n-1} - 1}{4} \\ R_{2n}^{bc*} &= \frac{8B_{2n-1} - C_{2n-1} - 1}{4} \end{aligned} \tag{12}$$

for $n \geq 1$, and proved in [24, Theorem 2.4] that the general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of second type are

$$\begin{aligned} B_n^{bc**} &= \frac{12B_{2n-1} + 3C_{2n-1} - 1}{4} \\ C_n^{bc**} &= 6B_{2n-1} + 3C_{2n-1} \\ R_n^{bc**} &= \frac{3C_{2n-1} - 1}{4} \end{aligned} \tag{13}$$

for $n \geq 1$.

2. Binet Formulas and Recurrence Relations.

Theorem 1. *Binet formulas for almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first type are*

$$\begin{aligned} B_{2n-1}^{bc*} &= \frac{(3-\sqrt{2})\alpha^{4n-2}+(3+\sqrt{2})\beta^{4n-2}-2}{8} \\ B_{2n}^{bc*} &= \frac{(3+\sqrt{2})\alpha^{4n-2}+(3-\sqrt{2})\beta^{4n-2}-2}{8} \\ C_{2n-1}^{bc*} &= \frac{(3-\sqrt{2})\alpha^{4n-2}-(3+\sqrt{2})\beta^{4n-2}}{2\sqrt{2}} \\ C_{2n}^{bc*} &= \frac{(3+\sqrt{2})\alpha^{4n-2}-(3-\sqrt{2})\beta^{4n-2}}{2\sqrt{2}} \\ R_{2n-1}^{bc*} &= \frac{(-5+4\sqrt{2})\alpha^{4n-2}-(5+4\sqrt{2})\beta^{4n-2}-2}{8} \\ R_{2n}^{bc*} &= \frac{(-1+2\sqrt{2})\alpha^{4n-2}-(1+2\sqrt{2})\beta^{4n-2}-2}{8} \end{aligned}$$

for $n \geq 1$, *and of second type are*

$$\begin{aligned} B_{n}^{bc**} &= \frac{(3+3\sqrt{2})\alpha^{4n-2}+(3-3\sqrt{2})\beta^{4n-2}-2}{8} \\ C_{n}^{bc**} &= \frac{(3+3\sqrt{2})\alpha^{4n-2}-(3-3\sqrt{2})\beta^{4n-2}}{2\sqrt{2}} \\ R_{n}^{bc**} &= \frac{3(\alpha^{4n-2}+\beta^{4n-2})-2}{8} \end{aligned}$$

for $n \geq 1$, *where* $\alpha = 1+\sqrt{2}$ *and* $\beta = 1-\sqrt{2}$.

Proof. Since $B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}$ and $C_n = \frac{\alpha^{2n}+\beta^{2n}}{2}$ by [14], we deduce from (12) that

$$\begin{aligned} B_{2n-1}^{bc*} &= \frac{-4B_{2n-1}+3C_{2n-1}-1}{4} \\ &= \frac{-4(\frac{\alpha^{4n-2}-\beta^{4n-2}}{4\sqrt{2}})+3(\frac{\alpha^{4n-2}+\beta^{4n-2}}{2})-1}{4} \\ &= \frac{\alpha^{4n-2}(\frac{-1}{\sqrt{2}}+\frac{3}{2})+\beta^{4n-2}(\frac{1}{\sqrt{2}}+\frac{3}{2})-1}{4} \\ &= \frac{(3-\sqrt{2})\alpha^{4n-2}+(3+\sqrt{2})\beta^{4n-2}-2}{8}. \end{aligned}$$

The others can be proved similarly. □

Recall that balancing numbers satisfy recurrence relation

$$B_n = 6B_{n-1} - B_{n-2}$$

for $n \geq 2$. Similarly we can give the following result.

Theorem 2. *Almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first type satisfy the recurrence relations*

$$\begin{aligned}
B_n^{bc*} &= B_{n-1}^{bc*} + 34B_{n-2}^{bc*} - 34B_{n-3}^{bc*} - B_{n-4}^{bc*} + B_{n-5}^{bc*} \\
C_n^{bc*} &= C_{n-1}^{bc*} + 34C_{n-2}^{bc*} - 34C_{n-3}^{bc*} - C_{n-4}^{bc*} + C_{n-5}^{bc*} \\
R_n^{bc*} &= R_{n-1}^{bc*} + 34R_{n-2}^{bc*} - 34R_{n-3}^{bc*} - R_{n-4}^{bc*} + R_{n-5}^{bc*}
\end{aligned}$$

for $n \geq 6$, and of second type satisfy the recurrence relations

$$\begin{aligned}
B_n^{bc**} &= 35B_{n-1}^{bc**} - 35B_{n-2}^{bc**} + B_{n-3}^{bc**} \\
C_n^{bc**} &= 35C_{n-1}^{bc**} - 35C_{n-2}^{bc**} + C_{n-3}^{bc**} \\
R_n^{bc**} &= 35R_{n-1}^{bc**} - 35R_{n-2}^{bc**} + R_{n-3}^{bc**}
\end{aligned}$$

for $n \geq 4$.

Proof. Let n be even, say $n = 2k$ for some positive integer k. Then from (12), we get

$$\begin{aligned}
B_{2k-1}^{bc*} + 34B_{2k-2}^{bc*} &- 34B_{2k-3}^{bc*} - B_{2k-4}^{bc*} + B_{2k-5}^{bc*} \\
&= \left(\frac{-4B_{2k-1} + 3C_{2k-1} - 1}{4}\right) + 34\left(\frac{4B_{2k-3} + 3C_{2k-3} - 1}{4}\right) \\
&\quad - 34\left(\frac{-4B_{2k-3} + 3C_{2k-3} - 1}{4}\right) - \left(\frac{4B_{2k-5} + 3C_{2k-5} - 1}{4}\right) \\
&\quad + \left(\frac{-4B_{2k-5} + 3C_{2k-5} - 1}{4}\right) \\
&= \frac{-4B_{2k-1} + 272B_{2k-3} - 8B_{2k-5} + 3C_{2k-1} - 1}{4}.
\end{aligned}$$

Here we notice that $-4B_{2k-1} + 272B_{2k-3} - 8B_{2k-5} = 4B_{2k-1}$. So we get

$$\begin{aligned}
B_{2k-1}^{bc*} + 34B_{2k-2}^{bc*} &- 34B_{2k-3}^{bc*} - B_{2k-4}^{bc*} + B_{2k-5}^{bc*} \\
&= \left(\frac{-4B_{2k-1} + 3C_{2k-1} - 1}{4}\right) + 34\left(\frac{4B_{2k-3} + 3C_{2k-3} - 1}{4}\right) \\
&\quad - 34\left(\frac{-4B_{2k-3} + 3C_{2k-3} - 1}{4}\right) - \left(\frac{4B_{2k-5} + 3C_{2k-5} - 1}{4}\right) \\
&\quad + \left(\frac{-4B_{2k-5} + 3C_{2k-5} - 1}{4}\right) \\
&= \frac{-4B_{2k-1} + 272B_{2k-3} - 8B_{2k-5} + 3C_{2k-1} - 1}{4} \\
&= \frac{4B_{2k-1} + 3C_{2k-1} - 1}{4} = B_{2k}^{bc*}.
\end{aligned}$$

Thus

$$B_n^{bc*} = B_{n-1}^{bc*} + 34B_{n-2}^{bc*} - 34B_{n-3}^{bc*} - B_{n-4}^{bc*} + B_{n-5}^{bc*}.$$

The others can be proved similarly. □

3. Balancing Numbers and Almost Balcobalancing Numbers

In this subsection, we give the general terms of all balancing numbers in terms of almost balcobalancing number of first and of second type.

