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A NOTE ON SOME WEIGHTED MAXIMAL OPERATORS OF PARTIAL SUMS OF WALSH-FOURIER SERIES

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Abstract. In this paper we introduce some new weighted maximal operators of the partial sums of the Walsh-Fourier series. We prove that for some "optimal" weights these new operators indeed are bounded from the martingale Hardy space $H_1(G)$ to the space weak $-\, L_1(G)$, but is not bounded from $H_1(G)$ to the space $L_1(G)$.

1. Introduction

All symbols used in this introduction can be found in Section 2.

It is well-known that the Walsh system does not form a basis in the space L_1 (see e.g. [2]). Moreover, there exists a function in the dyadic Hardy space $H_1(G)$, such that the partial sums of f are not bounded in the L_1-norm. Uniform and pointwise convergence and some approximation properties of partial sums in $L_1(G)$ norms were investigated by Avdispahić and Memić [1], Gát, Goginava and Tkebuchava [19, 20], Nagy [24, 25], Onneweer [26] and Persson, Schipp, Tephnadze and Weisz [29]. Fine [16] obtained sufficient conditions for the uniform convergence which are completely analogous to the Dini-Lipschits conditions. Gulicev [21] estimated the rate of uniform convergence of a Walsh-Fourier series by using Lebesgue constants and modulus of continuity. These problems for Vilenkin groups were investigated by Blahota, Nagy, Persson and Tephnadze [10] (see also [6, 7, 8, 9, 27]), Blahota and G. Tephnadze and Toledo [15] (see also [12, 13, 14]), Fridli [17], Gát [18] and Memić [23].

In study of convergence of subsequences of partial sums and their restricted maximal operators on the martingale Hardy spaces H_p for $0<p\leq 1$, the central role is played by the fact that any natural number $n\in\mathbb{N}$ can be uniquely expressed as

$$n=\sum_{k=0}^{\infty} n_j 2^j, \quad n_j\in Z_2 \ (j\in\mathbb{N}),$$

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where only a finite numbers of n_j differ from zero and their important characters $[n]$, $|n|$, $\rho(n)$ and $V(n)$ are defined by

$$[n] := \min\{j \in \mathbb{N}, n_j \neq 0\}, \ |n| := \max\{j \in \mathbb{N}, n_j \neq 0\}, \ \rho(n) = |n| - [n],$$

$$V(n) := n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|, \quad \text{for all} \quad n \in \mathbb{N}$$

Moreover, every $n \in \mathbb{N}$ can be also represented as

$$n = \sum_{i=1}^{r} 2^{n_i}, n_1 > n_2 > ... > n_r \geq 0$$

and for any $\{n_{s_j}\}, j = 1, 2, \ldots, r$, satisfying

$$2^s \leq n_{s_1} \leq n_{s_2} \leq ... \leq n_{s_r} < 2^{s+1}, \ s \in \mathbb{N},$$

we define numbers

$$s_- := \min\{[n_{s_j}]\}, \quad s_+ := \max\{|n_{s_j}|\} = s, \quad \rho_s(n_{s_j}) := s_+ - s_-. \tag{1}$$

In particular, (see [11], [22] and [30])

$$V(n)/8 \leq \|D_n\|_1 \leq V(n)$$

Hence, for any $f \in H_1$ (see [34])

$$\|S_n F\|_{H_1} \leq cV(n) \|F\|_{H_1}.$$

For $0 < p < 1$ in [32, 33] the weighted maximal operator $\tilde{S}^{*,p}$, defined by

$$\tilde{S}^{*,p} F := \sup_{n \in \mathbb{N}} \frac{|S_n F|}{(n+1)^{1/p-1}} \tag{2}$$

was investigated and it was proved that the following inequalities hold:

$$\left\| \tilde{S}^{*} F \right\|_p \leq C_p \|F\|_{H_p}$$

Moreover, it was also proved that the rate of the sequence $\{(n+1)^{1/p-1}\}$ given in the denominator of (2) can not be improved.

