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ON COMPLETE MEASURABILITY OF MULTIFUNCTIONS DEFINED ON PRODUCT SPACES

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1. Introduction

In the present note we occupy ourselves with the cases in which we can say that a multifunction of two variables is jointly measurable. In particular we generalize onto the case of multifunctions the Theorems 2 and 3 from paper [**6**, pp. 150–151]. We start with the concept of "measurable space with negligibles" which is intended as a common generalization of the two principal examples: $(S, \mathcal{M}(S), \mathcal{N}(S))$, where $(S, \mathcal{M}(S), \mu)$, is a measurable space and $\mathcal{N}(S)$ is the σ-ideal of μ-measure zero subsets of S and $(S, \mathcal{B}(S), \mathcal{I}(S))$, where $(S, \mathcal{T}(S))$ is a topological space, $\mathcal{B}(S)$ is the σ-algebra of subsets of S with the Baire property and $\mathcal{I}(S)$ is the σ-ideal of meager subsets of S.

2. Preliminaries

Definition 1 ([**1**, Definition 1])**.** A measurable space with negligibles is a triple $(S, \mathcal{M}(S), \mathcal{J}(S))$ where S is a set, $\mathcal{M}(S)$ is a σ-algebra of subsets of S and $\mathcal{J}(S) \subset \mathcal{P}(S)$ is a σ-ideal of the Boolean algebra $\mathcal{P}(S)$ generated by $\mathcal{J}(S) \cap \mathcal{M}(S)$.

Such space $(S, \mathcal{M}(S), \mathcal{J}(S))$ is said to be complete if $\mathcal{J}(S) \subset \mathcal{M}(S)$.

If $(S, \mathcal{M}(S), \mathcal{J}(S))$is an arbitrary measurable space with negligibles, we can determine its completion $(S, \widehat{\mathcal{M}}(S), \mathcal{J}(S))$ by putting:

$$\widehat{\mathcal{M}}(S) = \big\{H \subset S : \text{there exist two } \mathcal{M}(\mathcal{S})\text{-measurable sets } A \text{ and } B \text{ such that } A \subset H \subset B \text{ and } A \setminus B \in \mathcal{J}(\mathcal{S})\big\}.$$

Let $(X, \mathcal{M}(X), \mathcal{J}(X))$ and $(Y, \mathcal{M}(Y), \mathcal{J}(Y))$ be two measurable spaces with negligibles. Let $\mathcal{M}(X) \otimes \mathcal{M}(Y)$ be the σ-algebra generated by $\mathcal{M}(X) \times \mathcal{M}(Y)$ and let $\mathcal{J}(X) \otimes \mathcal{J}(Y)$ denotes the σ-ideal generated by all the sets of the form

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$X \times J$ and $I \times Y$ where $I \in \mathcal{J}(X)$ and $J \in \mathcal{J}(Y)$. Let the product $X \times Y$ be endowed with the σ-ideal $\mathcal{J}(X \otimes Y)$, including $\mathcal{J}(X) \otimes \mathcal{J}(Y)$, with the following "Fubini's property":

For any set $A \in \widehat{\mathcal{M}}(X \otimes Y)$ the following implications hold:

$$\{x \in X : A_x \notin \mathcal{J}(Y)\} \in \mathcal{J}(X) \implies A \in \mathcal{J}(X \otimes Y) \quad \text{and} \tag{1}$$

$$\{y \in Y : A^y \notin \mathcal{J}(X)\} \in \mathcal{J}(Y) \implies A \in \mathcal{J}(X \otimes Y), \tag{2}$$

where $A_x = \{y \in Y : (x, y) \in A\}$ and $A^y = \{x \in X : (x, y) \in A\}$ denote the x-section of A and y-section of A respectively and $\hat{M}(X \otimes Y)$ denotes $\mathcal{J}(X \otimes Y)$-completion of $\mathcal{M}(X) \otimes \mathcal{M}(Y)$.

3. The Abstract Baire Category Concepts

Following [**8**] a pair $(S, \mathcal{C}(S))$, where $\mathcal{C}(S) \subset \mathcal{P}(S)$ is a family of subsets of S, is called a category base, if the nonempty sets in $\mathcal{C}(S)$, called regions, satisfy the following axioms:

C.1. Every point of S belongs to some region, i.e. $S = \bigcup\{A : A \in \mathcal{C}(S)\}$.

C.2. Let A be a region and let $\mathcal{D}(S)$ be any nonempty family of disjoint regions which has cardinality less than the cardinality of $\mathcal{C}(S)$.

(a) If $A \cap (\bigcup\{D : D \in \mathcal{D}(S)\})$ contains a region, then there is a region $D_0 \in \mathcal{D}(S)$ such that $A \cap D_0$ contains a region.

(b) If $A \cap (\bigcup\{D : D \in \mathcal{D}(S)\})$ contains no region then there is a region $B \subset A$ which is disjoint from every region in $\mathcal{D}(S)$.