Theorem 3. *The general terms of all balancing numbers are*

$$\begin{aligned}
B_{2n-1} &= \frac{B_{2n}^{bc*} - B_{2n-1}^{bc*}}{2} \\
B_{2n} &= \frac{B_{2n+1}^{bc*} - B_{2n}^{bc*}}{6} \\
b_{2n-1} &= \frac{R_{2n}^{bc*} - R_{2n-1}^{bc*} - 1}{2} \\
b_{2n} &= \frac{R_{2n+1}^{bc*} - R_{2n}^{bc*} - 3}{6} \\
C_{2n-1} &= \frac{C_{2n}^{bc*} - C_{2n-1}^{bc*}}{2} \\
C_{2n} &= \frac{C_{2n+1}^{bc*} - C_{2n}^{bc*}}{6} \\
c_{2n-1} &= B_{2n}^{bc*} - B_{2n-1}^{bc*} - R_{2n}^{bc*} + R_{2n-1}^{bc*} \\
c_{2n} &= \frac{B_{2n+1}^{bc*} - B_{2n}^{bc*} - R_{2n+1}^{bc*} + R_{2n}^{bc*}}{3}
\end{aligned}$$

for $n \geq 1,$ *or*

$$\begin{aligned}
B_{2n-1} &= \frac{B_n^{bc**} - R_n^{bc**}}{3} \\
B_{2n} &= \frac{B_n^{bc**} + C_n^{bc**} - R_n^{bc**}}{3} \\
b_{2n-1} &= \frac{B_n^{bc**} - C_n^{bc**} + 5R_n^{bc**}}{3} \\
b_{2n} &= \frac{C_n^{bc**} - 3}{6} \\
C_{2n-1} &= \frac{4B_n^{bc**} - 2C_n^{bc**} + 8R_n^{bc**} + 3}{3} \\
C_{2n} &= \frac{2B_n^{bc**} + 3C_n^{bc**} - 2R_n^{bc**}}{3}
\end{aligned}$$

$$c_{2n-1} = \frac{2C_n^{bc**} - 12R_n^{bc**} - 3}{3}$$
$$c_{2n} = \frac{2B_n^{bc**} + C_n^{bc**} - 2R_n^{bc**}}{3}$$

for $n \geq 1$.

Proof. Since $B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}$, we get

$$\begin{aligned}
B_{2n-1} &= \frac{\alpha^{4n-2} - \beta^{4n-2}}{4\sqrt{2}} \\
&= \frac{\alpha^{4n-2}(2\sqrt{2}) - \beta^{4n-2}(2\sqrt{2})}{16} \\
&= \frac{\alpha^{4n-2}(3+\sqrt{2}-3+\sqrt{2}) + \beta^{4n-2}(3-\sqrt{2}-3-\sqrt{2})}{16} \\
&= \frac{\frac{(3+\sqrt{2})\alpha^{4n-2}+(3-\sqrt{2})\beta^{4n-2}-2}{8} - \frac{(3-\sqrt{2})\alpha^{4n-2}+(3+\sqrt{2})\beta^{4n-2}-2}{8}}{2} \\
&= \frac{B_{2n}^{bc*} - B_{2n-1}^{bc*}}{2}
\end{aligned}$$

by Theorem 1. The others can be proved similarly. □

4. Almost Balcobalancing Numbers and Balcobalancing Numbers.

In this section, we give the general terms of all almost balcobalancing numbers of first and of second type in terms of balcobalancing numbers and conversely we give the general terms of all balcobalancing numbers in terms of almost balcobalancing numbers of first and of second type.

Theorem 4. *The general terms of almost balcobalancing numbers, almost Lucas–balcobalancing numbers and almost balcobalancers of first type are*

$$B_{2n-1}^{bc*} = \frac{-B_n^{bc} + 5B_{n-1}^{bc} + 3C_{n-1}^{bc} + 2R_n^{bc} + 1}{2}$$
$$B_{2n}^{bc*} = \frac{B_n^{bc} + 7B_{n-1}^{bc} + 3C_{n-1}^{bc} - 2R_n^{bc} + 1}{2}$$
$$C_{2n-1}^{bc*} = 3B_n^{bc} - B_{n-1}^{bc} - 2C_{n-1}^{bc} - 6R_n^{bc} - 1$$
$$C_{2n}^{bc*} = 3B_n^{bc} + 7B_{n-1}^{bc} + 2C_{n-1}^{bc} - 6R_n^{bc} + 1$$
$$R_{2n-1}^{bc*} = \frac{4B_n^{bc} - 6B_{n-1}^{bc} - 5C_{n-1}^{bc} - 8R_n^{bc} - 3}{2}$$
$$R_{2n}^{bc*} = \frac{2B_n^{bc} - C_{n-1}^{bc} - 4R_n^{bc} - 1}{2}$$

for $n\geq 2$, and of second type are

$$\begin{aligned} B_n^{bc**} &= \frac{3B_n^{bc}+9B_{n-1}^{bc}+3C_{n-1}^{bc}-6R_n^{bc}+1}{2}\\ C_n^{bc**} &= 3B_n^{bc}+15B_{n-1}^{bc}+6C_{n-1}^{bc}-6R_n^{bc}+3\\ R_n^{bc**} &= \frac{6B_{n-1}^{bc}+3C_{n-1}^{bc}+1}{2}\end{aligned}$$

for $n\geq 2$.

Proof. Recall that $B_n^{bc}=\frac{\alpha^{4n+1}+\beta^{4n+1}}{8}-\frac{1}{4}$, $C_n^{bc}=\frac{\alpha^{4n+1}-\beta^{4n+1}}{2\sqrt{2}}$ and $R_n^{bc}=\frac{\alpha^{4n}+\beta^{4n}}{8}-\frac{1}{4}$ by [22, Theorem 3.6]. So we get from (12) that

