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DISTRIBUTIVE LATTICES WHOSE CONGRUENCE LATTICE IS STONE

Z. HELEYOVÁ

Abstract. Using Priestley's topological duality we characterize bounded distributive lattices with (L_n)- and relative (L_n)-congruence lattices. In particular, characterizations of bounded distributive lattices with Stone and relative Stone congruence lattices are obtained. Using these descriptions we derive some results of [**8**], [**9**] [**5**] and [**6**]. In the last section we discuss questions concerning the relation between completeness of a bounded distributive lattice and its minimal Boolean completition. This is connected with a problem of D. Thomas [**9**].

1. Introduction

In his well-known monograph [**4**], G. Grätzer posed a problem of characterizing lattices whose congruence lattices belong to the n-th subvariety ($n \geq 1$) of distributive p-algebras, i.e. are (L_n)-lattices when adopting the terminology of [**7**]. Lattices with Stone (i.e. (L_1)-) congruence lattices have first been characterized in [**8**]. In [**6**], lattices with relative Stone congruence lattices have been described and it was shown that distributive lattices have relative Stone congruence lattices iff they are discrete. In [**5**], lattices with (L_n)- and relative (L_n)-congruence lattices have been characterized for any $n \geq 1$; for semi-discrete lattices this has been done in [**7**]. In all the papers mentioned above the results are presented in terms of weak projectivity of quotients of the given lattice.

On the other hand, using Priestley's topological duality for bounded distributive lattices, D. Thomas in [**9**] showed that a bounded distributive lattice has the Stone congruence lattice iff the minimal Boolean extension $\boldsymbol{L}_B$ of the lattice $\boldsymbol{L}$ is a complete Boolean algebra (the original purely algebraic proof is due to T. Katriňák [**8**]). She also posed a problem as whether there is a bounded distributive lattice $\boldsymbol{L}$ which is not complete although $\boldsymbol{L}_B$ is complete.

In this paper, using Priestley's topological duality we characterize bounded distributive lattices with (L_n)- and relative (L_n)-congruence lattices for any $n \geq 1$.

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In particular, we get descriptions via their duals of those distributive lattices whose congruence lattices are Stone and relative Stone. Using these characterizations we derive the Katriňák and Thomas result mentioned above and some results of [**5**] and [**6**]. In particular, we prove that chains with (L_n)-congruence lattices must be discrete. From this we conclude that for every infinite complete chain $\boldsymbol{L}$ its minimal Boolean extension $\boldsymbol{L}_B$ is not complete. Concerning the question of D. Thomas, two examples of a bounded distributive lattice $\boldsymbol{L}$ which is not complete although $\boldsymbol{L}_B$ is complete are presented.

2. Preliminaries

A (distributive) p**-algebra** is an algebra $\mathbf{L} = \langle L; \vee, \wedge, ^\star, 0, 1\rangle$ where $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded (distributive) lattice and the unary operation * (of pseudocomplementation) is defined by

$$a^\star = \max\{x \in L \mid x \wedge a = 0\}.$$

It is well-known that the class B_ω of all distributive p-algebras is equational. K. B. Lee in [**10**] showed that the lattice of all equational subclasses of B_ω is a chain

$$B_{-1} \subset B_0 \subset B_1 \subset \cdots \subset B_n \subset \cdots \subset B_\omega$$

of type $\omega + 1$, where B_{-1}, B_0 and B_1 denote the classes of all trivial, Boolean and Stonean algebras, respectively. Moreover, he proved that for $n \geq 1$, $\mathbf{L} \in B_n$ if and only if $\mathbf{L}$ satisfies the identity

$$(L_n) \quad (x_1 \wedge \ldots \wedge x_n)^* \vee (x_1^* \wedge \ldots \wedge x_n)^* \vee \ldots \vee (x_1 \wedge \ldots \wedge x_n^*)^* = 1.$$

2.1. Definition ([7, Definition 1]). Let $\mathbf{L}$ be a distributive p-algebra and $n \geq 1$. $\mathbf{L}$ is said to be an (L_n)-lattice if $\mathbf{L} \in B_n$.