Notice (see [**9**]) that parts (a) and (b) of C.2 can be rewritten in the following form:

(c) If $A \cap B$ contains no region for each $B \in \mathcal{C}(S)$ then there is a subregion of A which is disjoint from $\bigcup \mathcal{D}(S)$.

A set E is singular if every region contains a subregion which is disjoint from E. A countable union of singular sets is called a meager set. A set which is not meager set is called an abundant set. The family of all singular sets forms an ideal and the family $\mathcal{J}_{\mathcal{C}}(S)$ of all meager sets forms a σ-ideal.

A set E is meager (resp. abundant) in a region A if $E \cap A$ is a meager (resp. abundant) set.

A set E has the abstract Baire property if every region $A \in \mathcal{C}(S)$ has a subregion $B \subset A$ in which either E or $X \setminus E$ is a meager set. The sets which have the abstract Baire property form a σ-algebra $\mathcal{B}_{\mathcal{C}}(S)$, which contains all regions and all meager sets. Thus the triple $(S, \mathcal{B}_{\mathcal{C}}(S), \mathcal{J}_{\mathcal{C}}(S))$ creates a complete measurable space with negligibles.

A family $\mathcal{A}$ of $\mathcal{C}(S)$-regions with the property that each abundant set is abundant everywhere in at least one region A in $\mathcal{A}$ (i.e. it is abundant in every subregion of A) is called a quasi-base. A category base is called separable if it has a countable quasi-base.

Let $(X, \mathcal{C}(X))$ and $(Y, \mathcal{C}(Y))$ be category bases. It is known that $(X \times Y, \mathcal{C}(X) \times \mathcal{C}(Y))$ is not necessarily a category base (see Example 2A, p. 110 in [**7**]).

If $(X \times Y, \mathcal{C}(X) \times \mathcal{C}(Y))$ is a category base, then it is called a product base. A general theorem concerning the existence of product bases and many examples are given on pp. 112–114 in [**7**].

Assume that $(X \times Y, \mathcal{C}(X) \times \mathcal{C}(Y))$ is a product base and $(Y, \mathcal{C}(Y))$ is separable. Among the properties involving separability there is the following:

(3) If $A \in \mathcal{B}_{\mathcal{C}}(X \otimes Y) \wedge \{x : A_x \notin \mathcal{J}_{\mathcal{C}}(Y)\} \in \mathcal{J}_{\mathcal{C}}(X)$, then $A \in \mathcal{J}_{\mathcal{C}}(X \otimes Y)$,

where $\mathcal{J}_{\mathcal{C}}(X \otimes Y)$ means the σ-ideal of meager sets with respect to $\mathcal{C}(X) \times \mathcal{C}(Y)$ and $\mathcal{B}_{\mathcal{C}}(X \otimes Y)$, including $\mathcal{B}_{\mathcal{C}}(X) \otimes \mathcal{B}_{\mathcal{C}}(Y)$, the σ-algebra of sets with the abstract Baire property with respect to $\mathcal{C}(X) \times \mathcal{C}(Y)$.

4. Main Results

Let S and Z be some sets and let $F \colon S \to Z$ be a multifunction (i.e. $F(s) \subset Z$ for $s \in S$). Then two counterimages of $G \subset Z$ may be defined:

(4) $$F^+(G) = \{s \in S : F(s) \subset G\} \quad \text{and} \quad F^-(G) = \{s \in S : F(s) \cap G \neq \emptyset\}.$$

It is clear that

(5) $$F^-(G) = S \setminus F^+(Z \setminus G) \quad \text{and} \quad F^-(G) = S \setminus F^+(Z \setminus G).$$

Definition 2. Let $(S, \mathcal{M}(S))$ be e measurable space and let $(Z, \mathcal{T}(Z))$ be a topological space. We say that a multifunction $F \colon X \to Z$ is lower (upper) $\mathcal{M}(S)$-measurable if the counterimage $F^-(G)$ $(F^+(G))$ is a $\mathcal{M}(S)$-measurable set for each $G \in \mathcal{T}(Z)$.

We describe the relationships between lower and upper $\mathcal{M}(S)$-measurability without any metrizability assumptions in contrast to the corresponding results from [**2**].

Proposition 1 (cf. [**2**, Theorem 3.1, p. 55] in the metric case). *Let $(S, \mathcal{M}(S))$ be a measurable space, $(Z, \mathcal{T}(Z))$ a topological space and let $F \colon S \to Z$ be a multifunction. Then*

(i) *If $(Z, \mathcal{T}(Z))$ is a perfect space and F is upper $\mathcal{M}(S)$-measurable, then it is lower $\mathcal{M}(S)$-measurable.*
(ii) *If $(Z, \mathcal{T}(Z))$ is perfectly normal and F is a compact-valued lower $\mathcal{M}(S)$-measurable multifunction, then it is upper $\mathcal{M}(S)$-measurable.*

Proof. Part (i) is obvious because we have

$$F^-(G) = \bigcup_{n\in N} F^-(B_n) \in \mathcal{M}(S), \tag{6}$$

where $Z \setminus B_n \in \mathcal{T}(Z)$, whenever $G \in \mathcal{T}(Z)$.