In [34] and [35] it was proved that if $F \in H_p$, then there exists an absolute constant c_p, depending only on p, such that

$$\|S_n F\|_{H_p} \leq C_p 2^{\rho(n)(1/p-1)} \|F\|_{H_p}.$$

In [5] it was proved that the maximal operator

$$\sup_{n \in \mathbb{N}} \frac{|S_n F|}{2^{\rho(n)(1/p-1)}} \tag{3}$$

is bounded from H_p to weak $-$ L_p. Moreover, it was also proved that the rate of the sequence $\{2^{\rho(n)(1/p-1)}\}$ given in the denominator of (2) can not be improved.

In [5] it was also proved that the weighted maximal operator (3) is not bounded from $H_p(G)$ to the Lebesgue space $L_p(G)$, for $0<p<1$.

In [3] it was proved that if $0<p<1$, $f\in H_p$, $\{n_k : k\geq 0\}$ is a sequence of positive integers and $\{n_{s_i}:\ 1\leq i\leq r\}\subset\{n_k:\ k\geq 0\}$ are integers such that $2^s\leq n_{s_1}\leq n_{s_2}\leq ...\leq n_{s_r}\leq 2^{s+1},\ s\in\mathbb{N}$, then the weighted maximal operator $\widetilde{S}^{*,\nabla,1}$, defined by

$$\widetilde{S}^{*,\nabla,1}F:=\sup_{s\in\mathbb{N}}\sup_{2^s\leq n_{s_i}<2^{s+1}}\frac{|S_nF|}{2^{\rho_s\left(n_{s_i}\right)(1/p-1)}},$$

where $\rho_s\left(n_{s_i}\right)$ are defined by (1), is bounded from the Hardy space H_p to the Lebesgue space L_p. Moreover, if $0<p<1$, $\{n_k : k\geq 0\}$ is a sequence of positive numbers and $\{n_{s_i}:\ 1\leq i\leq r\}\subset\{n_k:\ k\geq 0\}$ is a subsequence satisfying the condition

$$2^s\leq n_{s_1}\leq n_{s_2}\leq ...\leq n_{s_r}\leq 2^{s+1},\ s\in\mathbb{N},$$

then, for any nonnegative, nondecreasing function $\varphi:\mathbb{N}_+\to\mathbb{R}$ satisfying condition

$$\sup_{s\in\mathbb{N}}\sup_{2^s\leq n_{s_i}<2^{s+1}}\frac{2^{\rho_s\left(n_{s_i}\right)(1/p-1)}}{\varphi(n_{s_i})}=\infty,$$

the maximal operator, defined by

$$\sup_{s\in\mathbb{N}}\sup_{2^s\leq n_{s_i}<2^{s+1}}\frac{|S_nF|}{\varphi\left(n_{s_i}\right)},$$

is not bounded from the Hardy space H_p to the Lebesgue space L_p.

In [4] it was proved that if $0<p<1$, $F\in H_p\left(G\right)$ and $\varphi:\mathbb{N}_+\to\mathbb{R}_+$ be any nonnegative and nondecreasing function satisfying the condition

$$\sum_{n=1}^{\infty}\frac{1}{\varphi^p(n)}<c<\infty,$$

then, for any sequence $\{n_k : k\geq 0\}$ of positive integers, the weighted maximal operator $\widetilde{S}^{*,\nabla,2}$, defined by

$$\widetilde{S}^{*,\nabla,2}F=\sup_{k\in\mathbb{N}}\frac{|S_{n_k}F|}{2^{\rho(n_k)(1/p-1)}\varphi(\rho\left(n_k\right))},\tag{4}$$

is bounded from the Hardy space H_p to the Lebesgue space L_p. Moreover, for any $0<p<1$ and any sequence $\{n_k : k\geq 0\}$ of positive numbers and $\varphi:\mathbb{N}_+\to\mathbb{R}_+$ be any nonnegative and nondecreasing function satisfying the condition

$$\sum_{n=1}^{\infty}\frac{1}{\varphi^p(n)}=\infty,$$

the weighted maximal operator $\widetilde{S}^{*,\nabla,2}$, defined by (4), is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$. Hence, we get that if $0<p<1$ and $F\in H_p(G)$, then the weighted maximal operator $\widetilde{S}^{*,\nabla,\varepsilon,2}$, defined by

$$\widetilde{S}^{*,\nabla,\varepsilon,2}F:=\sup_{n\in\mathbb{N}}\frac{|S_nF|}{2^{\rho(n)(1/p-1)}\left(\rho(n)\log^{1+\varepsilon}\rho(n)\right)^{1/p}},\quad \varepsilon>0,$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$ for $\varepsilon>0$ and is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$ for $\varepsilon=0$.

