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A NOTE ON MIXING PROPERTIES OF INVERTIBLE EXTENSIONS

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ABSTRACT. The natural invertible extension $\tilde{T}$ of an $\mathbb{N}^d$-action T has been studied by Lacroix. He showed that $\tilde{T}$ may fail to be mixing even if T is mixing for $d \geq 2$. We extend this observation by showing that if T is mixing on $(k+1)$ sets then $\tilde{T}$ is in general mixing on no more than k sets, simply because $\mathbb{N}^d$ has a corner. Several examples are constructed when $d = 2$: (i) a mixing T for which $\tilde{T}^{(n,m)}$ has an identity factor whenever $n \cdot m < 0$; (ii) a mixing T for which $\tilde{T}$ is rigid but $\tilde{T}^{(n,m)}$ is mixing for all $(n,m) \neq (0,0)$; (iii) a T mixing on 3 sets for which $\tilde{T}$ is not mixing on 3 sets.

1. Invertible Extensions

Let T be a measure-preserving $\mathbb{N}^d$-action on the probability space $(X, \mathcal{B}, \mu)$. Such an action may be thought of as the natural shift-action on the space

$$\left\{(x_{\mathbf{n}}) \in X^{\mathbb{N}^d} \mid x_{\mathbf{n}} = T^{\mathbf{n}} x_0 \ \forall\ \mathbf{n} \in \mathbb{N}^d\right\};$$

the projection π_0 onto the zero coordinate shows that T is isomorphic to the shift action, so we identify them. The natural invertible extension of T is constructed in [**3**], and may be thought of as the natural shift action $\tilde{T}$ on

$$\tilde{X} = \left\{(x_{\mathbf{n}}) \in X^{\mathbb{Z}^d} \mid x_{\mathbf{n}+\mathbf{m}} = T^{\mathbf{n}} x_{\mathbf{m}} \ \forall\ \mathbf{m} \in \mathbb{Z}^d, \mathbf{n} \in \mathbb{N}^d\right\}.$$

For any sets $F \subset \mathbb{Z}^d$, $G \subset \mathbb{N}^d$ let $\tilde{\pi}_F : \tilde{X} \to X^F$, $\pi_G : X \to X^G$ denote the projections. The set $\tilde{X}$ is a probability space with σ-algebra $\tilde{\mathcal{B}}$ and measure $\tilde{\mu}$ defined as follows. The σ-algebra $\tilde{\mathcal{B}}$ is the smallest one containing all sets of the form

$$A_{\mathbf{m},C} = \left\{(x_{\mathbf{n}}) \in \tilde{X} \mid x_{\mathbf{m}} \in C\right\}$$

for $\mathbf{m} \in \mathbb{Z}^d$ and $C \in \mathcal{B}$, and $\tilde{\mu}$ is defined via the Daniell-Kolmogorov consistency theorem (see [**1**, Theorem 1, Chapter IV.6]) from the requirement that $\tilde{\mu}(A_{\mathbf{m},C}) =$

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$\mu(C)$. Notice that for $\{\mathbf{m}_1, \dots, \mathbf{m}_s\} \subset \mathbb{Z}^d$ and sets $C_1, \dots, C_s \in \mathcal{B}$, if $\boldsymbol{\ell} \in \mathbb{N}^d$ has $\boldsymbol{\ell} + \mathbf{m}_j \in \mathbb{N}^d$ for all j, then

$$\tilde{\mu}\left(\{(x_{\mathbf{n}}) \in \tilde{X} \mid x_{\mathbf{m}_j} \in C_j \text{ for } j = 1, \dots, s\}\right)$$

and

$$\mu\left(T^{-(\boldsymbol{\ell}+\mathbf{m}_1)}(C_1) \cap \dots \cap T^{-(\boldsymbol{\ell}+\mathbf{m}_s)}(C_s)\right)$$

coincide. We shall use the following notation: if $\tilde{B} \subset \tilde{X}$ is measurable with respect to $\tilde{\pi}_{\mathbb{N}^d}^{-1}(\mathcal{B})$ then let $B = \pi_{\mathbb{N}^d}(\tilde{B}) \subset X$. Let $\tilde{T}_+ = \tilde{T}|_{\mathbb{N}^d}$ be the $\mathbb{N}^d$-action obtained by restricting the invertible extension to $\mathbb{N}^d \subset \mathbb{Z}^d$. The projection $\tilde{\pi}_{\mathbb{N}^d} : \tilde{X} \to X^{\mathbb{N}^d}$ realizes T as a factor of $\tilde{T}_+$. If the generators of the original $\mathbb{N}^d$-action are invertible, then $\tilde{\pi}_{\mathbb{N}^d}$ is an isomorphism.