Let B be a closed subset of Z. By virtue of perfect normality of $(Z, \mathcal{T}(Z))$ there is a sequence $(G_n)_{n\in N}$ of $\mathcal{T}(Z)$-open sets such that

$$B = \bigcap_{n\in N} G_n = \bigcap_{n\in N} Cl(G_n) \qquad \text{and} \tag{7}$$

$$G_{n+1} \subset Cl(G_{n+1}) \subset G_n \quad \text{for } n = 1, 2, \ldots, . \tag{8}$$

By (7) and (5) we have

$$F^-(B) = S \setminus F^+\Big(\bigcup_{n\in N}(Z \setminus G_n)\Big) = S \setminus F^+\Big(\bigcup_{n\in N}(Z \setminus Cl(G_n)\Big). \tag{9}$$

The family $\{Z \setminus Cl(G_n) : n \in N\}$ forms an open covering of compact subset $F(s)$ for each fixed $s \in F^+(\bigcup_{n\in N}(Z \setminus G_n))$. By (8) this covering is increasing. Consequently we have

$$\begin{aligned} &F(s) \subset \bigcup_{n\in N}(Z \setminus G_n) \text{ if and only if there exists } n(s) \in N \text{ such that} \\ &F(s) \subset Z \setminus Cl(G_{n(s)} \subset Z \setminus G_{n(s)+1}. \end{aligned} \tag{10}$$

Applying (10) we infer that

$$F^+\Big(\bigcup_{n\in N}(Z \setminus G_n)\Big) = \bigcup_{n\in N} F^+(Z \setminus G_n). \tag{11}$$

So (9) completes the argument and proof is finished. □

Proposition 2. *Let $(S, \mathcal{M}(S))$ be a measurable space and let $(Z, \mathcal{T}(Z)$ be a second countable Hausdorff space. Let $F_1, F_2 \colon S \to Z$ be two compact-valued lower $\mathcal{M}(S)$-measurable multifunctions. Then*

$$\{s \in S : F_1(s) \neq F_2(s)\} \in \mathcal{M}(S). \tag{12}$$

Proof. Observe that

$$\begin{aligned} &F_1(s) \neq F_2(s) \text{ if and only if there exists } z \in Z \text{ such that} \\ &z \in F_1(s) \div F_2(s), \text{ where } \div \text{ denotes the symmetric difference.} \end{aligned} \tag{13}$$

Let $z_0 \in F_1(s)$ and $z_0 \notin F_2(s)$. For each $z \in F_2(s)$ let $U(z)$ and $V(z)$ denote open sets with the property:

(14) $$z \in U(z) \text{ and } z_0 \in V(z) \text{ and } U(z) \cap V(z) = \emptyset.$$

The family $\{U(z) : z \in F_2(s)\}$ forms an open covering of the compact set $F_2(s)$. Thus

(15) there exists $n \in N$ and $\{z_1, z_2, \dots, z_n\} \subset Z$ such that $F_2(s) \subset \bigcup_{i=1}^{n} U(z_i)$.

Moreover

(16) $$\bigcap_{i=1}^{n} V(z_i) \cap \bigcup_{i=1}^{n} U(z_i) = \emptyset.$$

There is a basic open set V in Y such that:

(17) $$z_0 \in V \subset V(z_1) \cap V(z_2) \cap \dots \cap V(z_n).$$

From the fact that $z_0 \in F_1(s) \cap V$ we infer that:

(18) $$s \in {F_1}^-(V) \quad \text{and} \quad s \in S \setminus {F_2}^-(V) = {F_2}^+(Z \setminus V).$$

Consequently,

(19) $$s \in {F_1}^-(V) \cap {F_2}^+(Z \setminus V) \in \mathcal{M}(S).$$

If on the contrary, $z_0 \in F_2(s) \setminus F_1(s)$, then symmetrically

$$s \in {F_2}^-(V) \cap {F_1}^+(Z \setminus V) \in \mathcal{M}(S)$$

for chosen in a suitable manner basic open set $V \subset Z$.

Thus, if $\{V_1, V_2, \dots\}$ create a countable basis of Z, we have

(20) $$\{s \in S : F_1(s) \neq F_2(s)\} = \bigcup_{i \in N} [({F_1}^-(V_i) \cap {F_2}^+(Z \setminus V_i)) \cup ({F_2}^-(V_i) \cap {F_1}^-(Z \setminus V_i))] \in \mathcal{M}(S)$$

completing the proof. □

Let as remark that:

(21) the equality (20) holds also in the case when F_1 and F_2 are closed-valued and Z is regular and second countable.