2.2. Proposition ([1, p. 163]). *Let* $\mathbf{L}$ *be a distributive p-algebra and* $n \geq 1$. *Then* $\mathbf{L}$ *is an* L_n*-lattice if and only if it satisfies the identity*

$$(x_0 \wedge \ldots \wedge x_n)^* = \bigvee\{(x_{i_1} \wedge \ldots \wedge x_{i_n})^* \mid 0 \leq i_1 < \ldots < i_n \leq n\}.$$

We recall some concepts and results of Priestley's duality theory. For a general background in Priestley's duality theory we refer the reader to [**11**] and [**12**] or [**13**].

The triple $\langle X; \mathcal{T}, \leq\rangle$ is said to be **an ordered topological space** if $\langle X; \mathcal{T}\rangle$ is a topological space and $\leq$ is a partial order relation defined on the set X. A subset $U \subseteq X$ is called a **downset** (**upset**) if $x \in U$ and $y \leq x$ ($y \geq x$) yields $y \in U$. The ordered topological space $\langle X; \mathcal{T}, \leq\rangle$ is said to be **totally-order disconnected** if for given $x, y \in X$ with $x \not\geq y$ there exists a clopen downset $U \subseteq X$ such that $x \in U$ and $y \notin U$. A compact totally-order disconnected space will be called a **CTOD-space**. If X is a CTOD-space, then the family $\mathcal{O}(X)$ of the clopen downsets of X is a bounded distributive lattice called the **dual lattice** of X.

The **dual space** of a bounded distributive lattice L is $\langle X_L; \mathcal{T}, \leq\rangle$, where $X_L = \mathcal{I}_P(L)$ is the set of all prime ideals of the lattice L, $\leq$ is given by $P \leq Q$ iff $Q \subseteq P$ for any $P, Q \in X_L$ and the base for $\mathcal{T}$ consists of the sets $X_a \cap (X \setminus X_b)$ $(a, b \in L)$, where $X_a = \{I \in \mathcal{I}_P(L) \mid a \notin I\}$. The dual space X_L is a CTOD-space and $\mathcal{O}(X_L) = \{X_a \mid a \in L\}$.

The following statement is known as Priestley's representation theorem for bounded distributive lattices.

2.3. Theorem ([11, Theorem 1]). *Let $\boldsymbol{L}$ be a bounded distributive lattice and let $\langle X; \mathcal{T}, \leq\rangle$ be its dual space. Then $\boldsymbol{L}$ is isomorphic to the dual lattice $\mathcal{O}(X)$ of $\langle X; \mathcal{T}, \leq\rangle$.*

Dual to this theorem is the following result.

2.4. Theorem ([11, Theorem 2]). *Let $\langle X; \mathcal{T}, \leq\rangle$ be a CTOD-space and $\boldsymbol{L}$ its dual lattice and denote by $\langle Y; \mathcal{T}', \leq'\rangle$ the dual space of $\boldsymbol{L}$. Then $\langle X; \mathcal{T}, \leq\rangle$, $\langle Y; \mathcal{T}', \leq'\rangle$ are homeomorphic as topological spaces and isomorphic as partially ordered sets.*

The Priestley duality theory also enables us to find a nice and concrete representation of the congruence lattice $\mathrm{Con}\,(\boldsymbol{L})$ of a bounded distributive lattice $\boldsymbol{L}$. If $\langle X; \mathcal{T}, \leq\rangle$ is a CTOD-space and $\boldsymbol{L} = \mathcal{O}(X)$ (i.e. $\boldsymbol{L}$ is a bounded distributive lattice with a dual space X), then $\mathrm{Con}\,(\boldsymbol{L})$ is isomorphic to the lattice $\mathcal{A}(X)$ of all open subsets of the space X (see e.g. [13]).