Definition. The $\mathbb{N}^d$-action T is mixing on $(k+1)$ sets if for any $A_0, A_1, \dots, A_k \in \mathcal{B}$,

$$\mu\left(A_0 \cap T^{-\mathbf{n}_1} A_1 \cap \dots \cap T^{-\mathbf{n}_k} A_k\right) \longrightarrow \mu(A_0) \dots \mu(A_k) \tag{1}$$

as $\mathbf{n}_i \to \infty$, $\mathbf{n}_i - \mathbf{n}_j \to \infty$ for $i \neq j$. Here $\to \infty$ means leaving finite subsets of $\mathbb{N}^d$, and $\mathbf{n}_i - \mathbf{n}_j \to \infty$ means that if $\mathbf{n}_i + \boldsymbol{\ell} = \mathbf{n}_j + \mathbf{m}$ for $\boldsymbol{\ell}, \mathbf{m} \in \mathbb{N}^d$ then $\boldsymbol{\ell}$ or $\mathbf{m} \to \infty$.

If $k = 1$ then mixing on $(k+1)$ sets is called mixing. A $\mathbb{Z}^d$-action T is said to be mixing on $(k+1)$ sets if (1) holds with the vectors $\mathbf{n}_j$ now allowed to lie in $\mathbb{Z}^d$.

Lacroix [**3**] has shown, **inter alia**, that T mixing does not imply that $\tilde{T}$ will be mixing, with an example in which $\tilde{T}^{\mathbf{n}}$ has an identity factor for some $\mathbf{n} \in \mathbb{Z}^d \backslash \mathbb{N}^d$. We extend this by proving the following theorem and illustrating it with several examples in $d = 2$, including one in which T is mixing but $\tilde{T}^{\mathbf{n}}$ has an identity factor for every $\mathbf{n} \in \mathbb{Z}^2 \backslash \left(\mathbb{N}^2 \cup -\mathbb{N}^2\right)$.

The "corner" $0 \in \mathbb{N}^d$ is distinguished because it must (unlike the $\mathbb{Z}^d$ case) appear in the expression (1) above. This forces the order of mixing to drop.

Theorem. *If the $\mathbb{N}^d$-action T is mixing on $(k+1)$ sets, then the invertible extension $\tilde{T}$ is mixing on k sets.*

Proof. Assume T is mixing on $(k+1)$ sets for some $k \geq 1$. Let $\tilde{B}_1, \dots, \tilde{B}_k$ be sets measurable with respect to $\tilde{\pi}_{S(N)}^{-1}(\mathcal{B})$ where $S(N) = [-N, N]^d \cap \mathbb{Z}^d$. Write $\mathbf{N} = (N, N, \dots, N)$. Let $\mathbf{m}_2(n), \dots, \mathbf{m}_k(n)$ be integer vectors with $\mathbf{m}_i(n) \to \infty$ and $\mathbf{m}_i(n) - \mathbf{m}_j(n) \to \infty$ as $n \to \infty$ for each $i \neq j$. For each $n = 1, 2, \dots$ let $\boldsymbol{\ell}(n) \in \mathbb{N}^d$ be chosen so that $\boldsymbol{\ell}(n) \to \infty$, $\mathbf{n}_j(n) = \mathbf{m}_j(n) + \boldsymbol{\ell}(n) \to \infty$ as $n \to \infty$, and $\mathbf{n}_j(n) \in \mathbb{N}^d$ for all n.

Notice by construction we have $\boldsymbol{\ell}(n) \to \infty$, $\mathbf{n}_j(n) \to \infty$, $\boldsymbol{\ell}(n) - \mathbf{n}_j(n) \to \infty$, and for each $i \neq j$, $\mathbf{n}_j(n) - \mathbf{n}_i(n) \to \infty$. It follows that if n is large enough to ensure that $\boldsymbol{\ell}(n) - \mathbf{N} \in \mathbb{N}^d$, then we have