Let $(S, \mathcal{M}(S), \mathcal{J}(S))$ be a measurable space with negligibles and let $\widehat{\mathcal{M}}(S)$ be the $\mathcal{J}(S)$-completion of $\mathcal{M}(S)$. Let Z be a topological space and let $\mathcal{B}_0(Z)$ denotes the Borel σ-algebra of the space Z.

We define the following "projection property":

$$\text{If } A \in \mathcal{M}(S) \otimes \mathcal{B}_0(Z), \text{ then} \quad \Pi_S(A) = \{s \in S : \text{there exists } z \in Z \text{ such that } (s,z) \in A\} \in \widehat{\mathcal{M}}(S). \tag{22}$$

Among examples of spaces fulfilling "projection property" (22) are complete measure spaces $(S, \mathcal{M}(S), \mu)$ (see [**1**, 6B(f)]) and the $(S, \mathcal{B}_0(S), \mathcal{J}(S))$, where S is an arbitrary topological space (see [**1**, 7C, p. 67]). Such spaces are proto-decomposable in the meaning of Definition 1B(h), in [**1**], and thus, by [**1**, 1D(b)(iii) and 1H, p. 10], their completions have σ-algebras closed under Souslin's operation, which in turn insures (22). Note that the notion of proto-decomposability of measurable spaces with negligibles were offered by D. H. Fremlin as a generalization of the localization principle of Banach, which also applies to the most important measure spaces. An abstract measurable space with negligibles $(S, \mathcal{M}(S), \mathcal{J}(S))$ which is also ω_1-saturated, that means:

$$\text{If for every } \mathcal{A} \subset \mathcal{M}(S) \text{ card } \mathcal{A} = \omega_1, \text{ then there exist two sets } A \in \mathcal{A} \text{ and } B \in \mathcal{A} \text{ such that } A \neq B \text{ and } A \cap B \notin \mathcal{J}(S), \tag{23}$$

is also known to be proto-decomposable and thus its completion $(S, \widehat{\mathcal{M}}(S), \mathcal{J}(S))$ has the required property (22).

Proposition 3. *Let $(S, \mathcal{M}(S), \mathcal{J}(S))$ be a measurable space with negligibles and Z let be a separable metrizable space. Assume that property* (22) *holds.Let for $n \in N$ $F_n \colon S \to Z$ be a sequence of closed-valued lower $\mathcal{M}(S)$-measurable multifunctions. Then multifunction $F \colon S \to Z$ given by formula:*

$$F(s) = \Big(\bigcap\nolimits_{n \in N} F_n\Big)(s) = \bigcap\nolimits_{n \in N} F_n(s) \tag{24}$$

is upper $\widehat{\mathcal{M}}(S)$-measurable.

Proof. Define functions $f_n \colon S \times Z \to R$ as follows:

$$f_n(s,z) = \operatorname{dist}(z, F_n(s)) \quad \text{for} \quad (s,z) \in S \times Z, \tag{25}$$

where (denoting by d a metric on Z) $\operatorname{dist}(z, B) = \inf\{d(z,b) : b \in B\}$.

Then observe, that the graph of F_n is the kernel of f_n:

$$\operatorname{Gr} F_n = \{(s,z) \in S \times Z : z \in F(s)\} = {f_n}^{-1}(0).$$

All the sections $(f_n)^z$, $z \in Z$, are $\mathcal{M}(S)$-measurable and all the sections $(f_n)_s$, $s \in S$, are continuous on Z. Thus, by virtue of the known theorem (see e.g. Theorem 2, p. 65 in [**5**]) f_n is $\mathcal{M}(S)\otimes\mathcal{B}_0(Z)$-measurable, so that, by (26) we have:

$$\operatorname{Gr} F_n \in \mathcal{M}(S) \otimes \mathcal{B}_0(Z). \tag{27}$$

Hence

$$\operatorname{Gr}\, F = \operatorname{Gr}\left(\bigcap\nolimits_{n\in N} F_n\right) = \bigcap\nolimits_{n\in N}(\operatorname{Gr}\, F_n) \in \mathcal{M}(S) \otimes \mathcal{B}_0(Z). \tag{28}$$

Let $B \in \mathcal{B}_0(Z)$ be a closed subset of Z. By virtue of (22) we obtain

$$F^-(B) = \Pi_S(GrF \cap (S \times B)) \in \widehat{\mathcal{M}}(S) \tag{29}$$

as a projection of the intersection of two $\mathcal{M}(S) \otimes \mathcal{B}_0(Z)$-measurable subsets of $S \times Z$, which finishes the proof. □

In context of Proposition 3 let us remark that there is an example (see Example 2, p. 166 in [**3**]) showing, that the intersection of two lower $\mathcal{M}(S)$-measurable multifunctions F_1 and F_2 with closed values may fail to be lower $\mathcal{M}(S)$-measurable, even if Z is Polish, S is the unit interval endowed with the Borel σ-algebra and $\operatorname{dom}(F_1 \cap F_2) = \{s \in S : F_1 \cap F_2 \neq \emptyset\} = S$.