Let $\boldsymbol{L}$ be a bounded distributive lattice. The results above enable us to simplify our investigations by assuming that $\boldsymbol{L} = \mathcal{O}(X)$ for some CTOD-space X and that $\langle \mathrm{Con}\,(\boldsymbol{L}); \vee, \wedge, ^\star, 0, 1\rangle \cong \langle \mathcal{A}(X); \vee, \wedge, ^\star, \emptyset, X\rangle$, where the operation * of pseudocomplementation is defined in $\mathcal{A}(X)$ as follows:

$$U^\star = X \setminus \overline{U} \quad \text{for every} \quad U \in \mathcal{A}(X).$$

Here (and throughout the paper) $\overline{U}$ denotes the $\mathcal{T}$-closure of U in $\langle X; \mathcal{T}, \subseteq\rangle$. Further, we shall denote by $\mathrm{Int}\,(A)$ the interior of a set A, $A \subseteq X$.

In [12] it was shown that $\boldsymbol{L}$ is complete iff its dual X_L is **extremally-order disconnected**, i.e. for every open downset U in X_L, the smallest closed downset containing U is open. An **extremally disconnected** topological space is defined such that the closure of each of its open subsets is open.

It is well-known that for every bounded distributive lattice $\boldsymbol{L}$ there is exactly one minimal Boolean extension $\boldsymbol{L}_B$ of $\boldsymbol{L}$ such that $\mathrm{Con}\,(\boldsymbol{L}_B)$ is isomorphic to $\mathrm{Con}\,(\boldsymbol{L})$ (see e.g. [4, II.4]). The dual space of $\boldsymbol{L}_B$ can be obtained from $\langle X_L; \mathcal{T}, \leq\rangle$ by replacing the partial order $\leq$ with the trivial order. Hence from the previous paragraph we get the following statement:

2.5. Proposition. *Let L be a bounded distributive lattice. Then X_L is extremally disconnected if and only if the minimal Boolean extension $\boldsymbol{L}_B$ of $\boldsymbol{L}$ is complete.*

3. Lattices with (L_n)-congruence lattices

3.1. Theorem. *Let $\boldsymbol{L}$ be a bounded distributive lattice. Then* $\mathrm{Con}\,(\boldsymbol{L})$ *is an (L_n)-lattice $(n \geq 1)$ if and only if for any $n+1$ pairwise disjoint $\mathcal{T}$-open subsets $U_0, U_1, \dots, U_n$ of X_L*

$$\bigcap_{i=0}^{n} \overline{U}_i = \emptyset.$$

Proof. We can assume that $\mathrm{Con}\,(\boldsymbol{L}) = \mathcal{A}(X)$ where X is the dual space of $\boldsymbol{L}$ and $\mathcal{A}(X)$ is the lattice of all open subsets of X. Let $\mathrm{Con}\,(\boldsymbol{L})$ be an (L_n)-lattice and let $U_0, U_1, \dots, U_n$ be pairwise disjoint subsets of X. Let $U := U_0 \cup \ldots \cup U_n$ and $V_i := U \setminus U_i$, $i = 0, 1, \dots, n$. From 2.2 it follows that

$$\begin{aligned} X = (\emptyset)^\star = (V_0 \cap \ldots \cap V_n)^\star &= \bigvee \{(V_{i_1} \cap \ldots \cap V_{i_n})^\star \mid 0 \leq i_1 < \ldots < i_n \leq n\} \\ = \bigvee_{i=0}^{n} U_i^\star &= \bigcup_{i=0}^{n} X \setminus \overline{U}_i = X \setminus \bigcap_{i=0}^{n} \overline{U}_i\,. \end{aligned}$$

Therefore $\bigcap_{i=0}^{n} \overline{U}_i = \emptyset$.