$$\begin{aligned} B_{2n-1}^{bc*} &= \frac{-4B_{2n-1}+3C_{2n-1}-1}{4}\\ &= \frac{-4(\frac{\alpha^{4n-2}-\beta^{4n-2}}{4\sqrt{2}})+3(\frac{\alpha^{4n-2}+\beta^{4n-2}}{2})-1}{4}\\ &= \frac{\alpha^{4n-2}(\frac{-1}{\sqrt{2}}+\frac{3}{2})+\beta^{4n-2}(\frac{1}{\sqrt{2}}+\frac{3}{2})-1}{4}\\ &= \alpha^{4n-2}(\frac{3-\sqrt{2}}{8})+\beta^{4n-2}(\frac{3-\sqrt{2}}{8})-\frac{1}{4}\\ &= \frac{\left\{\begin{array}{c}\alpha^{4n-2}(\frac{-\alpha^3}{8}+\frac{5\alpha^{-1}}{8}+\frac{3\alpha^{-1}}{2\sqrt{2}}+\frac{2\alpha^2}{8})\\ +\beta^{4n-2}(\frac{-\beta^3}{8}+\frac{5\beta^{-1}}{8}-\frac{3\beta^{-1}}{2\sqrt{2}}+\frac{2\alpha^2}{8})-\frac{1}{2}\end{array}\right\}}{2}\\ &= \frac{\left\{\begin{array}{c}-(\frac{\alpha^{4n+1}+\beta^{4n+1}}{8}-\frac{1}{4})+5(\frac{\alpha^{4n-3}+\beta^{4n-3}}{8}-\frac{1}{4})+3(\frac{\alpha^{4n-3}-\beta^{4n-3}}{2\sqrt{2}})\\ +2(\frac{\alpha^{4n}+\beta^{4n}}{8}-\frac{1}{4})+1\end{array}\right\}}{2}\\ &= \frac{-B_n^{bc}+5B_{n-1}^{bc}+3C_{n-1}^{bc}+2R_n^{bc}+1}{2}.\end{aligned}$$

The others can be proved similarly. □

Conversely, we give the general terms of all balcobalancing numbers in terms of almost balcobalancing numbers of first and of second type as follows.

Theorem 5. *The general terms of all balcobalancing numbers are*

$$\begin{aligned} B_n^{bc} &= B_{2n-1}^{bc*}+C_{2n}^{bc*}\\ C_n^{bc} &= R_{2n+2}^{bc*}-R_{2n+1}^{bc*}\\ R_n^{bc} &= \frac{C_{2n+1}^{bc*}-C_{2n}^{bc*}-6}{24}\end{aligned}$$

for $n \geq 1$, *or*

$$B_n^{bc} = \frac{C_{n+1}^{bc**} - 6R_{n+1}^{bc**} - 3}{6}$$
$$C_n^{bc} = \frac{2B_{n+1}^{bc**} - 2C_{n+1}^{bc**} + 10R_{n+1}^{bc**} + 3}{3}$$
$$R_n^{bc} = \frac{2B_n^{bc**} + 3C_n^{bc**} - 2R_n^{bc**} - 3}{12}$$

for $n \geq 1$.

Proof. Applying [22, Theorem 3.6], we get

$$\begin{aligned}
B_n^{bc} &= \frac{\alpha^{4n+1} + \beta^{4n+1}}{8} - \frac{1}{4} \\
&= \frac{\alpha^{4n-2}(7+5\sqrt{2}) + \beta^{4n-2}(7-5\sqrt{2})}{8} - \frac{1}{4} \\
&= \frac{\alpha^{4n-2}(3-\sqrt{2}+4+6\sqrt{2}) + \beta^{4n-2}(3+\sqrt{2}+4-6\sqrt{2})}{8} - \frac{1}{4} \\
&= \frac{(3-\sqrt{2})\alpha^{4n-2} + (3+\sqrt{2})\beta^{4n-2} - 2}{8} + \frac{(3+\sqrt{2})\alpha^{4n-2} - (3-\sqrt{2})\beta^{4n-2}}{2\sqrt{2}} \\
&= B_{2n-1}^{bc*} + C_{2n}^{bc*}
\end{aligned}$$

by Theorem 1. The others can be proved similarly. □

5. Relationship with Pell and Pell-Lucas Numbers.

Recall that general terms of all balancing numbers can be given in terms of Pell numbers

$$B_n = \frac{P_{2n}}{2},\ b_n = \frac{P_{2n-1} - 1}{2},\ C_n = P_{2n} + P_{2n-1} \text{ and } c_n = P_{2n-1} + P_{2n-2}.$$

Similarly we can give general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first and of second type in terms of Pell numbers as follows.

Theorem 6. *The general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first type are*

$$\begin{aligned}
B_{2n-1}^{bc*} &= \frac{P_{4n-2} + 3P_{4n-3} - 1}{4} \\
B_{2n}^{bc*} &= \frac{5P_{4n-2} + 3P_{4n-3} - 1}{4} \\
C_{2n-1}^{bc*} &= 2P_{4n-2} - P_{4n-3} \\
C_{2n}^{bc*} &= 4P_{4n-2} + P_{4n-3}
\end{aligned}$$

$$\begin{aligned} R^{bc*}_{2n-1} &= \frac{3P_{4n-2}-5P_{4n-3}-1}{4} \\ R^{bc*}_{2n} &= \frac{3P_{4n-2}-P_{4n-3}-1}{4} \end{aligned}$$

for $n \geq 1$, and of second type are

$$\begin{aligned} B^{bc**}_{n} &= \frac{9P_{4n-2}+3P_{4n-3}-1}{4} \\ C^{bc**}_{n} &= 6P_{4n-2}+3P_{4n-3} \\ R^{bc**}_{n} &= \frac{3P_{4n-2}+3P_{4n-3}-1}{4} \end{aligned}$$

for $n \geq 1$.

Proof. From Theorem 1, we deduce that

$$\begin{aligned} B^{bc*}_{2n-1} &= \frac{\alpha^{4n-2}(3-\sqrt{2})+\beta^{4n-2}(3+\sqrt{2})-2}{8} \\ &= \frac{\alpha^{4n-2}(-2+3\sqrt{2})+\beta^{4n-2}(2+3\sqrt{2})}{8\sqrt{2}}-\frac{1}{4} \\ &= \frac{\alpha^{4n-2}(\frac{1+3\alpha^{-1}}{2\sqrt{2}})+\beta^{4n-2}(\frac{-1-3\beta^{-1}}{2\sqrt{2}})}{4}-\frac{1}{4} \\ &= \frac{\frac{\alpha^{4n-2}-\beta^{4n-2}}{2\sqrt{2}}+3\left(\frac{\alpha^{4n-3}-\beta^{4n-3}}{2\sqrt{2}}\right)-1}{4} \\ &= \frac{P_{4n-2}+3P_{4n-3}-1}{4} \end{aligned}$$

since $P_n = \frac{\alpha^n-\beta^n}{2\sqrt{2}}$. The others can be proved similarly. □

Conversely, we can give the general terms of Pell numbers in terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first and of second type as follows.