Definition 3. Let $(S, \mathcal{T}(S))$ and $(Z, \mathcal{T}(Z))$ be two topological spaces and $F\colon S \to Z$ let be a multifunction. F is called lower semicontinuous at a point $s_0 \in S$ when for each $U \in \mathcal{T}(Z)$ we have:

(30) If $F(s_0) \cap U \neq \emptyset$, then there exists a set $G \in \mathcal{T}(S)$ such that $s_0 \in G$ and $F(s) \cap U \neq \emptyset$ for each $s \in G$.

Dualy, F is called upper semicontinuous at a point $s_0 \in S$ when for each $U \in \mathcal{T}(Z)$ we have:

(31) If $F(s_0) \subset U$, then there exists a set $G \in \mathcal{T}(S)$ such that $s_0 \in G$ and $F(s) \subset U$ for each $s \in G$.

F is lower (resp. upper) semicontinuous if it is lower (resp. upper) semicontinuous at each point $s_0 \in S$.

Let $\mathcal{U}(s_0)$ denotes a filterbase of open neighborhoods of the point $s_0 \in S$. The grill of $\mathcal{U}(s_0)$, denoted here by $\mathcal{U}''(s_0)$, is defined as follows:

$$\mathcal{U}''(s_0) = \big\{A(s_0) \subset S : A(s_0) \cap U \neq \emptyset \ \text{ for each } \ U \in \mathcal{U}(s_0)\big\}. \tag{32}$$

Observe that:

$$\text{If } A \in \mathcal{U}''(s_0), \text{ then } s_0 \in Cl(A). \tag{33}$$

Following [**4**] we define the upper and lower limit of a multifunction $F\colon S \to Z$ as follows:

$$\operatorname*{p\text{-}Lim\,sup}_{s\to s_0} F(s) = \bigcap\nolimits_{U\in\mathcal{U}(s_0)} Cl\Big(\bigcup\nolimits_{s\in U} F(s)\Big), \tag{34}$$

$$\operatorname*{p\text{-}Lim\,inf}_{s\to s_0} F(s) = \bigcap\nolimits_{A\in\mathcal{U}"(s_0)} Cl\Big(\bigcup\nolimits_{s\in A} F(s)\Big). \tag{35}$$

Our multifunction F is lower semicontinuous at $s_0 \in S$ if and only if $F(s_0) \subset \text{p-Lim}\inf_{s\to s_0} F(s)$. If the space Z is regular, F has closed values and it is continuous at $s_0 \in S$ (that means it is simultaneously lower and upper semicontinuous), then

$$\operatorname*{p\text{-}Lim\,inf}_{s\to s_0} F(s) = F(s_0) = \operatorname*{p\text{-}Lim\,sup}_{s\to s_0} F(s). \tag{36}$$

Let $\mathcal{B}$ be a basis for S. Let us replace $\mathcal{U}''(s_0)$ in (35) by equality:

$$\mathcal{U}''(s_0) \cap \mathcal{B} = \{U \in \mathcal{B} : s_0 \in Cl(U)\} \tag{37}$$

and denote the resulting operation by q-Lim inf . We have:

$$\text{p-Lim}\inf \subset \text{q-Lim}\inf \subset \text{p-Lim}\sup . \tag{38}$$

At each continuity point s_0 of F we have also

$$\operatorname*{q\text{-}Lim\,inf}_{s\to s_0} F(s) = F(s_0) = \operatorname*{p\text{-}Lim\,sup}_{s\to s_0} F(s). \tag{39}$$

Thus the set $\{s_0 \in S : \text{q-Lim}\inf_{s\to s_0} F(s) \neq \text{p-Lim}\sup_{s\to s_0} F(s)\}$ is contained in the set $D(F)$ of all discontinuity points of F.

Propositon 4. *Let $(X, \mathcal{M}(X))$ be a measurable space, $(Y, \mathcal{T}(Y))$ a second countable topological space and let $(Z, \mathcal{T}(Z))$ be a second countable perfectly normal topological space. Let $F\colon X \times Y \to Z$ be a closed-valued multifunction with lower $\mathcal{M}(X)$)-measurable all sections F^y, $y \in Y$.Denote by $\mathcal{J}(X \otimes Y)$ a σ-ideal in $X \times Y$ including $\mathcal{J}(X) \otimes \mathcal{J}(Y)$ such that* (22) *holds. Let P be a countable dense subset of Y. Then multifunction $G_*\colon X \times Y \to Z$ defined by formula:*

$$G_*(x,y) = \operatorname*{q\text{-}Lim\,inf}_{t\to y\wedge t\in P} (F_x)(t) \tag{40}$$

is upper $\widehat{\mathcal{M}}(X \otimes Y)$-measurable, where $\widehat{\mathcal{M}}(X \otimes Y)$ denotes $\mathcal{J}(X \otimes Y)$-completion of $\mathcal{M}(X) \otimes \mathcal{B}_0(Y)$.