In this paper we introduce some new weighted maximal operators of the partial sums of the Walsh-Fourier series. We prove that for some "optimal" weights these new operators indeed are bounded from the martingale Hardy space $H_1(G)$ to the space weak $-L_1(G)$, but is not bounded from $H_1(G)$ to the space $L_1(G)$.

2. Preliminaries

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N}:=\mathbb{N}_+\cup\{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2:=\{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given so that the measure of a singleton is 1/2. Define the group G as the complete direct product of the group Z_2, with the product of the discrete topologies of Z_2's. The elements of G are represented by sequences $x:=(x_0,x_1,\cdots,x_j,\cdots)$, where $x_j=0\vee 1$.

It is easy to give a base for the neighborhood of $x\in G$:

$$I_0(x):=G,\quad I_n(x):=\{y\in G:y_0=x_0,...,y_{n-1}=x_{n-1}\}\ (n\in\mathbb{N}).$$

Denote $I_n:=I_n(0)$, $\overline{I_n}:=G\backslash I_n$. Then, it is easy to prove that

$$\overline{I_M}=\bigcup_{s=0}^{M-1}I_s\backslash I_{s+1}. \tag{5}$$

Let define the Walsh system $w:=(w_n:n\in\mathbb{N})$ on G by

$$w_n(x):=\prod_{k=0}^{\infty}r_k^{n_k}(x)=(-1)^{\sum_{k=0}^{|n|}n_kx_k}\qquad(n\in\mathbb{N}).$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see e.g. [30] and [36, 28]).

If $f\in L_1$ we define Fourier coefficients, partial sums of the Fourier series, Dirichlet kernels with respect to the Walsh system by

$$\widehat{f}(k):=\int_G fw_kd\mu\ \ (k\in\mathbb{N}),\quad S_nf:=\sum_{k=0}^{n-1}\widehat{f}(k)\,w_k,\quad D_n:=\sum_{k=0}^{n-1}w_k\ \ (n\in\mathbb{N}_+).$$

Recall that (see [28] and [30])

$$D_{2^n}(x)=\begin{cases}2^n, & \text{if } x\in I_n,\\ 0, & \text{if } x\notin I_n,\end{cases}\tag{6}$$

and

$$D_n=w_n\sum_{k=0}^{\infty}n_kr_kD_{2^k}=w_n\sum_{k=0}^{\infty}n_k\left(D_{2^{k+1}}-D_{2^k}\right),\ \text{for}\ \ n=\sum_{i=0}^{\infty}n_i2^i.\tag{7}$$

The σ-algebra generated by the intervals $\{I_n(x):x\in G\}$ will be denoted by $\zeta_n\,(n\in\mathbb{N})$. Denote by $F=(F_n,n\in\mathbb{N})$ martingale with respect to $\zeta_n\ (n\in\mathbb{N})$ (see e.g. [36, 37]). The maximal function F^* of a martingale F is defined by

$$F^*:=\sup_{n\in\mathbb{N}}|F_n|.$$

For $0<p<\infty$ the Hardy martingale spaces $H_p(G)$ consists of all martingales for which

$$\|F\|_{H_p}:=\|F^*\|_p<\infty.$$

It is easy to prove that for every martingale $F=(F_n,n\in\mathbb{N})$ and every $k\in\mathbb{N}$ the limit $\widehat{F}(k):=\lim_{n\to\infty}\int_G F_n(x)\,w_k(x)\,d\mu(x)$ exists and it is called the k-th Walsh-Fourier coefficients of F.

If $F:=(S_{2^n}f:n\in\mathbb{N})$ is a regular martingale, generated by $f\in L_1$, then $\widehat{F}(k)=\widehat{f}(k),\ k\in\mathbb{N}$.

