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WEAK ISOMETRIES IN PARTIALLY ORDERED GROUPS

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Abstract. In this paper the author gives necessary and sufficient conditions under which to a stable weak isometry f in a directed group G there exists a direct decomposition $G = A \times B$ of G such that $f(x) = x(A) - x(B)$ for each $x \in G$. Further, some results on weak isometries in partially ordered groups are established.

Isometries in an abelian lattice ordered group (l-group) have been introduced and investigated by Swamy [**16**], [**17**]. Jakubík [**4**] proved that for every stable isometry f in an l-group G there exists a direct decomposition $G = A \times B$ of G such that $f(x) = x(A) - x(B)$ for each $x \in G$. Isometries in non-abelian l-groups were also studied in [**2**] and [**5**]. Weak isometries in l-groups were introduced by Jakubík [**6**]. Rachůnek [**14**] generalized the notion of the isometry for any partially ordered group (po-group). Isometries and weak isometries in some types of po-groups have been investigated in [**7**], [**8**], [**9**], [**10**], [**12**], [**13**], [**14**]. In [**11**] it was proved that each stable weak isometry in a directed group is an involutory group automorphism (hence each weak isometry in a directed group is an isometry).

First we recall some notions and notations used in the paper.

Let G be a po-group. The group operation will be written additively. We denote $G^+ = \{x \in G; x \geq 0\}$. If a, b are elements of G, then we denote by $U(a,b)$ and $L(a,b)$ the set of all upper bounds and the set of all lower bounds of the set $\{a,b\}$ in G, respectively. If for a, $b \in G$ there exists the least upper bound (greatest lower bound) of the set $\{a,b\}$ in G, then it will be denoted by $a \vee b$ ($a \wedge b$). For each $a \in G$, $|a| = U(a,-a)$.

A partially ordered semigroup (po-semigroup) P with a neutral element is said to be the direct product of its po-subsemigroups P_1 and P_2 (notation: $P = P_1 \times P_2$) if the following conditions are fulfilled:

(1) If $a \in P_1$, $b \in P_2$, then $a + b = b + a$.
(2) Each element $c \in P$ can be uniquely represented in the form $c = c_1 + c_2$ where $c_1 \in P_1$, $c_2 \in P_2$.
(3) If $g, h \in P$, $g = g_1 + g_2$, $h = h_1 + h_2$ where $g_1, h_1 \in P_1$, $g_2, h_2 \in P_2$, then $g \leq h$ if and only if $g_1 \leq h_1$, $g_2 \leq h_2$.

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In this case it is also spoken about the direct decomposition of the po-semigroup P.

The direct decomposition of a po-group is defined analogously.

If $G = P \times Q$ is a direct decomposition of a po-group G, then for $x \in G$ we denote by $x(P)$ and $x(Q)$ the components of x in the direct factors P and Q, respectively. Analogous denotation of components is also applied for the direct decomposition of the semigroup G^+.

If G is a po-group, then a mapping $f\colon G \to G$ is called a weak isometry if $|f(x) - f(y)| = |x - y|$ for each x, $y \in G$. A weak isometry f is called a stable weak isometry if $f(0) = 0$. A weak isometry f is called an isometry if f is a bijection.

A po-group G is called directed if $U(x, y) \neq \emptyset$ and $L(x, y) \neq \emptyset$ for each $x, y \in G$.

In [**12**] and [**13**] the above mentioned Jakubík's result concerning stable isometries and directed decompositions of l-groups was extended to Riesz and distributive multilattice groups.

The following example shows that this result cannot be extended to all directed groups.

Example. Let G be the additive group of all complex numbers $x+iy$ such that x and y are integers. An element $z = x + iy \in G$ is positive if and only if $y \geq x$ and $y \geq -x$. Then G is an abelian non-distributive multilattice group. If we put $f(x+iy) = y+ix$ for each $x+iy \in G$, then f is a stable weak isometry in G. If there exists a direct decomposition $G = A \times B$ of G such that $f(z) = z(A) - z(B)$ for each $z \in G$ then $z + f(z) = 2z(A)$ for each $z \in G$. For the element $a = 1$ we have $a+f(a) = 1+i$, but there does not exist an element b in G such that $a+f(a) = 2b$. Hence there does not exist the above mentioned direct decomposition of G.