Theorem 7. *The general term of Pell numbers is*

$$P_{2n} = \begin{cases} \dfrac{B^{bc*}_{n+1}-B^{bc*}_{n}}{3} & n \geq 2 \text{ even} \\ B^{bc*}_{n+1}-B^{bc*}_{n} & n \geq 1 \text{ odd} \end{cases}$$

$$P_{2n-1} = \begin{cases} \dfrac{R^{bc*}_{n+1}-R^{bc*}_{n}}{3} & n \geq 2 \text{ even} \\ R^{bc*}_{n+1}-R^{bc*}_{n} & n \geq 1 \text{ odd} \end{cases}$$

or

$$P_{2n}=\begin{cases}\dfrac{2(B^{bc**}_{\frac{n}{2}}+C^{bc**}_{\frac{n}{2}}-R^{bc**}_{\frac{n}{2}})}{3} & n\geq 2 \ even\\[2ex] \dfrac{2(B^{bc**}_{\frac{n+1}{2}}-R^{bc**}_{\frac{n+1}{2}})}{3} & n\geq 1 \ odd\end{cases}$$

$$P_{2n-1}=\begin{cases}\dfrac{C^{bc**}_{\frac{n}{2}}}{3} & n\geq 2 \ even\\[2ex] \dfrac{2B^{bc**}_{\frac{n+1}{2}}-2C^{bc**}_{\frac{n+1}{2}}+10R^{bc**}_{\frac{n+1}{2}}+3}{3} & n\geq 1 \ odd.\end{cases}$$

Proof. Let n be even, say $n=2k$ for some positive integer k. Then

$$\begin{aligned}P_{4k}&=\frac{\alpha^{4k}-\beta^{4k}}{2\sqrt{2}}\\&=\frac{6\sqrt{2}(\alpha^{4k}-\beta^{4k})}{24}\\&=\frac{\alpha^{4k}[(3-\sqrt{2})\alpha^2-(3+\sqrt{2})\alpha^{-2}]+\beta^{4k}[(3+\sqrt{2})\beta^2-(3-\sqrt{2})\beta^{-2}]}{24}\\&=\frac{\frac{(3-\sqrt{2})\alpha^{4k+2}+(3+\sqrt{2})\beta^{4k+2}-2}{8}-\frac{(3+\sqrt{2})\alpha^{4k-2}+(3-\sqrt{2})\beta^{4k-2}-2}{8}}{3}\\&=\frac{B^{bc*}_{2k+1}-B^{bc*}_{2k}}{3}\end{aligned}$$

by Theorem 1. So $P_{2n}=\frac{B^{bc*}_{n+1}-B^{bc*}_{n}}{3}$. The others can be proved similarly. □

Similarly the general terms of all balancing numbers can be given in terms of Pell-Lucas numbers

$$B_n=\frac{Q_{2n}+Q_{2n-1}}{8},\ b_n=\frac{Q_{2n}-Q_{2n-1}-4}{8},\ C_n=\frac{Q_{2n}}{2} \text{ and } c_n=\frac{Q_{2n-1}}{2}.$$

As in Theorem 6, we can give general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first and of second type in terms of Pell-Lucas numbers as follows.

Theorem 8. *The general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first type are*

$$\begin{aligned} B_{2n-1}^{bc*} &= \frac{2Q_{4n-2} - Q_{4n-3} - 2}{8} \\ B_{2n}^{bc*} &= \frac{4Q_{4n-2} + Q_{4n-3} - 2}{8} \\ C_{2n-1}^{bc*} &= \frac{Q_{4n-2} + 3Q_{4n-3}}{4} \end{aligned}$$

$$\begin{aligned} C_{2n}^{bc*} &= \frac{5Q_{4n-2} + 3Q_{4n-3}}{4} \\ R_{2n-1}^{bc*} &= \frac{-Q_{4n-2} + 4Q_{4n-3} - 2}{8} \\ R_{2n}^{bc*} &= \frac{Q_{4n-2} + 2Q_{4n-3} - 2}{8} \end{aligned}$$

for $n \geq 1$, and of second type are

$$\begin{aligned} B_{n}^{bc**} &= \frac{6Q_{4n-2} + 3Q_{4n-3} - 2}{8} \\ C_{n}^{bc**} &= \frac{9Q_{4n-2} + 3Q_{4n-3}}{4} \\ R_{n}^{bc**} &= \frac{3Q_{4n-2} - 2}{8} \end{aligned}$$

for $n \geq 1$.

Proof. It can be proved in the same way that Theorem 6 was proved. □

Conversely, we can give the general terms of Pell-Lucas numbers in terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first and of second type as follows.

Theorem 9. *The general term of Pell-Lucas numbers is*

$$Q_{2n} = \begin{cases} \dfrac{2(B_{n+1}^{bc*} - B_{n}^{bc*} + R_{n+1}^{bc*} - R_{n}^{bc*})}{3} & n \geq 2 \text{ even} \\ 2(B_{n+1}^{bc*} - B_{n}^{bc*} + R_{n+1}^{bc*} - R_{n}^{bc*}) & n \geq 1 \text{ odd} \end{cases}$$

$$Q_{2n-1} = \begin{cases} \dfrac{2(B_{n+1}^{bc*} - B_{n}^{bc*} - R_{n+1}^{bc*} + R_{n}^{bc*})}{3} & n \geq 2 \text{ even} \\ 2(B_{n+1}^{bc*} - B_{n}^{bc*} - R_{n+1}^{bc*} + R_{n}^{bc*}) & n \geq 1 \text{ odd} \end{cases}$$

or

$$Q_{2n} = \begin{cases} \dfrac{4B^{bc**}_{\frac{n}{2}} + 6C^{bc**}_{\frac{n}{2}} - 4R^{bc**}_{\frac{n}{2}}}{3} & n \geq 2 \ \textit{even} \\ \dfrac{8B^{bc**}_{\frac{n+1}{2}} - 4C^{bc**}_{\frac{n+1}{2}} + 16R^{bc**}_{\frac{n+1}{2}} + 6}{3} & n \geq 1 \ \textit{odd} \end{cases}$$

$$Q_{2n-1} = \begin{cases} \dfrac{4B^{bc**}_{\frac{n}{2}} + 2C^{bc**}_{\frac{n}{2}} - 4R^{bc**}_{\frac{n}{2}}}{3} & n \geq 2 \ \textit{even} \\ \dfrac{4C^{bc**}_{\frac{n+1}{2}} - 24R^{bc**}_{\frac{n+1}{2}} - 6}{3} & n \geq 1 \ \textit{odd.} \end{cases}$$

Proof. It can be proved as in the same way that Theorem 7 was proved. □

Thus we construct one-to-one correspondence between all almost balcobalancing numbers and Pell and Pell-Lucas numbers.

6. Relationship with Triangular and Square Triangular Numbers.

Recall that triangular numbers denoted by T_n are the numbers of the form

$$T_n = \frac{n(n+1)}{2}.$$

It is known that there is a correspondence between balancing numbers and triangular numbers. Indeed from (1), we note that n is a balancing number if and only if n^2 is a triangular number since

$$\frac{(n+r)(n+r+1)}{2} = n^2.$$

So

$$T_{B_n+R_n} = B_n^2.$$

For triangular and balcobalancing numbers, we proved in [22, Theorem 5.12] that

$$T_{B_n^{bc}+R_n^{bc}} = (B_n^{bc})^2 + \frac{B_n^{bc}}{2}. \tag{14}$$

As in (14), we can give the following result.