Conversely, assume that the given condition holds. Let $V_1, V_2, \dots, V_n$ be arbitrary open subsets of X and let

$$\begin{aligned} U_0 &= V_1 \cap \ldots \cap V_n, \\ U_1 &= V_1^\star \cap V_2 \cap \ldots \cap V_n, \\ &\cdots\cdots \\ U_n &= V_1 \cap \ldots \cap V_{n-1} \cap V_n^\star. \end{aligned}$$

Clearly, $U_0, U_1, \dots, U_n$ are pairwise disjoint subsets of X and

$$\begin{aligned} (V_1 \cap \ldots \cap V_n)^\star \cup (V_1^\star \cap \ldots \cap V_n) \cup \ldots \cup (V_1 \cap \ldots \cap V_n^\star)^\star \\ = U_0^\star \cup \ldots \cup U_n^\star = X \setminus \bigcap_{i=0}^{n} \overline{U}_i = X, \end{aligned}$$

which implies that $\mathrm{Con}\,(\boldsymbol{L})$ is an (L_n)-lattice. □

3.2. Corollary. *Let $\boldsymbol{L}$ be a bounded distributive lattice. Then the following are equivalent:*

(1) $\mathrm{Con}\,(\boldsymbol{L})$ *is a Stone lattice;*
(2) $\overline{U} \cap \overline{V} = \emptyset$ *for any disjoint $\mathcal{T}$-open subsets U, V of X_L;*
(3) X_L *is extremally disconnected.*

Proof. The equivalence of (1) and (2) follows immediatelly from 3.1 if one puts $n = 1$. To show the equivalence of (1) and (3), we assume that $\boldsymbol{L} = \mathcal{O}(X)$ for some CTOD-space X. Let $\mathrm{Con}\,(\boldsymbol{L})$ be a Stone lattice. Then for arbitrary open subset $U \subseteq X$ we have $U^\star \cup U^{\star\star} = X$. Therefore, $\overline{U} = X \setminus U^\star = U^{**} \in \mathcal{A}(X)$ is an open set in X.

Conversely, let $\overline{U}$ be an open set for every open $U \subseteq X$. Then

$$\begin{aligned} U^\star \cup U^{\star\star} &= \mathrm{Int}\,(X \setminus U) \cup \mathrm{Int}\,(X \setminus U^\star) = \mathrm{Int}\,(X \setminus U) \cup \mathrm{Int}\,(X \setminus \mathrm{Int}\,(X \setminus U)) \\ &= \mathrm{Int}\,(X \setminus U) \cup \mathrm{Int}\,(\overline{U}) = \mathrm{Int}\,(X \setminus U) \cup \overline{U} = X. \end{aligned}$$

Therefore $\mathrm{Con}\,(\boldsymbol{L})$ is a Stone lattice. □

From 3.2 and 2.5 we immediately get the mentioned result of T. Katriňák [**8**] and D. Thomas [**9**]: $\mathrm{Con}\,(\boldsymbol{L})$ is Stone iff the minimal Boolean extension of $\boldsymbol{L}$ is complete.

The previous results can be extended to the class of distributive lattices which do not have zero and unit elements.

Let L be an arbitrary distributive lattice and let $\tilde{L}$ be obtained from L by adjoining a unit and zero to L. Let $X = \mathcal{I}_p(L)$. Then the dual space of $\tilde{L}$ is $\langle \tilde{X}; \mathcal{T}; \leq \rangle$ where $\tilde{X} = \mathcal{I}_p(\tilde{L}) = \{0\} \cup X \cup \{\{0\} \cup L\}$ (if we identify $i \in X$ with $i \cup \{0\} \in \tilde{X}$). The isomorphism $\alpha\colon \mathrm{Con}\,(\tilde{L}) \to \mathcal{A}(\tilde{X})$ can be described in the followig way (see [**13**]).

For every $\theta \in \mathrm{Con}\,(\tilde{L})$

$$\alpha\colon \theta \mapsto U_\theta = \tilde{X} \setminus f(\mathcal{I}_p(\tilde{L}/\theta)),$$

where $f(x) = \varphi^{-1}(x)$ $(x \in \mathcal{I}_p(\tilde{L}/\theta))$ and $\varphi\colon \tilde{L} \to \tilde{L}/\theta$ is the canonical homomorphism.