Proof. Let $\mathcal{B}$ denotes a countable base of Y. We have

$$G_*(x,y) = \bigcap_{U \in \mathcal{B} \wedge y \in Cl(U)} Cl\Big(\bigcup_{t \in U \cap P} F(x,t)\Big). \tag{41}$$

Define for each $U \in \mathcal{B}$ a multifunction H_U by formula:

$$H_U(x,y) = \bigcup_{t \in U \cap P} F(x,t) \subset Z, \tag{42}$$

and observe that for $V \in \mathcal{T}(Z)$ we have:

$$\begin{aligned} H_U{}^-(V) &= \big\{(x,y) : \text{there is } t \in U \cap P \text{ such that } F(x,t) \cap V \neq \emptyset\big\} \\ &= \bigcup_{t \in U \cap P} (\{x \in X : F(x,t) \cap V \neq \emptyset\} \times Y) \\ &= \bigcup_{t \in U \cap P} ((F^t)^-(V) \times Y) \in \mathcal{M}(X) \otimes \mathcal{B}_0(Y) \end{aligned} \tag{43}$$

since $U \cap P$ is countable and F^t are lower $\mathcal{M}(X)$-measurable. So by the well known fact (see [**2**, Prop. 2.6, p. 55]) multifunction $\overline{H}_U : X \times Y \to Z$ defined by equality:

$$\overline{H}_U(x,y) = Cl(H_U(x,y)) \tag{44}$$

is also $\mathcal{M}(X) \otimes \mathcal{B}_0(Y)$-measurable.

Observe that

$$G_*(x,y) = \bigcap \{\overline{H}_U(x,y) : U \in \mathcal{B} \wedge y \in Cl(U)\}. \tag{45}$$

Define multifunction $G_U : X \times Y \to Z$ by formula:

$$G_U(x,y) = \begin{cases} \overline{H}_U(x,y) & \text{if } y \in Cl(U), \\ \mathrm{Z} & \text{if } y \notin Cl(U) \end{cases}$$

and observe that

$$G_U{}^-(V) = H_U{}^-(V) \cap (X \times Cl(U)) \cup (X \times (Y \setminus Cl(U)) \in \mathcal{M}(X) \otimes \mathcal{B}_0(Y). \tag{46}$$

We have

$$G_*(x,y) = \bigcap_{U \in \mathcal{B}} G_U(x,y). \tag{47}$$

By virtue of Proposition 3 multifunction G_* is upper $\widehat{\mathcal{M}}(X \otimes Y)$-measurable (according to Proposition 1 it is also lower $\widehat{\mathcal{M}}(X \otimes Y)$-measurable) and the proof is finished. □

Proposition 5. *Let X, Y, Z, P and F be the same as in Proposition 4. Define multifunction $G^*\colon X \times Y \to Z$ as follows:*

$$G^*(x,y) = \operatorname{p-Lim\,sup}_{t\to y \wedge t\in P} (F_x)(t). \tag{48}$$

Then multifunction G^* is upper $\widehat{\mathcal{M}}(X\otimes Y)$-measurable.

Proof. The proof of this proposition is very similar to the preceding one. □

We are now in position to state and prove our main theorem, serving as a unification and generalization in several aspects of Theorems 2 on p. 150 and 3 on p. 151 from famous paper [**6**].

Theorem 1. *Let $(X, \mathcal{M}(X), \mathcal{J}(X))$ be a measurable space with negligibles, $(Y, \mathcal{T}(Y))$ a second countable topological space and $(Z, \mathcal{T}(Z))$ a second countable perfectly normal topological space. Let $\mathcal{J}(Y) \subset \mathcal{B}_0(Y)$ be a Borel σ-ideal in Y such that there exists a σ-ideal $\mathcal{J}(X\otimes Y)$ including $\mathcal{J}(X)\otimes\mathcal{J}(Y)$ fulfilling* (1) *and* (22)*. Assume that $F\colon X\times Y \to Z$ is a closed valued multifunction with the following three properties:*

- (i) *All the sections F^y, $y\in Y$, are lower $\widehat{\mathcal{M}}(X)$-measurable.*
- (ii) *For all $x\in X$ the set $D(F_x)$ of discontinuity points of the section F_x is $\mathcal{J}(Y)$-negligible.*
- (iii) *For all $(x,y)\in X\times Y$ the inclusions*

$$G_*(x,y)\subset F(x,y)\subset G^*(x,y) \tag{49}$$

hold, where G_* and G^* are multifunctions constructed from F according to (40) *and* (48) *by using some fixed countable dense subset $P\subset Y$, the existence of which we assume. Then F is lower measurable with respect to the $\mathcal{J}(X\otimes Y)$-completion of $\mathcal{M}(X)\otimes\mathcal{B}_0(Y)$.*