$$\begin{aligned}
&\tilde{\mu}\left(\tilde{B}_1 \cap \tilde{T}^{-\mathbf{m}_2(n)}\tilde{B}_2 \cap \dots \cap \tilde{T}^{-\mathbf{m}_k(n)}\tilde{B}_k\right) \\
&\quad = \tilde{\mu}\left(\tilde{T}^{-\boldsymbol{\ell}(n)}\tilde{B}_1 \cap \tilde{T}^{-\mathbf{n}_2(n)}\tilde{B}_2 \cap \dots \cap \tilde{T}^{-\mathbf{n}_k(n)}\tilde{B}_k\right) \\
&\quad = \tilde{\mu}\left(\tilde{X} \cap \tilde{T}^{-\boldsymbol{\ell}(n)}\tilde{B}_1 \cap \tilde{T}^{-\mathbf{n}_2(n)}\tilde{B}_2 \cap \dots \cap \tilde{T}^{-\mathbf{n}_k(n)}\tilde{B}_k\right) \\
&\quad = \tilde{\mu}\Big(\tilde{X} \cap \tilde{T}^{-(\boldsymbol{\ell}(n)-\mathbf{N})}\left(\tilde{T}^{-\mathbf{N}}\tilde{B}_1\right) \cap \tilde{T}^{-(\mathbf{n}_2(n)-\mathbf{N})}\left(\tilde{T}^{-\mathbf{N}}\tilde{B}_2\right) \cap \dots \\
&\qquad\qquad \cap \tilde{T}^{-(\mathbf{n}_k(n)-\mathbf{N})}\left(\tilde{T}^{-\mathbf{N}}\tilde{B}_k\right)\Big) \\
&\quad = \mu\left(X \cap T^{-(\boldsymbol{\ell}(n)-\mathbf{N})}C_1 \cap T^{-(\mathbf{n}_2(n)-\mathbf{N})}C_2 \cap \dots \cap T^{-(\mathbf{n}_k(n)-\mathbf{N})}C_k\right) \\
&\quad \to \mu(C_1)\dots\mu(C_k) \\
&\quad = \tilde{\mu}(\tilde{T}^{-\mathbf{N}}\tilde{B}_1)\dots\tilde{\mu}(\tilde{T}^{-\mathbf{N}}\tilde{B}_k) = \tilde{\mu}(\tilde{B}_1)\dots\mu(\tilde{B}_k),
\end{aligned}$$

where $C_j = \tilde{\pi}_{\mathbb{N}^d}(\tilde{T}^{-\mathbf{N}}\tilde{B}_j)$ for each j. It follows that $\tilde{T}$ is mixing on k sets. □

2. Examples

Example 1. If $X = \mathbb{T}$, the additive group, and the $\mathbb{N}^2$-action T is generated by $T^{(1,0)}x = T^{(0,1)}x = 2x \mod 1$, then it is clear that T is mixing while $\tilde{T}$ cannot be mixing since $\tilde{T}^{(1,-1)}$ is the identity map on $\tilde{X} = \widehat{\mathbb{Z}[\frac{1}{2}]}$.

This example is of course not a faithful action — in [**3**] a faithful example is given, generated by the toral endomorphisms dual to the matrices $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$.

Example 2. The previous example may be refined to produce a mixing $\mathbb{N}^2$-action T with the property that $\tilde{T}^{(n,m)}$ has an identity factor for every pair n, m with opposite signs. Let X be the infinite torus $\mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}} \times \dots$. Let $S_a : \mathbb{T} \to \mathbb{T}$ denote the map $S_a(x) = ax \mod 1$, and let

$$S^* = S_2 \times S_4 \times S_8 \times S_{16} \times \dots,$$

and

$$S_a^\infty = S_a \times S_a \times S_a \times S_a \times \dots.$$

Throughout the indicated correspondence between positions in infinite products holds. Define the $\mathbb{N}^2$-action T by the two generators

$$T^{(1,0)} = S^* \times S^* \times S^* \times S^* \times \dots$$

and

$$T^{(0,1)} = S_2^\infty \times S_4^\infty \times S_8^\infty \times S_{16}^\infty \times \dots .$$

Then T is a mixing $\mathbb{N}^2$-action on X: it is enough to check that for any pair of non-trivial characters $\chi_0, \chi_1 \in \widehat{X}$ the character $\chi_0 + \widehat{T}^{(n,m)}\chi_1$ is non-trivial for large $(n,m) \in \mathbb{N}^2$ and this is clear since each character is finitely supported.

The invertible extension $\tilde{T}$ is obtained as follows. Let $\Sigma = \widehat{\mathbb{Z}[\frac{1}{2}]}$ be the solenoid, and $\tilde{S}_a : \Sigma \to \Sigma$ the endomorphism dual to multiplication by a, invertible if a is a power of 2. Then the generators of $\tilde{T}$ are simply given by placing tildes on the definition of the generators of T, and they act on $\tilde{X} = \Sigma^\infty$. For any pair $(n,m) \in \mathbb{Z}^2 \setminus \left(\mathbb{N}^2 \cup -\mathbb{N}^2\right)$, the map $\tilde{T}^{(n,m)}$ has a non-trivial identity factor and therefore cannot be mixing: to see this, notice that $\tilde{T}^{(|n|,0)}$ acts in the $|m|$th position in each of the indicated factors as $\times 2^{|nm|}$, while $\tilde{T}^{(0,|m|)}$ acts in the $|n|$th position in the $S_{2^{|m|}}^\infty$ factor as $\times 2^{|nm|}$ in each copy of Σ.