For every $U \in \mathcal{A}(\tilde{X})$ the congruence $\theta_U = \alpha^{-1}(U)$ is defined:

$$x \equiv y(\theta_U) \Leftrightarrow x \setminus U = y \setminus U \quad (x, y \in \mathcal{O}(\tilde{X}) = \tilde{L}).$$

It is easy to verify that $\mathrm{Con}\,(L) \cong C_L = \{\theta \in \mathrm{Con}\,(\tilde{L}) \mid [0]\theta = \{0\} \text{ and } [1]\theta = \{1\}\}$. We will show that C_L is isomorphic with the lattice $\mathcal{A}(X)$.

Let $\theta \in C_L$. The CTOD spaces $\mathcal{I}_p(\tilde{L})$ and $\mathcal{I}_p(\tilde{L}/\theta)$ have both the minimal elements $\omega = \{0\}$, $\omega_1 = \{[0]\theta\}$ and the maximal elements $\iota = \{0\} \cup L$, $\iota_1 = \{[0]\theta\} \cup \{[a]\theta : a \in L\}$, respectively. Then we have

$$f(\omega_1) = \varphi^{-1}(\omega_1) = \varphi^{-1}(\{[0]\theta\}) = \{0\} = \omega$$

and
$$f(\iota_1) = \varphi^{-1}(\iota_1) = \phi^{-1}(\{[0]\theta\} \cup \{[a]\theta : a \in L\}) = \{0\} \cup L = \iota.$$

Therefore $U_\theta = \tilde{X} \setminus f(\mathcal{I}_p(\tilde{L}/\theta)) = X \setminus f(\mathcal{I}_p(\tilde{L}/\theta))$ i.e. U_θ is an open subset of X with induced topology.

Conversely let $U \subseteq \tilde{X}$ be an open subset and $U \subseteq X$.
Then $0 \equiv a(\theta_U) \Leftrightarrow a \setminus U = 0 \setminus U = \emptyset$ for $a \in \mathcal{O}(\tilde{X}) = \tilde{L}$. Since $\omega \notin U$ and $\omega \in a$ for arbitrary $a \in \mathcal{O}(\tilde{X})$, $a \neq \emptyset$, we conclude $[0]\theta_U = \{0\}$. Similarly $1 \equiv a(\theta_U) \Leftrightarrow a \setminus U = 1 \setminus U = \tilde{X} \setminus U$. But $\iota \in \tilde{X} \setminus U$ and $\iota \in a \in \mathcal{O}(\tilde{X})$ imply that $a = \tilde{X}$. Hence $[1]\theta_U = \{1\}$. Therefore $\theta_U \in C_L$.

So we can conclude that Con(L) is isomorphic with the lattice $\mathcal{A}(X)$ of open subsets of $X = \mathcal{I}_p(L)$ with induced topology (i.e. the base of this topology consists of the sets $X \cap (\tilde{X}_a \cap (\tilde{X} \setminus \tilde{X}_b)) = X_a \cap (X \setminus X_b)$, $a, b \in L$).

Using this idea, analogously as in 3.1 one can prove:

3.3 Theorem. *Let* $\boldsymbol{L}$ *be a distributive lattice and the dual space* $\langle X_L; \mathcal{T}, \leq\rangle$. *Then* Con (L) *is an* (L_n)*-lattice* ($n \geq 1$) *if and only if for any* $n+1$ *pairwise disjoint* $\mathcal{T}$*-open subsets* $U_0, U_1, \dots, U_n$ *of* X_L
$$\bigcap_{i=0}^{n} \overline{U}_i = \emptyset.$$

4. Lattices with Relative (L_n)-congruence Lattices

4.1. Definition ([7, Definition 2]). Let L be a distributive lattice. L is said to be a relative (L_n)-lattice if every interval $[a, b]$ in L is an (L_n)-lattice.

4.2. Proposition. ([**7**], Theorem 1) Let L be a distributive lattice with 1. The following conditions are equivalent:

(i) L is a relative (L_n)-lattice;
(ii) for every $a \in L, [a, 1]$ is an (L_n)-lattice.