Proof. Let us consider the set:

$$A = \{(x,y) : G_*(x,y) \neq G^*(x,y)\}. \tag{50}$$

Both multifunction G_* and G^* are upper $\widehat{\mathcal{M}}(X\otimes Y)$-measurable by virtue of Propositions 4 and 5 respectively. Therefore by the remark (21) (Z being perfectly normal is also regular) we infer that $A\in\widehat{\mathcal{M}}(X\otimes Y)$. Observe that by the assumption (ii) all x-sections of the set A are $\mathcal{J}(Y)$-negligibles:

$$A_x = \{y\in Y : G_*(x,y)\neq G^*(x,y)\}\subset D(F_x)\in\mathcal{J}(Y). \tag{51}$$

Consequently we have

$$\{x\in X : A_x\notin\mathcal{J}(Y)\} = \emptyset\in\mathcal{J}(X)\}, \tag{52}$$

which, by using (1), insures the appartenancy of A to the $\mathcal{J}(X \otimes Y)$. The double inclusion (49) entrains, by transitivity, implication:

$$\text{If } G_*(x,y) = G^*(x,y), \text{ then } G_*(x,y) = F(x,y), \tag{53}$$

which, in tour, guarantees the $\mathcal{J}(X \otimes Y)$-negligibility of A_1:

$$A_1 = \{(x,y) : G_*(x,y) \neq F(x,y)\} \subset A \in \mathcal{J}(X \otimes Y). \tag{54}$$

Next, let U be an arbitrary open subset of Z. G_* is upper $\widehat{\mathcal{M}}(X \otimes Y)$-measurable and thus, by Proposition 1 it is also lower $\widehat{\mathcal{M}}(X \otimes Y)$-measurable. So that we have:

$$G_*{}^-(U) = (B \setminus A_2) \cup A_3 \in \widehat{\mathcal{M}}(X \otimes Y) \tag{55}$$

for some $B \in \mathcal{M}(X) \otimes \mathcal{B}_0(Y)$ and A_2, $A_3 \in \mathcal{J}(X \otimes Y)$.

Next let us remark that by (53) and (54):

$$\begin{aligned} F^-(U) &= (F^-(U) \cap (X \times Y \setminus A_1)) \cup (F^-(U) \cap A_1) \\ &= (G_*{}^-(U) \cap (X \times Y \setminus A_1)) \cup A_4 = (B \setminus A_5) \cup A_4, \end{aligned}$$

where $A_4 = F^-(U) \cap A_1$, $A_5 = A_1 \cup [A_2 \cap (X \times Y \setminus A_1)]$.

All the sets A_i, $i = 1, 2, \ldots, 5$, are $\mathcal{J}(X \otimes Y)$-negligibles members of $\widehat{\mathcal{M}}(X \otimes Y)$. Therefore F is lower measurable with respect to the $\mathcal{J}(X \otimes Y)$-completion of $\mathcal{M}(X) \otimes \mathcal{B}_0(Y)$ and if it is moreover compact-valued also upper $\widehat{\mathcal{M}}(X \otimes Y)$-measurable. The proof of theorem is finished. □

The same proof works in the case of multifunctions defined on product category base $(X \times Y, \mathcal{C}(X) \times \mathcal{C}(Y))$ (cf. [**7**]), where $(Y, \mathcal{C}(Y))$ is a separable category base. We use (3) instead of (1), we take as P a subset obtained by selecting a point from each member of countable quasi-base of Y, and we generalize the notion of continuity by taking in all Definitions 3 the set all members of this quasi-base containing $y_0 \in Y$ instead of the filter base $\mathcal{U}(y_0)$. The thesis of Proposition 3 holds in the presence of proto-decomposability of product space $X \times Y$ endowed with the σ-algebra of sets with the abstract Baire property. This version seems to be new even in the single-valued case of real functions defined on the product of category bases.