Example 3. The opposite extreme to the previous example is given by the Gaussian construction of Ferenci and Kaminski [**2**]: for numbers $\alpha > 0$, $\beta > 0$ with $1, \alpha, \beta$ rationally independent they construct a two-dimensional Gaussian action T with covariance function

$$R(n,m) = \frac{\sin(2\pi(n\alpha + m\beta))}{2\pi(n\alpha + m\beta)}.$$

If (n_j, m_j) is a sequence with $n_j\alpha + m_j\beta \to 0$ as $j \to \infty$ then for large j we must have $n_j \cdot m_j < 0$. Along such a sequence $R(n_j, m_j) \to 1$ so the action is rigid, showing that the $\mathbb{Z}^2$-action is not mixing. On the other hand, if $(n_j, m_j) \to \infty$ in $\mathbb{N}^2$ or $-\mathbb{N}^2$ then it is clear that $R(n_j, m_j) \to 0$ showing that the $\mathbb{N}^2$-action T_+ is mixing.

For the next example, recall that a finite set F with $(0,0) \in F \subset \mathbb{Z}^2$ (or $\mathbb{N}^2$) is a **mixing shape** for a λ-preserving $\mathbb{Z}^2$-action $\tilde{T}$ (resp. $\mathbb{N}^2$-action T) if

$$\lim_{k\to\infty} \lambda\left(\bigcap_{\mathbf{n}\in F} T^{-k\mathbf{n}} B_{\mathbf{n}}\right) = \prod_{\mathbf{n}\in F} \lambda(B_{\mathbf{n}})$$

for all measurable sets $B_{\mathbf{n}}$.

Example 4. Using ideas from algebraic dynamical systems, as described for example in [**5**], we exhibit an $\mathbb{N}^2$-action T which is mixing on three sets for which the extension $\tilde{T}$ is not mixing on three sets. The example is a modification of Ledrappier's original example, [**4**]. Let $\mathbb{F}_2$ denote the field with two elements, let

$$X = \left\{ \mathbf{x} \in \mathbb{F}_2^{\mathbb{N}^2} \mid x_{(n-1,m+1)} + x_{(n,m)} + x_{(n+1,m)} \ = 0 \ \forall \ (n,m) \in \mathbb{N}^2 \right\},$$

and define the $\mathbb{N}^2$-action T to be the shift action on X. We claim that T is mixing on three sets. To see this, work in the dual group $\widehat{X} = \mathbb{F}_2/\langle y + x + x^2\rangle$, with the $\mathbb{N}^2$ action being generated by the endomorphisms dual to multiplication by x and y. The map $f(x,y) \mapsto f(x, x+x^2)$ identifies $\widehat{X}$ with $\mathbb{F}_2[x]$, with the generators now being multiplication by x and by $x+x^2$. Using Fourier analysis on the group X (see for example [**5**, Section 27]) it is enough to show that for any $a,b,c \in \mathbb{N}$ and $\epsilon_a, \epsilon_b, \epsilon_c \in \mathbb{F}_2$ the equation

$$\epsilon_a x^a + \epsilon_b x^{n_1} y^{m_1} x^b + \epsilon_c x^{n_2} y^{m_2} = 0$$

for $(n_1,m_1),(n_2,m_2) \in \mathbb{N}^2$ requires that the points $(n_1,m_1),(n_2,m_2),(0,0)$ cannot be far apart or the coefficients $\epsilon_a, \epsilon_b, \epsilon_c$ are zero. Using the identity $y = x + x^2$, the equation becomes

$$\epsilon_a x^a + \epsilon_b (x^{b+n_1+m_1} + \cdots + x^{b+n_1+2m_1}) + \epsilon_c (x^{b+n_2+m_2} + \cdots + x^{b+n_2+2m_2}) = 0.$$

If (n_1,m_1) and (n_2,m_2) are far from the origin then we see that $\epsilon_a = 0$, and if (n_1,m_1) and (n_2,m_2) are far from each other then we see that $\epsilon_b = \epsilon_c = 0$.

The natural extension $\tilde{T}$ has $\{(-1,1),(0,0),(1,0)\}$ as a non–mixing shape since in the group

$$\tilde{X} = \left\{ \mathbf{x} \in \mathbb{F}_2^{\mathbb{N}^2} \mid x_{(n-1,m+1)} + x_{(n,m)} + x_{(n+1,m)} = 0 \ \forall \ (n,m) \in \mathbb{Z}^2 \right\}$$

the relation $x_{(-2^n,2^n)} = x_{(0,0)} + x_{(2^n,0)}$ holds for all n.

It is not clear how to construct examples along the lines of Example 4 with the property that T is mixing on k sets while $\tilde{T}$ is not mixing on k sets for each $k \geq 1$: see Remark 28.12 in [**5**] for what is known.

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