Now we are going to establish necessary and sufficient conditions under which to a stable weak isometry f in a directed group G there exists a direct decomposition $G = A \times B$ of G such that $f(x) = x(A) - x(B)$ for each $x \in G$.

First we establish the following theorem.

1. Theorem. *Let f be a stable weak isometry in a* po-*group G, $A_1 = \{x \in G^+, f(x) = x\}$, $B_1 = \{x \in G^+, f(x) = -x\}$. Then the following conditions are equivalent:*

- (i) *For each $x \in G^+$ there exists the least upper bound of the set $\{0, f(x)\}$ in G^+.*
- (ii) *For each $x \in G^+$, there exists $x_1 \in G^+$ such that $0 \leq x_1 \leq x$, $f(x) \leq x_1 \leq f(x) + x$.*
- (iii) *G^+ is the direct product of the* po-*semigroup A_1 and the commutative* po-*semigroup B_1 and $f(x) = x(A_1) - x(B_1)$ for each $x \in G^+$.*

Proof. (i) $\implies$ (ii). Let $x \in G^+$ and let $x_1 = 0 \vee f(x)$. From $|x| = |f(x)|$ we get $x \geq f(x) \geq -x$. Thus $0 \leq x_1 \leq x$. Further, we have $-x \leq 0 \wedge f(x)$. Hence $-f(x) \vee 0 = -(f(x) \wedge 0) \leq x$. This implies $0 \vee f(x) \leq f(x) + x$. Therefore $f(x) \leq x_1 \leq f(x) + x$.

(ii) $\implies$ (iii). Let $x \in G^+$ and let $0 \leq x_1 \leq x$. $f(x) \leq x_1 \leq f(x) + x$ for some $x_1 \in G^+$. From $|x| = |f(x)|$ we get $x = -f(x) \vee f(x)$. Let $x_2 = x - x_1$. Then $x = x_2 + x_1$, $x \geq x_2 \geq 0$. Since $x_1 \leq f(x) + x$, we have $-f(x) \leq x_2$. Hence $x_1 \in U(0, f(x))$, $x_2 \in U(0, -f(x))$. Let $t \in U(0, f(x))$. Then $x_2 + t \in U(f(x), -f(x)) = |f(x)| = |x| = U(x_2 + x_1)$. This implies $t \geq x_1$. Therefore $x_1 = f(x) \vee 0$. Analogously we can show that $x_2 = -f(x) \vee 0$.

Let $z \in U(x_1, x_2)$. Then $z \in U(f(x), -f(x)) = |f(x)| = |x| = U(x)$. Since $x \in U(x_1, x_2)$, we have $x = x_1 \vee x_2$. Then clearly $x_1 \wedge x_2 = 0$ and $x_1 + x_2 = x_2 + x_1$.

Since $-x_2 = f(x) \wedge 0$, $f(x) = f(x) \vee 0 + f(x) \wedge 0$, we have $f(x) = x_1 - x_2$. From $|x_1| = |f(x_1)|$ we obtain $x_1 \geq f(x_1)$, $x_1 \geq -f(x_1)$. Then $f(x_1) + x_2 \geq -x_1 + x_2 = x_2 - x_1 = -f(x)$. From $|x_2| = |x - x_1| = |f(x) - f(x_1)| = |x_1 - x_2 - f(x_1)|$ we get $x_2 \geq x_1 - x_2 - f(x_1)$. This implies $f(x_1) + x_2 \geq -x_2 + x_1 = x_1 - x_2 = f(x)$. Hence $f(x_1) + x_2 \geq -f(x) \vee f(x) = x = x_1 + x_2$. This yields $f(x_1) \geq x_1$. Therefore $f(x_1) = x_1$.