Theorem 10. *B_n^{bc*} is a almost balcobalancing number of first type if and only if $2B_n^{bc*}R_n^{bc*} + (R_n^{bc*})^2 + \frac{B_n^{bc*}}{2} + R_n^{bc*} + \frac{1}{2}$ is a triangular number, that is,*

$$T_{B_n^{bc*}+R_n^{bc*}} = 2B_n^{bc*}R_n^{bc*} + (R_n^{bc*})^2 + \frac{B_n^{bc*}}{2} + R_n^{bc*} + \frac{1}{2}$$

and B_n^{bc**} *is a almost balcobalancing number of second type if and only if* $2B_n^{bc**}R_n^{bc**}+(R_n^{bc**})^2+\frac{B_n^{bc**}}{2}+R_n^{bc**}-\frac{1}{2}$ *is a triangular number, that is,*

$$T_{B_n^{bc**}+R_n^{bc**}}=2B_n^{bc**}R_n^{bc**}+(R_n^{bc**})^2+\frac{B_n^{bc**}}{2}+R_n^{bc**}-\frac{1}{2}.$$

Proof. From (7), we get $n^2-2nr-r^2-r-1=0$ and hence

$$\frac{(n+r)(n+r+1)}{2}=2nr+r^2+\frac{n}{2}+r+\frac{1}{2}.$$

So

$$T_{B_n^{bc*}+R_n^{bc*}}=2B_n^{bc*}R_n^{bc*}+(R_n^{bc*})^2+\frac{B_n^{bc*}}{2}+R_n^{bc*}+\frac{1}{2}$$

as we claimed. The other case can be proved similarly. □

There are infinitely many triangular numbers that are also square numbers which are called square triangular numbers and is denoted by S_n. Notice that

$$S_n=s_n^2=\frac{t_n(t_n+1)}{2},$$

where s_n and t_n are the sides of the corresponding square and triangle. We can give the general terms of S_n, s_n and t_n in terms of balancing and cobalancing numbers, namely, $S_n=B_n^2, s_n=B_n$ and $t_n=B_n+b_n$. Their Binet formulas are

$$\begin{aligned} S_n &= \frac{\alpha^{4n}+\beta^{4n}-2}{32}, \\ s_n &= \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}, \quad \text{and} \\ t_n &= \frac{\alpha^{2n}+\beta^{2n}-2}{4} \end{aligned} \tag{15}$$

for $n\geq 1$. We can give the general terms of almost balcobalancing numbers of first and of second type in terms of s_n and t_n as follows.

Theorem 11. *The general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first type are*

$$\begin{aligned} B_{2n-1}^{bc*} &= \frac{-2s_{2n-1}+3t_{2n-1}+1}{2} \\ B_{2n}^{bc*} &= \frac{2s_{2n-1}+3t_{2n-1}+1}{2} \\ C_{2n-1}^{bc*} &= 6s_{2n-1}-2t_{2n-1}-1 \\ C_{2n}^{bc*} &= 6s_{2n-1}+2t_{2n-1}+1 \\ R_{2n-1}^{bc*} &= \frac{8s_{2n-1}-5t_{2n-1}-3}{2} \\ R_{2n}^{bc*} &= \frac{4s_{2n-1}-t_{2n-1}-1}{2} \end{aligned}$$

for $n \geq 1$*, and of second type are*

$$\begin{aligned} B_{n}^{bc**} &= \frac{6s_{2n-1}+3t_{2n-1}+1}{2} \\ C_{n}^{bc**} &= 6s_{2n-1}+6t_{2n-1}+3 \\ R_{n}^{bc**} &= \frac{3t_{2n-1}+1}{2} \end{aligned}$$

for $n \geq 1$.

Proof. From (12), we get

$$\begin{aligned} B_{2n-1}^{bc*} &= \frac{-4B_{2n-1}+3C_{2n-1}-1}{4} \\ &= \frac{-4(\frac{\alpha^{4n-2}-\beta^{4n-2}}{4\sqrt{2}})+3(\frac{\alpha^{4n-2}+\beta^{4n-2}}{2})-1}{4} \\ &= \frac{\alpha^{4n-2}(3-\sqrt{2})+\beta^{4n-2}(3+\sqrt{2})-2}{8} \\ &= \frac{\alpha^{4n-2}(-2+3\sqrt{2})+\beta^{4n-2}(2+3\sqrt{2})-2\sqrt{2}}{8\sqrt{2}} \\ &= \frac{-2(\frac{\alpha^{4n-2}-\beta^{4n-2}}{4\sqrt{2}})+3(\frac{\alpha^{4n-2}+\beta^{4n-2}-2}{4})+1}{2} \\ &= \frac{-2s_{2n-1}+3t_{2n-1}+1}{2} \end{aligned}$$

by (15). The others can be proved similarly. □

Conversely, we can give the general terms of S_n, s_n and t_n in terms of almost balcobalancing numbers of first and of second type as follows.

Theorem 12. *The general terms of S_n, s_n and t_n are*

$$S_n = \begin{cases} (\frac{B^{bc*}_{n+1} - B^{bc*}_n}{6})^2 & n \geq 2 \ \textit{even} \\ \\ (\frac{B^{bc*}_{n+1} - B^{bc*}_n}{2})^2 & n \geq 1 \ \textit{odd} \end{cases}$$

$$s_n = \begin{cases} \frac{B^{bc*}_{n+1} - B^{bc*}_n}{6} & n \geq 2 \ \textit{even} \\ \\ \frac{B^{bc*}_{n+1} - B^{bc*}_n}{2} & n \geq 1 \ \textit{odd} \end{cases}$$

$$t_n = \begin{cases} \frac{B^{bc*}_{n+1} - B^{bc*}_n + R^{bc*}_{n+1} - R^{bc*}_n - 3}{6} & n \geq 2 \ \textit{even} \\ \\ \frac{B^{bc*}_{n+1} - B^{bc*}_n + R^{bc*}_{n+1} - R^{bc*}_n - 1}{2} & n \geq 1 \ \textit{odd} \end{cases}$$

or

$$S_n = \begin{cases} (\frac{B^{bc**}_{\frac{n}{2}} + C^{bc**}_{\frac{n}{2}} - R^{bc**}_{\frac{n}{2}}}{3})^2 & n \geq 2 \ \textit{even} \\ \\ (\frac{B^{bc**}_{\frac{n+1}{2}} - R^{bc**}_{\frac{n+1}{2}}}{3})^2 & n \geq 1 \ \textit{odd} \end{cases}$$

$$s_n = \begin{cases} \frac{B^{bc**}_{\frac{n}{2}} + C^{bc**}_{\frac{n}{2}} - R^{bc**}_{\frac{n}{2}}}{3} & n \geq 2 \ \textit{even} \\ \\ \frac{B^{bc**}_{\frac{n+1}{2}} - R^{bc**}_{\frac{n+1}{2}}}{3} & n \geq 1 \ \textit{odd} \end{cases}$$

$$t_n = \begin{cases} \frac{2B^{bc**}_{\frac{n}{2}} + 3C^{bc**}_{\frac{n}{2}} - 2R^{bc**}_{\frac{n}{2}} - 3}{6} & n \geq 2 \ \textit{even} \\ \\ \frac{2B^{bc**}_{\frac{n+1}{2}} - C^{bc**}_{\frac{n+1}{2}} + 4R^{bc**}_{\frac{n+1}{2}}}{3} & n \geq 1 \ \textit{odd.} \end{cases}$$