Let L be a bounded distributive lattice with the dual space X_L. According to 4.2, Con (L) is a relative (L_n)-lattice if and only if for every $\mathcal{T}$-open set $U \subseteq X_L$, $[U, X]$ is an (L_n)-lattice. One can easily verify that $[U, X_L]$ is isomorphic with the lattice of $\mathcal{T}_Y$-open subsets of the space $Y = X_L \setminus U$ with induced topology $\mathcal{T}_Y$. Therefore from 3.1 we immediatelly get the following characterization which can (similarly as in the previous section) be extended to the class of all distributive lattices:

4.3. Theorem. *Let* L *be a distributive lattice. Then* Con (L) *is a relative* (L_n)*-lattice* ($n \geq 1$) *if and only if for every closed subset* $Y \subseteq X_L$*, the space*

$\langle Y; \mathcal{T}_Y \rangle$ with the induced topology $\mathcal{T}_Y$ satisfies

$$\bigcap_{i=0}^{n} \overline{U}_i = \emptyset$$

for any $n+1$ pairwise disjoint $\mathcal{T}_Y$-open sets $U_0, U_1, \ldots, U_n$ of Y. (Here $\overline{U}_i$ denotes the $\mathcal{T}_Y$-closure of U_i.)

4.4. Corollary. *Let L be a distributive lattice. Then* Con(L) *is a relative Stone lattice if and only if for every $\mathcal{T}$-closed subset $Y \subseteq X$, the space $\langle Y; \mathcal{T}_Y \rangle$ is extremally disconnected.*

Ph. Dwinger in [**3**] showed that every infinite complete Boolean algebra has a quotient which is not complete. Hence any dual space X_L of an infinite Boolean lattice $\boldsymbol{L}$ which is extremally disconnected has a closed subspace which is not extremally disconnected. From this fact and 3.4 and 4.4 we easily derive the following result (see [**6**, Theorem 7]).

4.5. Corollary. *Let $\boldsymbol{L}$ be a distributive lattice. Then* Con$(\boldsymbol{L})$ *is a relative Stone lattice if and only if* Con$(\boldsymbol{L})$ *is Boolean, i.e. $\boldsymbol{L}$ is discrete (meaning that bounded chains in $\boldsymbol{L}$ are finite).*

Proof. It is well-known that the congruence lattice of a discrete lattice is Boolean. For the converse, let $\boldsymbol{L}$ be infinite. We want to show that Con$(\boldsymbol{L})$ is not relative Stone. If Con$(\boldsymbol{L})$ is not Stone, we are ready. Now let Con$(\boldsymbol{L})$ be Stone. By 3.2, which by 3.3 can be extended to unbounded lattices, $\langle X_L; \mathcal{T}, \leq \rangle$ is extremally disconnected. Since $\langle X_L; \mathcal{T} \rangle$ is the dual of $\boldsymbol{L}_B$ and $\boldsymbol{L}_B$ is infinite, by Dwinger's result, X_L has a cloSed subspace which is not extremally disconnected. By 4.4 this means that Con$(\boldsymbol{L})$ is not relative Stone. □

5. Completeness of $\boldsymbol{L}$ Versus $\boldsymbol{L}_B$

In this section we deal with the relation between completeness of a bounded distributive lattice $\boldsymbol{L}$ and its minimal Boolean extension $\boldsymbol{L}_B$. We first apply our result 3.1 to derive the following result (see [**5**, Theorem 3]):

5.1. Theorem. *Let $\boldsymbol{L}$ be a chain. Then* Con$(\boldsymbol{L})$ *is an (L_n)-lattice ($n \geq 1$) if and only if $\boldsymbol{L}$ is discrete.*

Proof. Assume that Con$(\boldsymbol{L})$ is (L_n)-lattice ($n \geq 1$) and suppose to the contrary that $\boldsymbol{L}$ is not discrete. Then there exist $a, b \in L$, $a < b$ such that $[a, b]$ contains an infinite chain $\boldsymbol{C}$.