Question 1. *Let $(Y, \mathcal{T}(Y), \mathcal{S}(Y), \mathcal{M}(Y), \mathcal{J}(Y))$ be a bitopological space which is simultaneously a measurable space with negligibles $\mathcal{J}(Y)$.*

Two topologies $\mathcal{T}(Y)$ and $\mathcal{S}(Y)$ are assumed to be related modulo σ-ideal $\mathcal{J}(Y)$ namely the symmetric difference $Cl_{\mathcal{T}}(A) \div Cl_{\mathcal{S}}(A)$ is $\mathcal{J}(Y)$-negligible for each subset $A \subset Y$. From Th. 2 in [**10**] *it follows that for each multifunction $H \colon Y \to Z$ (where Z is a second countable Hausdorff space) which is at every point $y \in Y$*

either $\mathcal{T}(Y)$-continuous or $\mathcal{S}(Y)$-continuous the set $\mathcal{T}\cap\mathcal{S}$-$D(H)\in\mathcal{J}(Y)$. Under what conditions imposed on $\mathcal{T}(Y)$ and $\mathcal{S}(Y)$ we have

$$\mathcal{T}\cap\mathcal{S}\text{-}\operatorname*{q\text{-}Lim\,inf}_{t\to y\wedge t\in P} H(t)\subset H(y)\subset \mathcal{T}\cap\mathcal{S}\text{-}\operatorname*{p\text{-}Lim\,sup}_{t\to y\wedge t\in P} H(t)? \tag{56}$$

Let $(X,\mathcal{M}(X),\mathcal{J}(X))$ be a (complete) measurable space with negligibles. Under what conditions a multifunction $F\colon X\times Y\to Z$, whose all x-sections are at each $y\in Y$ either $\mathcal{T}(Y)$-continuous or $\mathcal{S}(Y)$-continuous and all y-section are $\widehat{\mathcal{M}}(X)$-measurable, is $\widehat{\mathcal{M}}(X\otimes Y)$-measurable. Evidently $(Y,\mathcal{T}(Y)\cap\mathcal{S}(Y))$ is assumed to be second countable Baire space and $\mathcal{M}(Y)$ is related with the $\mathcal{T}(Y)\cap\mathcal{S}(Y)$-Borel σ-algebra $\mathcal{B}_0(Y,\mathcal{T}(Y)\cap\mathcal{S}(Y))$.Is the condition $\mathcal{T}-\operatorname{Lim}H(y)\subset H(y)\subset\mathcal{S}-\operatorname{Lim}H(y)$ sufficient for the double inclusion (56)*?*

In that manner we have the possibility to obtain many applications of Theorem 1, e.g. for multifunctions whose x-sections are monotone in certain generalized meaning.

Question 2. *Let Y be a (finite dimensional) Euclidean space with the scalar product $\langle\cdot\mid\cdot\rangle$. Let us consider the unit sphere*

$$S^1=\left\{y\in Y:\|y\|=\sqrt{<y\mid y>}=1\right\} \tag{57}$$

endowed with the metric $\varrho(y_1,y_2)=\arccos\langle y_1\mid y_2\rangle$. By an angular region in Y is called the subset of the form

$$\Omega(y_0,V)=y_0+\left\{y\in Y:\frac{y}{\|y\|}\in V\right\}, \tag{58}$$

where $V\subset S^1$ is a ϱ-open subset of the unit sphere S^1 (cf. [**11**, p. 318]*).*

A multifunction $H\colon Y\to Z$, where Z is an arbitrary topological space, is called lower semicontinuous at $y_0\in Y$ from the angular region $\Omega(y_0,V)$ if for each open subset $G\subset Z$ such that $G\cap H(y_0)\neq\emptyset$ the big inverse image $H^-(G)$ contains $\Omega(y_0,V)\cap\{y\in Y:\|y-y_0\|<r\}$ for some $r>0$.

Analogously, H is called upper semicontinuous at $y_0\in Y$ from $\Omega(y_0,V)$ if for each subset $G\subset Z$ such that $H(y_0)\subset G$ the small inverse image $H^+(G)$ contains $\Omega(y_0,V)\cap\{y\in Y:\|y-y_0\|<r\}$ for some $r>0$.

A multifunction H is said to be Ω-lower (resp. upper) semicontinuous on Y, if for each $y\in Y$ there is an angular region $\Omega(y,V(y))$ such that H is lower (resp. upper) semicontinuous at y from $\Omega(y,V(y))$.

A multifunction simultaneously Ω-lower and Ω-upper semicontinuous is called Ω-continuous.

Under what conditions a multifunction $F\colon X\times Y\to Z$, where Y is as above, X is a complete measurable space with negligibles and Z is a second countable perfectly normal topological space, with Ω-continuous x-sections and $\mathcal{M}(X)$-measurable y-sections is $\widehat{\mathcal{M}}(X\otimes Y)$-measurable with respect to the $\mathcal{J}(X\otimes Y)$-completion *of* $\mathcal{M}(X) \otimes \mathcal{B}_0(Y)$*. The* σ*-ideal* $\mathcal{J}(X \otimes Y)$ *including* $\mathcal{J}(X) \otimes \mathcal{J}(Y)$ *means here an* σ*-ideal fulfilling the conditions of Theorem* 1*, where* $\mathcal{J}(Y)$ *is a Borel* σ*-ideal in* Y*, e.g. of subsets of Lebesgue measure zero or of the first category. Is the finite-dimensionality of* Y *essential?*

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