From $|x_2| = |f(x_2)|$ we have $x_2 \geq f(x_2)$, $x_2 \geq -f(x_2)$. Then $-f(x_2) + x_1 \geq -x_2 + x_1 = x_1 - x_2 = f(x)$. From $|x_1| = |x - x_2| = |f(x) - f(x_2)|$ we obtain $x_1 \geq f(x_2) - f(x)$. Thus $-f(x_2) + x_1 \geq -f(x)$. Hence $-f(x_2) + x_1 \geq x = x_2 + x_1$. This implies $-f(x_2) \geq x_2$. Therefore $f(x_2) = -x_2$.

Thus $x = x_1 + x_2$, where $x_1 \in A_1$, $x_2 \in B_1$. By Lemmas 1.8 and 1.9 [**13**], A_1 is a semigroup and B_1 is a commutative semigroup. Let $x = a + b$, where $a \in A_1$, $b \in B_1$. From Theorem 1.13 [**13**] it follows that $f(a + b) = a - b$. Then $b \in U(0, -f(x))$. Hence $b \geq x_2$. Since $a + b = x_1 + x_2$, $a - b = x_1 - x_2$, we obtain $x_2 - b = -x_2 + b \geq 0$. Therefore $x_2 = b$. Then $x_1 = a$. From this also follows that $c + d = d + c$ for each $c \in A_1$, $d \in B_1$.

Let $u, v \in G^+$, $u \leq v$. Let $u = u_1 + u_2$, $v = v_1 + v_2$, $v - u = (v - u)_1 + (v - u)_2$ where u_1, v_1, $(u - v)_1 \in A_1$, u_2, v_2, $(u - v)_2 \in B_1$. From $v - u = v_1 + v_2 - u_2 - u_1$ we get $(v - u)_1 + u_1 + (v - u)_2 + u_2 = v_1 + v_2$. Then we have $v_1 - u_1 = (v - u)_1 \geq 0$, $v_2 - u_2 = (v - u)_2 \geq 0$. Hence $v_1 \geq u_1$, $v_2 \geq u_2$. Therefore G^+ is the direct product of partially ordered semigroups A_1 and B_1.

(iii) $\implies$ (i). Let G^+ be the direct product of the po-semigroup A_1 and the commutative po-semigroup B_1 and $f(z) = z(A_1) - z(B_1)$ for each $z \in G^+$. Let $x \in G^+$. Then $x(A_1) \in U(0, f(x))$. Let $y \geq 0$, $y \geq f(x)$. Thus $y(B_1) \geq 0$ $y(A_1) + y(B_1) + x(B_1) \geq x(A_1)$. Since $G^+ = A_1 \times B_1$, from the last inequality we get $y(A_1) \geq x(A_1)$ Then $y = y(A_1) + y(B_1) \geq x(A_1)$. Therefore $x(A_1) = 0 \vee f(x)$.□

2. Theorem. *Let f be a stable isometry in a directed group G. Let $A_1 = \{x \in G^+, f(x) = x\}$, $B_1 = \{x \in G^+, f(x) = -x\}$, $A = A_1 - A_1$, $B = B_1 - B_1$. Then the following conditions are equivalent:*

(i) *For each* $x \in G^+$ *there exists the least upper bound of* $\{0, f(x)\}$ *in* G^+.
(ii) *For each* $x \in G^+$ *there exists* $x_1 \in G^+$ *such that* $0 \le x_1 \le x$, $f(x) \le x_1 \le f(x) + x$.
(iii) G *is the direct product of the* po-*group* A *and the abelian* po-*group* B *and* $f(z) = z(A) - z(B)$ *for each* $z \in G$.

Proof. In view of 1 it suffices to verify that (ii) implies (iii) and (iii) implies (i).

(ii) $\Longrightarrow$ (iii). In [**13**] it was proved that A is a group, $A^+ = A_1$ [Lemma 1.8], B is an abelian group, $B^+ = B_1$ [Lemma 1.9] and $f(a+b) = a - b$ for each $a \in A$, $b \in B$ [Theorem 1.13]. By 1, $G^+ = A_1 \times B_1$. Then from Theorem 2.3 [**3**] it follows that $G = A \times B$.