(i) Assume that $\boldsymbol{C}$ is an infinite increasing chain. We denote the elements of C by $a_{10} < a_{11} < \ldots < a_{1n} < a_{20} < \ldots$ Let $X = \mathcal{I}_p(\boldsymbol{L})$. Set

$$U_i = \bigcup_{j=1}^{\infty} (X_{a_{j,i+1}} \cap X \setminus X_{a_{j,i}}) \quad (i = 0, \ldots n-1),$$
$$U_n = \bigcup_{j=1}^{\infty} (X_{a_{j+1,0}} \cap X \setminus X_{a_{j,n}}).$$

Clearly $U_0, \ldots U_n$ are pairwise disjoint open sets.
Let $i_c = \downarrow C = \{e \in L : e \leq a_{ji}$ for some $a_{ji} \in C\}$. Now we will prove that $i_c \in \overline{U_i}$ $(i = 0, \ldots n)$.

Take arbitrary base set $X_g \cap X \setminus X_h$, $g, h \in L$ such that $i_c \in X_g \cap X \setminus X_h$. Since $i_c \in X_g$, for every j, i we have $X_{a_{j,i}} \subseteq X_g$, thus $U_i \subseteq X_g$. On the other hand, $i_c \in X \setminus X_h$ implies that $h \leq a_{k,i}$ for some $k \geq 1$ and $i \in \{0, \ldots, n\}$. Then $X \setminus X_{a_{j,i}} \subseteq X \setminus X_h$ for any $j \geq k$ and $i \in \{0, \ldots, n\}$. Therefore $U_i \cap (X_g \cap X \setminus X_h) \neq \emptyset$ $(i = 0, \ldots, n)$. It implies that $i_c \in \bigcap_{i=0}^{n} \overline{U_i}$.

(ii) Assume that $\boldsymbol{C}$ is an infinite decreasing chain with elements
$b_{10} > b_{11} > \ldots > b_{1n} > b_{20} > \ldots$. Set

$$U_i = \bigcup_{j=1}^{\infty} (X_{b_{j,i}} \cap X \setminus X_{b_{j,i+1}}) \ (i = 0, \ldots, n-1),$$
$$U_n = \bigcup_{j=1}^{\infty} (X_{b_{j,n}} \cap X \setminus X_{b_{j+1,0}}).$$

Again $U_0, \ldots, U_n$ are pairwise disjoint open subsets of X. Let $j_c = \bigcap_{i,j} \downarrow b_{ji} = \{f \in L : f \leq b_{ji}$ for every $b_{ji} \in C\}$ and $X_g \cap X \setminus X_h$ be any base set containing j_c. Since $j_c \in X \setminus X_h$, we have $X \setminus X_{b_{ji}} \subseteq X \setminus X_h$ for $j \geq 1$ and $i \in \{0, \ldots n\}$, thus $U_i \subseteq X \setminus X_h$. Further, $j_c \in X_g$ yields that there exists $k \geq 1$ such that $g > b_{ji}$ for every $j > k$ and $i = 0, \ldots, n$. Then $X_{b_{j,i}} \subseteq X_g$ for $j > k$. Hence $U_i \cap (X_g \cap X \setminus X_h) \neq \emptyset$ $(i = 0, \ldots, n)$. It implies that $j_c \in \bigcap_{i=0}^{n} \overline{U_i}$

Conversely assume that the chain $\boldsymbol{L}$ is discrete.
Then $X = I_p(\boldsymbol{L}) = \{(a] \mid a \in L\} \cong L$ is also a discrete chain. Clearly, $\{(a]\} = X_b \cap X \setminus X_a$ is an open subset in X for arbitrary $a, b \in L$ where b covers a in $\boldsymbol{L}$. It implies that X (with induced topology) is extremally disconnected and the statement follows immediately from 3.2. □

This gives us an example of a complete distributive lattice $\boldsymbol{L}$ whose minimal Boolean extension $\boldsymbol{L}_B$ is not complete:

5.2. Corollary. *For every infinite complete chain $\boldsymbol{L}$ its minimal Boolean extension $\boldsymbol{L}_B$ is not complete.*

Proof. Since $\boldsymbol{L}$ is complete, it is bounded, hence not discrete. Therefore by 5.1, $\mathrm{Con}\,(L)$ is not Stone. □

Now we present via its dual an example of a non-complete bounded distributive lattice $\boldsymbol{L}$ whose minimal Boolean extension $\boldsymbol{L}_B$ is complete. This answers the question posed by D. Thomas in [**9**]. This example was motivated by a work of W. Bowen [**2**].