Proof. Let n be even, say $n = 2k$ for some positive integer $k \geq 1$. Then from (15)

$$S_{2k} = \frac{\alpha^{8k} + \beta^{8k} - 2}{32}$$

$$= (\frac{\alpha^{4k}-\beta^{4k}}{4\sqrt{2}})^2$$
$$= \left[\frac{6\sqrt{2}(\alpha^{4k}-\beta^{4k})}{48}\right]^2$$
$$= \left[\left\{\begin{array}{c}\alpha^{4k}[(3-\sqrt{2})\alpha^2-(3+\sqrt{2})\alpha^{-2}]+\\ \beta^{4k}[(3+\sqrt{2})\beta^2-(3-\sqrt{2})\beta^{-2}]\end{array}\right\}/48\right]^2$$
$$= \left[\frac{\frac{(3-\sqrt{2})\alpha^{4k+2}+(3+\sqrt{2})\beta^{4k+2}-2}{8}-\frac{(3+\sqrt{2})\alpha^{4k-2}+(3-\sqrt{2})\beta^{4k-2}-2}{8}}{6}\right]^2$$
$$= (\frac{B^{bc*}_{2k+1}-B^{bc*}_{2k}}{6})^2.$$

So $S_n = (\frac{B^{bc*}_{n+1}-B^{bc*}_{n}}{6})^2$ for even $n \geq 2$. The other cases can be proved similarly. □

Thus we construct one-to-one correspondence between all almost balcobalancing numbers and square triangular numbers.

Finally, we want to construct a correspondence between triangular and square triangular numbers via almost balcobalancing numbers, that is, we want to find out that for which almost balcobalancing numbers m, the equation

$$T_m = S_n$$

holds. The answer is given below.

Theorem 13. *For triangular numbers T_n and square triangular numbers S_n, we have*

(1) $T_{3B^{bc*}_{2n}-B^{bc*}_{2n-1}} = S_{2n}$
(2) $T_{C^{bc*}_{2n-1}-R^{bc*}_{2n}-R^{bc*}_{2n-1}-1} = S_{2n-1}$
(3) $T_{2B^{bc*}_{2n-1}+2C^{bc*}_{2n}+R^{bc*}_{2n+2}-R^{bc*}_{2n+1}} = S_{2n+1}$
(4) $T_{-2B^{bc*}_{2n-1}-2C^{bc*}_{2n}+R^{bc*}_{2n+2}-R^{bc*}_{2n+1}-1} = S_{2n}$
(5) $T_{\frac{2B^{bc**}_{n+1}-C^{bc**}_{n+1}+4R^{bc**}_{n+1}}{3}} = S_{2n+1}$
(6) $T_{\frac{2B^{bc**}_{n+1}-3C^{bc**}_{n+1}+16R^{bc**}_{n+1}+3}{3}} = S_{2n}$

for $n \geq 1$.

Proof. **(1)** Notice that $3B^{bc*}_{2n}-B^{bc*}_{2n-1} = \frac{(3+2\sqrt{2})\alpha^{4n-2}+(3-2\sqrt{2})\beta^{4n-2}-2}{4}$. Since $(3+2\sqrt{2})^2=\alpha^4$ and $(3-2\sqrt{2})^2=\beta^4$, we get from (15) that

$$T_{3B^{bc*}_{2n}-B^{bc*}_{2n-1}} = \frac{(3B^{bc*}_{2n}-B^{bc*}_{2n-1})(3B^{bc*}_{2n}-B^{bc*}_{2n-1}+1)}{2}$$
$$= \frac{\left[\frac{(3+2\sqrt{2})\alpha^{4n-2}+(3-2\sqrt{2})\beta^{4n-2}-2}{4}\right]\left[\frac{(3+2\sqrt{2})\alpha^{4n-2}+(3-2\sqrt{2})\beta^{4n-2}-2}{4}+1\right]}{2}$$

$$
\begin{aligned}
&= \frac{\left[(3+2\sqrt{2})\alpha^{4n-2}+(3-2\sqrt{2})\beta^{4n-2}\right]^2-4}{32} \\
&= \frac{(3+2\sqrt{2})^2\alpha^{8n-4}+(3-2\sqrt{2})^2\beta^{8n-4}-2}{32} \\
&= \frac{\alpha^{8n}+\beta^{8n}-2}{32} \\
&= S_{2n}.
\end{aligned}
$$

The others can be proved similarly. □

7. Sums of Almost Balcobalancing Numbers.

Recall that the sum of the first n-terms of all balancing numbers can be given in terms of same balancing numbers, that is,

$$
\sum_{i=1}^{n} B_i = \frac{5B_n - B_{n-1} - 1}{4}, \quad \sum_{i=1}^{n} b_i = \frac{5b_n - b_{n-1} + 2 - 2n}{4}
$$

$$
\sum_{i=1}^{n} C_i = \frac{5C_n - C_{n-1} - 2}{4}, \quad \sum_{i=1}^{n} c_i = \frac{5c_n - c_{n-1} - 2}{4}.
$$

Similarly we can give the following result.

Theorem 14. *The sum of the first n-terms of B_n^{bc*}, C_n^{bc*} and R_n^{bc*} is*

$$
\begin{aligned}
\sum_{i=1}^{n} B_i^{bc*} &= \frac{33B_n^{bc*} + 33B_{n-1}^{bc*} - B_{n-2}^{bc*} - B_{n-3}^{bc*} - 8n + 16}{32} \\
\sum_{i=1}^{n} C_i^{bc*} &= \frac{33C_n^{bc*} + 33C_{n-1}^{bc*} - C_{n-2}^{bc*} - C_{n-3}^{bc*} - 24}{32} \\
\sum_{i=1}^{n} R_i^{bc*} &= \frac{33R_n^{bc*} + 33R_{n-1}^{bc*} - R_{n-2}^{bc*} - R_{n-3}^{bc*} - 8n + 4}{32}
\end{aligned}
$$

for $n \geq 4$, and of B_n^{bc**}, C_n^{bc**} and R_n^{bc**} is

$$
\begin{aligned}
\sum_{i=1}^{n} B_i^{bc**} &= \frac{33B_n^{bc**} - B_{n-1}^{bc**} - 8n + 2}{32} \\
\sum_{i=1}^{n} C_i^{bc**} &= \frac{33C_n^{bc**} - C_{n-1}^{bc**} - 12}{32} \\
\sum_{i=1}^{n} R_i^{bc**} &= \frac{33R_n^{bc**} - R_{n-1}^{bc**} - 8n + 8}{32}
\end{aligned}
$$

for $n \geq 2$.

Proof. It can be proved easily from (12) and Theorem 1. □

We also note that

$$\sum_{i=1}^{n}(-1)^i B_i = \begin{cases} 2B_{\frac{n}{2}}^2 + B_{\frac{n}{2}}C_{\frac{n}{2}} & n \geq 2 \text{ even} \\ -2B_{\frac{n+1}{2}}(b_{\frac{n+1}{2}} + \frac{1}{2}) & n \geq 1 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i b_i = \begin{cases} 2B_{\frac{n}{2}}^2 & n \geq 2 \text{ even} \\ -2b_{\frac{n+1}{2}}^2 - 2b_{\frac{n+1}{2}} & n \geq 1 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i C_i = \begin{cases} B_n + 8B_{\frac{n}{2}}^2 & n \geq 2 \text{ even} \\ -B_n - 8(b_{\frac{n+1}{2}} + \frac{1}{2})^2 & n \geq 1 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i c_i = \begin{cases} B_n & n \geq 2 \text{ even} \\ -B_n & n \geq 1 \text{ odd.} \end{cases}$$

Similarly we can give the following theorem.