A bounded measurable function a is called p-atom, if there exists a dyadic interval I, such that

$$\int_I a\,d\mu=0,\qquad \|a\|_\infty\le\mu(I)^{-1/p},\qquad \operatorname{supp}(a)\subset I.$$

The dyadic Hardy martingale spaces $H_p(G)$ for $0<p\le 1$ have an atomic characterization. Namely, the following holds (see [28], [36, 37]):

Lemma 1. *A martingale $F=(F_n,n\in\mathbb{N})$ belongs to $H_p(G)$ $(0<p\le1)$ if and only if there exists a sequence $(a_k,\ k\in\mathbb{N})$ of p-atoms and a sequence $(\mu_k,k\in\mathbb{N})$ of real numbers such that for every $n\in\mathbb{N}$,*

$$\sum_{k=0}^{\infty}\mu_kS_{2^n}a_k=F_n,\qquad \sum_{k=0}^{\infty}|\mu_k|^p<\infty,\tag{8}$$

Moreover, $\|F\|_{H_p}\backsim\inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p}$, where the infimum is taken over all decomposition of F of the form (8).

3. The main result

Our main result reads:

Theorem 2. *a) Let $F \in H_1$. Then the weighted maximal operator $\widetilde{S}^{*,\nabla}$, defined by*

$$\widetilde{S}^{*,\nabla}F := \sup_{n\in\mathbb{N}}\frac{|S_nF|}{V(n)} \tag{9}$$

is bounded from the Hardy space H_1 to the space weak $-L_1$.

b) Let $F \in H_1$. Then the weighted maximal operator $\widetilde{S}^{*,\nabla}$, defined by (9) *is not bounded from the Hardy space H_1 to the space L_1.*

Proof. Since $S_n/V(n)$ is bounded from L_∞ to L_∞, by Lemma 1, the proof of Theorem 2 will be complete, if we prove that

$$t\mu\left\{x \in \overline{I_M} : \widetilde{S}^{*,\nabla}a(x) \geq t\right\} \leq C < \infty, \qquad t \geq 0 \tag{10}$$

for every 1-atom a. In this paper C denotes a positive constant but which can be different in different places.

We may assume that a is an arbitrary 1-atom, with support I, $\mu(I) = 2^{-M}$ and $I = I_M$. It is easy to see that $S_na(x) = 0$, when $n < 2^M$. Therefore, we can suppose that $n \geq 2^M$. Since $\|a\|_\infty \leq 2^M$, we obtain that

$$\frac{|S_na(x)|}{V(n)} \leq \frac{1}{V(n)}\|a\|_\infty \int_{I_M} |D_n(x+t)|\,\mu(t) \leq 2^M \int_{I_M} |D_n(x+t)|\,\mu(t).$$

Let $x \in I_s \backslash I_{s+1}$. Then, it is easy to see that $x + t \in I_s \backslash I_{s+1}$ for $t \in I_M$ and if we again combine (6) and (7) we find that $D_n(x+t) \leq c2^s$, for $t \in I_M$ and

$$\frac{|S_na(x)|}{V(n)} \leq C2^M 2^{s-M} \leq C2^s. \tag{11}$$

By applying (11) for any $x \in I_s \backslash I_{s+1}$, $0 \leq s < M$, we find that

$$\widetilde{S}^{*,\nabla}a(x) \leq C2^s. \tag{12}$$

It immediately follows that for $s \leq M$ we have the following estimate

$$\widetilde{S}^{*,\nabla}a(x) \leq C2^M \quad \text{for any } \; x \in I_s \backslash I_{s+1}, \quad s = 0, 1, \cdots, M$$

and also that

$$\mu\left\{x \in I_s \backslash I_{s+1} : \widetilde{S}^{*,\nabla}a(x) > C2^k\right\} = 0, \quad k = M, M+1, \ldots \tag{13}$$

By combining (5) and (12) we get that

$$\left\{x \in \overline{I_N} : \widetilde{S}^{*,\nabla}a(x) \geq C2^k\right\} \subset \bigcup_{s=k}^{M-1}\left\{x \in I_s \backslash I_{s+1} : \widetilde{S}^{*,\nabla}a(x) \geq C2^k\right\}$$

and

$$\mu\left\{x\in\overline{I_M}:\widetilde{S}^{*,\nabla}a\left(x\right)\geq C2^k\right\}\leq\sum_{s=k}^{M-1}\frac{1}{2^s}\leq\frac{2}{2^k}. \tag{14}$$

In view of (13) and (14) we can conclude that

$$2^k\mu\left\{x\in\overline{I_N}:\widetilde{S}^{*,\nabla}a\left(x\right)\geq C2^k\right\}<C<\infty,$$

which shows that (10) holds and the proof of part a) is complete.