(iii) $\Longrightarrow$ (i). Since $A^+ = A_1$ and $B^+ = B_1$, we get $G^+ = A_1 \times B_1$. Then the desired result follows from 1. □

3. Theorem. *Let* G *be a directed group. Let for each* $x \in G^+$ *there exists* $y \in G^+$ *such that* $x = 2y$. *Then for each stable isometry* f *in* G *there exists a direct decomposition* $G = A \times B$ *of* G *with* B *abelian such that* $f(z) = z(A) - z(B)$ *for each* $z \in G$.

Proof. Let $x \in G^+$. Let $y \in G^+$ such that $x = 2y$. From Theorem 1 [**11**] it follows that $y + f(y) = 0 \vee f(x)$. Then the required statement follows from 2. □

Throughout the rest of this paper let f be a stable weak isometry in a po-group G and let S be the subgroup of G generated by G^+. It is clear that S is a directed convex subgroup of G^+. In [**15**] Shimbireva proved that S is a normal subgroup of G.

4. Theorem. (i) $f(x+y) = f(x) + f(y)$ *for each* $x, y \in S$.
(ii) $f^2(x) = x$ *for each* $x \in S$.
(iii) $f(S) = S$.

Proof. The proof of (i) and (ii) is the same as the proof of analogous propositions in Theorem 3 [**11**] concerning directed groups.

(iii) Let $x \in S$. Then $x = a - b$ for some $a, b \in G^+$. By (i), $f(x) = f(a) - f(b)$. Since $a, b \in G^+$, from the relations $|a| = |f(a)|$, $|b| = |f(b)|$ we get $a \ge f(a)$, $b \ge -f(b)$. Hence $a + b \ge f(a) - f(b) = f(x)$, $a + b \ge 0$. Then $f(x) = (a+b) - [-f(x) + (a+b)] \in S$. In view of (ii) we have $f(S) = S$. □

5. Corollary. *Restriction of a stable weak isometry* f *in a* po-*group* G *to the subgroup* S *is an involutory group automorphism.*

Proof. It is easy to see that each stable weak isometry in G is an injection. Then 4 ends the proof. □

6. Theorem. *Let $x \in G^+$. Then*

(i) $f([0,x]) \subseteq [-x,x] \cap [-x+f(x), x+f(x)]$,

(ii) *if there exists the least upper bound of the set* $\{0, f(x)\}$ *in* G, *then* $f([0,x]) \subseteq [0 \wedge f(x), 0 \vee f(x)]$.

Proof. (i) Let $y \in [0,x]$. From $|x-y| = |f(x)-f(y)|$ we get $x-y \geq f(y)-f(x)$, $x-y \geq f(x)-f(y)$. Then $f(x)-f(y)+x \geq y \geq 0$, $f(y)-f(x)+x \geq y \geq 0$. Hence $x+f(x) \geq f(y) \geq -x+f(x)$. Since $x \in |y| = |f(y)|$, we have $x \geq f(y) \geq -x$.

(ii) From the assumption it follows that there exist $0 \wedge f(x)$ and $0 \vee -f(x)$ in G, too. From $|x| = |f(x)|$ we have $x = -f(x) \vee f(x)$. Then $[0 \wedge f(x)] + [0 \vee -f(x)] = 0 \vee [f(x) \vee -f(x)] = 0 \vee x = x$. This implies $0 \vee f(x) = x - [0 \vee -f(x)] = x + [0 \wedge f(x)] = x \wedge (x+f(x))$, $0 \wedge f(x) = -[0 \vee -f(x)] = -x + [0 \vee f(x)] = -x \vee [-x+f(x)]$. Then (i) ends the proof. □

7. Theorem. *Let $x \in G^+$. Then*

(i) $[-x,x] \cap [-x+f(x), x+f(x)] \subseteq f([0,x])$,

(ii) *if there exists the least upper bound of the set* $\{0, f(x)\}$ *in* G, *then* $[0 \wedge f(x), 0 \vee f(x)] \subseteq f([0,x])$.