Theorem 15. *For all almost balcobalancing numbers of first and of second type, we get*

$$\sum_{i=1}^{n}(-1)^i B_i^{bc*} = \frac{1}{32}\begin{cases} 33B_n^{bc*} - 33B_{n-1}^{bc*} - B_{n-2}^{bc*} + B_{n-3}^{bc*} - 4 & n \geq 4 \text{ even} \\ -33B_n^{bc*} + 33B_{n-1}^{bc*} + B_{n-2}^{bc*} - B_{n-3}^{bc*} + 4 & n \geq 5 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i C_i^{bc*} = \frac{1}{32}\begin{cases} 33C_n^{bc*} - 33C_{n-1}^{bc*} - C_{n-2}^{bc*} + C_{n-3}^{bc*} & n \geq 4 \text{ even} \\ -33C_n^{bc*} + 33C_{n-1}^{bc*} + C_{n-2}^{bc*} - C_{n-3}^{bc*} & n \geq 5 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i R_i^{bc*} = \frac{1}{32}\begin{cases} 33R_n^{bc*} - 33R_{n-1}^{bc*} - R_{n-2}^{bc*} + R_{n-3}^{bc*} + 4 & n \geq 4 \text{ even} \\ -33R_n^{bc*} + 33R_{n-1}^{bc*} + R_{n-2}^{bc*} - R_{n-3}^{bc*} + 12 & n \geq 5 \text{ odd} \end{cases}$$

and

$$\sum_{i=1}^{n}(-1)^i B_i^{bc**} = \frac{1}{36}\begin{cases} 35B_n^{bc**} - B_{n-1}^{bc**} + 4 & n \geq 2 \text{ even} \\ -35B_n^{bc**} + B_{n-1}^{bc**} - 4 & n \geq 3 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i C_i^{bc**} = \frac{1}{36}\begin{cases} 35C_n^{bc**} - C_{n-1}^{bc**} - 18 & n \geq 2 \text{ even} \\ -35C_n^{bc**} + C_{n-1}^{bc**} - 18 & n \geq 3 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i R_i^{bc**} = \frac{1}{36}\begin{cases} 35R_n^{bc**} - R_{n-1}^{bc**} + 4 & n \geq 2 \text{ even} \\ -35R_n^{bc**} + R_{n-1}^{bc**} - 4 & n \geq 3 \text{ odd.} \end{cases}$$

Proof. It can be proved similarly. □

8. Sums of Pell and Balancing Numbers.

Panda and Ray proved in [9, Theorem 3.4] that the sum of first $2n-1$ Pell numbers is equal to the sum of n^{th} balancing number and its balancer, that is,

$$\sum_{i=1}^{2n-1} P_i = B_n + b_n. \tag{16}$$

Later Gözeri, Özkoç and Tekcan proved in [3, Theorem 2.5] that the sum of Pell-Lucas numbers from 0 to $2n-1$ is equal to the the sum of n^{th} Lucas-balancing and the n^{th} Lucas-cobalancing number, that is,

$$\sum_{i=0}^{2n-1} Q_i = C_n + c_n.$$

Since $R_n = b_n$, (16) becomes

$$\sum_{i=1}^{2n-1} P_i = B_n + R_n. \tag{17}$$

As in (17), we can give the following result.

Theorem 16. *For the sum of Pell numbers, we have*

$$\sum_{i=1}^{2n-1} P_{2i} = B_{2n-1}^{bc*} + R_{2n}^{bc*}$$

for $n \geq 1$.

Proof. Notice that $P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$. So we get

$$\begin{aligned}
\sum_{i=1}^{2n-1} P_{2i} &= \sum_{i=1}^{2n-1} \left(\frac{\alpha^{2i} - \beta^{2i}}{2\sqrt{2}}\right) \\
&= \frac{\frac{\alpha^{4n}-\alpha^2}{\alpha^2-1} - \frac{\beta^{4n}-\beta^2}{\beta^2-1}}{2\sqrt{2}} \\
&= \frac{\frac{\alpha(\alpha^{4n-2}-1)}{2} - \frac{\beta(\beta^{4n-2}-1)}{2}}{2\sqrt{2}} \\
&= \frac{\alpha^{4n-1} - \beta^{4n-1} - 2\sqrt{2}}{4\sqrt{2}} \\
&= \frac{(\alpha-1)\alpha^{4n-1} - (1-\beta)\beta^{4n-1} - 4}{8} \\
&= \frac{(\alpha^2-\alpha)\alpha^{4n-2} + (\beta^2-\beta)\beta^{4n-2} - 4}{8}
\end{aligned}$$

$$= \frac{(2+\sqrt{2})\alpha^{4n-2} + (2+\sqrt{2})\beta^{4n-2} - 4}{8}$$
$$= \frac{(3-\sqrt{2})\alpha^{4n-2} + (3+\sqrt{2})\beta^{4n-2} - 2}{8}$$
$$+ \frac{(-1+2\sqrt{2})\alpha^{4n-2} - (1+2\sqrt{2})\beta^{4n-2} - 2}{8}$$
$$= B^{bc*}_{2n-1} + R^{bc*}_{2n}$$

as we wanted. □

Apart from Theorem 16, we can give the following theorem which can be proved similarly.

Theorem 17. *For the sum of Pell numbers, we have*

$$\sum_{i=1}^{2n} P_{2i-1} = R^{bc*}_{2n+2} - B^{bc*}_{2n+1}$$

and also

$$P_{4n-1} + \sum_{i=1}^{2n-1} P_{2i} = B^{bc**}_{n} + R^{bc**}_{n}$$
$$P_{4n-2} + \sum_{i=1}^{2n-1} P_{2i-1} = B^{bc**}_{n} - R^{bc**}_{n}$$

for $n \geq 1$.

For the sums of Pell-Lucas number, we can give the following theorem.

Theorem 18. *For the sum of Pell-Lucas numbers, we have*

$$\sum_{i=0}^{4n+1} Q_i = \frac{C^{bc*}_{2n+2} + C^{bc*}_{2n+1}}{3}$$
$$\sum_{i=1}^{4n} Q_i = 2R^{bc*}_{2n+2} - 2R^{bc*}_{2n+1} - 2$$

or

$$\sum_{i=0}^{4n+1} Q_i = \frac{4(B^{bc**}_{n+1} - R^{bc**}_{n+1})}{3}$$
$$\sum_{i=1}^{4n} Q_i = \frac{12R^{bc**}_{n+1} - 4B^{bc**}_{n+1} - 4}{3}$$

for $n \geq 1$.

In [16, Lemma 1], Santana and Diaz Barrero proved that the sum of first nonzero $4n+1$ terms of Pell numbers is a perfect square, that is,

$$\sum_{i=1}^{4n+1} P_i = \left[\sum_{i=0}^{n} \left(\begin{array}{c} 2n+1 \\ 2i \end{array} \right) 2^i\right]^2.$$

In fact this sum equals to c_{n+1}^2, that is,

$$\sum_{i=1}^{4n+1} P_i = c_{n+1}^2.$$

Similarly we can give the following result.