5.3. Example. Take two copies X_1, X_2 of the dual space of an infinite complete Boolean algebra. By 2.5, X_1, X_2 are extremally disconnected. Take elements $x_i \in X_i$, $i = 1, 2$ such that $\{x_i\}$ is not open. As X_i is Hausdorff, we have that $\{x_i\}$ is closed. Let X_L be the disjoint union of X_1 and X_2 with the order $\leq$ such that $x_2 \leq x_1$ are the only distinct comparable elements in $\leq$. Clearly, X_L is an extremally disconnected CTOD space which is not extremally-order disconnected: for the open downset $U = X_1 \setminus \{x_1\}$, the smallest closed downset containing U is $X_1 \cup \{x_2\}$ which is not open. Hence $\boldsymbol{L}$ is not complete, but the minimal Boolean extention of $\boldsymbol{L}$ is complete.

5.4. Remark. M. Ploščica recently suggested the following (purely algebraic) example: take the Boolean algebra $\mathbf{B} = \mathcal{P}(N)$ of all subsets of the set of natural numbers N. Let F be a non-trivial ultrafilter of $\mathbf{B}$. Further, let I be the ideal of all finite subsets of N. Define $L := I \cup F$. Obviously, $\boldsymbol{L}$ is a sublattice of $\mathbf{B}$ and L generates $\mathbf{B}$. While $\mathbf{B}$ is clearly complete, the lattice $\boldsymbol{L}$ is not complete: for any subset S of L, the set of all finite subsets of S is an ideal of $\boldsymbol{L}$ which does not have a supremum in $\boldsymbol{L}$.

References

1. Balbes R. and Dwinger Ph., *Distributive lattices*, Univ. Miss. Press, 1975.

2. Bowen W., *Lattice theory and topology*, Ph.D. thesis, Oxford, 1981.

3. Dwinger Ph., *On the completeness of the quotient algebras of a complete Boolean algebra. I*, Indag. Math. **20** (1958), 448–456.

4. Grätzer G., *General lattice theory*, Birkhäuser-Verlag, Basel, 1978.

5. Haviar M., *Lattices whose congruence lattices satisfy Lee's identities*, Demonstratio Math. **XXIV**, no. 1-2 (1991), 247–261.

6. Haviar M. and Katriňák T., *Lattices whose congruence lattice is relative Stone*, Acta Sci. Math. (Szeged) **51** (1987), 81–91.

7. ______, *Semi-discrete lattices with* (L_n)*-congruence lattices*, Contributions to General Algebra **7** (1991), 189–195, Verlag Hölder-Pichler-Tempsky, Wien.

8. Katriňák T., *Notes on Stone lattices II*, Mat. Časopis Sloven. Akad. Vied **17** (1967), 20–37.

9. Thomas D., *Problems in functional analysis*, Ph.D. thesis, Oxford, 1976.

10. Lee K. B., *Equational classes of distributive pseudocomplemented lattices*, Canad. J. Math. **22** (1970), 20–37.

11. Priestley H. A., *Representation of distributive lattices by means of ordered Stone spaces*, Bull. London Math. Soc. **2** (1970), 186–190.

12. ______, *Ordered topological spaces and the representation of distributive lattices*, Proc. London Math. Soc. (3) **24** (1972), 507–530.

13. ———, *Ordered sets and duality for distributive lattices*, Annals of Discrete Mathematics **23** (1984), 39–60.
14. Ploščica M., (private communication).

Z. Heleyová, Slovak Technical University, Dept. of Mathematics, Radlinského 11, 813 68 Bratislava, Slovakia; *e-mail*: heleyova@cvtstu.cvt.stuba.sk