Proof. (i) Let $z \in G$ such that $-x \leq z \leq x$, $-x+f(x) \leq z \leq x+f(x)$. Then $0 \leq z+x \leq 2x$, $0 \leq z-f(x)+x \leq 2x$. According to 4, $f(x) \in S$. By Theorem 1 [**11**], $x+f(x) = 0 \vee f(2x)$, $-x+f(x) = 0 \wedge f(2x)$. Since x, $f(x)$, z are elements of S, from 4 and 6(ii) we get $f(z)+f(x) = f(z+x) \leq x+f(x)$, $f(z)-x+f(x) = f(z-x+f(x)) \geq -x+f(x)$. Thus $0 \leq f(z) \leq x$. In view of 4 we have $f^2(z) = z \in f([0,x])$.

(ii) By the same way as in the proof of 6(ii) we can prove that $0 \vee f(x) = x \wedge [x+f(x)]$, $0 \wedge f(x) = -x \vee [-x+f(x)]$. Then (i) completes the proof. □

From 6 and 7 we immediately obtain

8. Theorem. *Let $x \in G^+$. Then*

(i) $f([0,x]) = [-x,x] \cap [-x+f(x), x+f(x)]$,

(ii) *if there exists the least upper bound of the set* $\{0, f(x)\}$ *in* G, *then* $f([0,x]) = [0 \wedge f(x), 0 \vee f(x)]$.

If C is a normal convex subgroup of a po-group H then the factor group H/C can be partially ordered by the induced order. See [**1**, p. 20].

9. Theorem. *The factor group G/S of a po-group G with respect to S is trivially ordered with regard to induced order.*

Proof. Let $S+a \leq S+b$ for some $a, b \in G$. Then there exist $h, g \in S$ such that $h+a \leq g+b$. This yields $0 \leq -h+g+b-a$. Thus $-h+g+b-a \in S$ and hence $b-a \in S$. This implies that $b = (b-a)+a \in S+a$. Therefore $S+a = S+b$. □

10. Theorem. *Let $a \in G$. Then $f(x) = \overline{h}(x-a) + f(a)$ for each $x \in S + a$, where $\overline{h}$ is a stable weak isometry in S.*

Proof. Let $h(z) = f(z+a) - f(a)$ for each $z \in G$. Then h is a stable weak isometry in G. By 4, $h(S) = S$. Thus the restriction $\overline{h}$ of h to S is a stable weak isometry in S. Let $x \in S + a$. Hence $x = y + a$ for some $y \in S$. Then $\overline{h}(y) = \overline{h}(x-a) = f(x) - f(a)$. From this we get $f(x) = \overline{h}(x-a) + f(a)$. □

11. Theorem. *Let $a \in G$.Then $f(S+a) = S + f(a)$.*

Proof. From 10 it follows that $f(S+a) \subseteq S + f(a)$. Let $z = t + f(a)$ where $t \in S$. By 10, $f(x) = \overline{h}(x-a) + f(a)$ for each $x \in S+a$ where $\overline{h}$ is a stable weak isometry in S. Since $z - f(a) \in S$, then there exists $u \in S$ such that $\overline{h}(u) = z - f(a)$. In view of 10 we have $f(u+a) = \overline{h}(u) + f(a) = z$. Therefore $S + f(a) \subseteq f[S+a]$.□

12. Theorem. *Let $S+a,\ S+b \in G/S$ such that $S+a \neq S+b$. Then $f(S+a) \neq f(S+b)$.*

Proof. Since f is an injection, it follows from 11. □

13. Theorem. *If the factor group G/S is finite, then f is a bijection.*

Proof. Since f is an injection, the desired assertion follows from 12. □

Remark. There exist a nontrivially ordered group H and a stable weak isometry in H which is not a bijection.

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