Set

$$f_n\left(x\right)=D_{2^{n+1}}\left(x\right)-D_{2^n}\left(x\right),\qquad n\geq 3.$$

In [5] it was proved that $\|f_n\|_{H_1}\leq 1$ and $|S_{2^n+2^s}f_n\left(x\right)|\geq C2^s$, for $x\in I_{s+1}\left(e_s\right)$, $s=0,\cdots,n-1$. Hence,

$$\int_G\sup_{n\in\mathbb{N}}\frac{|S_nf_n|}{V\left(n\right)}d\mu\geq\sum_{s=0}^{n_k-1}\int_{I_{s+1}(e_s)}\frac{|S_{2^n+2^s}f_n|}{V\left(2^n+2^s\right)}d\mu\geq C\sum_{s=0}^{n-1}\frac{1}{2^s}2^s\geq Cn.$$

Finally, we get that

$$\frac{\int_G\left(\sup_{n\in\mathbb{N}}\frac{|S_nf_n(x)|}{V(n)}\right)d\mu\left(x\right)}{\|f_n\|_{H_1}}\geq Cn\to\infty,\quad\text{as}\quad n\to\infty,$$

so also part b) is proved and the proof is complete. □

4. Further results

We finalize this paper with another natural conjecture related to Theorem 2. For this we need some new characterization of $n\in\mathbb{N}$.

Let

$$2^s\leq n_{s_1}\leq n_{s_2}\leq...\leq n_{s_r}\leq 2^{s+1},\ s\in\mathbb{N}.$$

For such n_{s_j}, which can be written as $n_{s_j}=\sum_{i=1}^{r_{s_j}}\sum_{k=l_i^{s_j}}^{t_i^{s_j}}2^k$, where

$$0\leq l_1^{s_j}\leq t_1^{s_j}\leq l_2^{s_j}-2<l_2^{s_j}\leq t_2^{s_j}\leq...\leq l_{r_{s_j}}^{s_j}-2<l_{r_{s_j}}^{s_j}\leq t_{r_{s_j}}^{s_j},$$

we define

$$A_s\ :=\ \left\{l_1^s,l_2^s,...,l_{r_s^1}^s\right\}\bigcup\left\{t_1^s,t_2^s,...,t_{r_s^2}^s\right\}=\left\{u_1^s,u_2^s,...,u_{r_s^3}^s\right\},$$

where $u_1^s<u_2^s<...<u_{r_s^3}^s$. We note that $t_{r_{s_j}}^{s_j}=s\in A_s,\quad$ for $\quad j=1,2,...,r$.

We denote the cardinality of the set A_s by $|A_s|$, that is

$$\text{card}(A_s):=|A_s|.$$

By this definition we can conclude that

$$|A_s| = r_s^3 \leq r_s^1 + r_s^2.$$

It is evident that $\sup_{s\in\mathbb{N}} |A_s| < \infty$ if and only if the sets $\{n_{s_1}, n_{s_2}, ..., n_{s_r}\}$ are uniformly finite for all $s \in \mathbb{N}_+$ and each n_{s_j} has bounded variation

$$V(n_{s_j}) < c < \infty, \text{ for each } j = 1, 2, \ldots, r.$$

Conjecture 3. a) Let $f \in H_1$ and $\{n_k : k \geq 0\}$ is a sequence of positive numbers. Then the weighted maximal operator $\widetilde{S}^{*,\nabla}$, defined by

$$S^{*,\nabla,3}F := \sup_{k\in\mathbb{N}} \frac{|S_{n_k}F|}{A_{|n_k|}},$$

is bounded from the Hardy space H_1 to the Lebesgue space L_1.

b) (Sharpness) Let

$$\sup_{k\in\mathbb{N}} |A_{n_k}| = \infty$$

and $\{\varphi_n\}$ is a nondecreasing sequence satisfying the condition

$$\overline{\lim}_{k\to\infty} \left(A_{|n_k|}/\varphi_{|n_k|}\right) = \infty.$$

Then there exists a martingale $f \in H_1$, such that the maximal operator, defined by

$$\sup_{k\in\mathbb{N}} \frac{|S_{n_k}F|}{\varphi_{|n_k|}}$$

is not bounded from the Hardy space H_1 to the Lebesgue space L_1.

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