Theorem 19. *The sum of Pell numbers from 1 to $8n+1$ is a perfect square and*

$$\sum_{i=1}^{8n+1} P_i = (4B_{2n-1}^{bc*} + 4C_{2n}^{bc*} + 1)^2$$

or

$$\sum_{i=1}^{8n+1} P_i = (\frac{2C_{n+1}^{bc**} - 12R_{n+1}^{bc**} - 3}{3})^2$$

for $n \geq 1$.

Proof. Since $P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$, we get

$$\begin{aligned}
\sum_{i=1}^{8n+1} P_i &= \sum_{i=1}^{8n+1} (\frac{\alpha^i - \beta^i}{2\sqrt{2}}) \\
&= \frac{\frac{\alpha^{8n+2}-\alpha}{\alpha-1} - \frac{\beta^{8n+2}-\beta}{\beta-1}}{2\sqrt{2}} \\
&= \frac{\alpha^{8n+2} + \beta^{8n+2} - 2}{4} \\
&= (\frac{\alpha^{4n+1} + \beta^{4n+1}}{2})^2 \\
&= (\frac{\alpha^3\alpha^{4n-2} + \beta^3\beta^{4n-2}}{2})^2 \\
&= \left(\frac{(7+5\sqrt{2})\alpha^{4n-2} + (7-5\sqrt{2})\beta^{4n-2}}{2}\right)^2 \\
&= \left(\begin{array}{c} \frac{(3-\sqrt{2})\alpha^{4n-2}+(3+\sqrt{2})\beta^{4n-2}-2}{2} \\ +\frac{(4+6\sqrt{2})\alpha^{4n-2}+(4-6\sqrt{2})\beta^{4n-2}+2}{2} \end{array} \right)^2
\end{aligned}$$

$$= \left[\begin{array}{c} 4\left(\frac{(3-\sqrt{2})\alpha^{4n-2}+(3+\sqrt{2})\beta^{4n-2}-2}{8}\right) \\ +4\left(\frac{(3+\sqrt{2})\alpha^{4n-2}-(3-\sqrt{2})\beta^{4n-2}}{2\sqrt{2}}\right)+1 \end{array} \right]^2$$
$$= (4B^{bc*}_{2n-1} + 4C^{bc*}_{2n} + 1)^2.$$

The other case can be proved similarly. □

As in Theorem 19, we can give the following theorem which can be proved similarly.

Theorem 20. *For the sums of Pell, Pell-Lucas, balancing and Lucas-cobalancing numbers, we have*

$$1 + \sum_{i=1}^{8n+3} P_i = \left(\frac{2B^{bc*}_{2n+2} + 2B^{bc*}_{2n+1} + 1}{3}\right)^2$$
$$\sum_{i=1}^{4n+2} Q_{2i-1} = \left(\frac{C^{bc*}_{2n+2} + C^{bc*}_{2n+1}}{3}\right)^2$$
$$\frac{\sum_{i=0}^{4n} Q_{2i+1}}{2} = \left(\frac{2R^{bc*}_{2n+2} + 2R^{bc*}_{2n+1} + 1}{3}\right)^2$$
$$\sum_{i=1}^{2n+1} B_{2i-1} = \left(\frac{C^{bc*}_{2n+2} + C^{bc*}_{2n+1}}{12}\right)^2$$
$$1 + \sum_{i=1}^{4n+2} c_i = \left(\frac{C^{bc*}_{2n+2} - C^{bc*}_{2n+1}}{2}\right)^2$$

or

$$1 + \sum_{i=1}^{8n+3} P_i = \left(\frac{4R^{bc**}_{n+1} + 1}{3}\right)^2$$
$$\sum_{i=1}^{4n+2} Q_{2i-1} = \left(\frac{4B^{bc**}_{n+1} - 4R^{bc**}_{n+1}}{3}\right)^2$$
$$\frac{\sum_{i=0}^{4n} Q_{2i+1}}{2} = \left(\frac{4B^{bc**}_{n+1} - 8R^{bc**}_{n+1} - 1}{3}\right)^2$$
$$\sum_{i=1}^{2n+1} B_{2i-1} = \left(\frac{C^{bc**}_{n+1} - 4R^{bc**}_{n+1} - 1}{6}\right)^2$$
$$1 + \sum_{i=1}^{4n+2} c_i = \left(\frac{4R^{bc**}_{n+1} + 1}{3}\right)^2$$

for $n \geq 1$.

9. Concluding Remark.

In [24], we said that a positive integer n is called an almost balcobalancing number if the Diophantine equation

$$\left|\begin{array}{c} 1+2+\cdots+(n-1)+1+2+\cdots+(n-1)+n \\ -2[(n+1)+(n+2)+\cdots+(n+r)] \end{array}\right|=1 \tag{18}$$

holds for some positive integer r which is called almost balcobalancer and deduced all results by considering this Diophantine equation. One can also consider the Diophantine equation

$$\left|\begin{array}{c} 2[(n+1)+(n+2)+\cdots+(n+r)]- \\ [1+2+\cdots+(n-1)+1+2+\cdots+(n-1)+n] \end{array}\right|=1 \tag{19}$$

instead of (18). Then there is no problem. If we consider the Diophantine equation in (19), then (only) all almost balcobalancing numbers of first type can be of second type, that is,

$$B_n^{bc*} \to B_n^{bc**}, C_n^{bc*} \to C_n^{bc**} \quad \text{and} \quad R_n^{bc*} \to R_n^{bc**},$$

and all almost balcobalancing numbers of second type can be of first type, that is,

$$B_n^{bc**} \to B_n^{bc*}, C_n^{bc**} \to C_n^{bc*} \quad \text{and} \quad R_n^{bc**} \to R_n^{bc*}.$$

For instance, if we consider the Diophantine equation in (19), then the general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first type are

$$\begin{aligned} B_n^{bc*} &= \frac{12B_{2n-1}+3C_{2n-1}-1}{4} \\ C_n^{bc*} &= 6B_{2n-1}+3C_{2n-1} \\ R_n^{bc*} &= \frac{3C_{2n-1}-1}{4} \end{aligned}$$

by (13), and of second type are

$$\begin{aligned} B_{2n-1}^{bc**} &= \frac{-4B_{2n-1}+3C_{2n-1}-1}{4} \\ B_{2n}^{bc**} &= \frac{4B_{2n-1}+3C_{2n-1}-1}{4} \\ C_{2n-1}^{bc**} &= 6B_{2n-1}-C_{2n-1} \\ C_{2n}^{bc**} &= 6B_{2n-1}+C_{2n-1} \\ R_{2n-1}^{bc**} &= \frac{16B_{2n-1}-5C_{2n-1}-1}{4} \\ R_{2n}^{bc**} &= \frac{8B_{2n-1}-C_{2n-1}-1}{4} \end{aligned}$$

by (12). So all results obtained in [24] and also in this paper are correct. One can take $B_n^{bc**} \leftrightarrow B_n^{bc*}, C_n^{bc**} \leftrightarrow C_n^{bc*}$ and $R_n^{bc**} \leftrightarrow R_n^{bc*}$ if necessary.

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