Acta Math. Univ. Comenianae
Vol. LXVII, 1(1998), pp. 17–42

SIMULATION OF ANISOTROPIC MOTION BY MEAN CURVATURE —— COMPARISON OF PHASE FIELD AND SHARP INTERFACE APPROACHES

M. BENEŠ AND K. MIKULA

Abstract. Motion by mean curvature is a problem arising in multi-phase thermomechanics and pattern formation. The article presents a numerical comparison of two approaches to the dynamics of closed curves, namely sharp-interface description leading to a degenerate-diffusion equation of slow and fast diffusion types related to anisotropic curve shortening flow, and a diffusive-interface description in the form of a recently improved version of the phase-field model. Numerical scheme used for simulation of (isotropic) phase field equation is analyzed with regards on the convergence, and simultaneously existence, uniqueness and asymptotical behaviour if the diffusive interface becomes sharp. The presented computational results indicate a consistent relation of both approaches and demonstrate the behaviour of both models in different situations of curve dynamics.

1. Introduction

The paper summarizes results of an intercomparing study in numerical simulation of closed-curve motion in the plane. The equation governing the anisotropic curve dynamics is assumed in the following form

$$\alpha(\theta)v_\Gamma = g(\theta)\kappa - F, \tag{1}$$

where $v_\Gamma \equiv v_\Gamma(\mathbf{x}, t)$ is the normal velocity and $\kappa(\mathbf{x}, t)$ the mean curvature at a point $\mathbf{x}$ of a curve $\Gamma(t)$ at a time t. As the equation (1) arises in the description of phase transitions, the following convention is natural. Let solid occupies a subset $\Omega_s(t) \subset \mathbb{R}^2$, and liquid occupies a subset $\Omega_l(t) \subset \mathbb{R}^2$ at the time t. The phase interface is then a set $\Gamma(t) = \partial\Omega_s(t) \cap \partial\Omega_l(t)$ assumed to be a curve. If $\mathbf{n}_\Gamma$ denotes the outer normal to Ω_s, then $\kappa = \operatorname{div} \mathbf{n}_\Gamma$, and $v_\Gamma = -\vec{v}_\Gamma \cdot \mathbf{n}_\Gamma$. The variable θ denotes the angle between $\mathbf{n}_\Gamma$ and x_1-axis. The range of θ is $J \in \mathbb{R}$. The strictly positive functions α, g, F are given by the constitutive description of interface. The term $F \equiv$ const represents a driving force proportional to the difference in one of thermodynamical potentials for both phases. In the case when $g = 1$, $\alpha =$ const.,

---

Received November 17, 1997.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 80A22, 82C26, 35A40, 35K65, 65N40, 53C80.

the motion is called **isotropic**. The curve evolution governed by equation (1) with $F \equiv 0$ is also called the curve shortening flow.

The curve evolution equation (1) arises in the study of first-order phase transitions on a microscopic level. Here, the pattern formation is governed by the heat exchange or particle diffusion processes and is a consequence of the interplay between undercooling and capillary effects. The sharp-interface description is covered by the Stefan problem with surface tension ([**8**], [**2**])

$$\rho c \frac{\partial u}{\partial t} = \lambda \nabla^2 u \qquad \text{in } \Omega_s \text{ and } \Omega_l, \tag{2}$$

$$u \mid_{\partial\Omega} = u_\Omega, \tag{3}$$

$$u \mid_{t=0} = u_0, \tag{4}$$

$$\lambda \frac{\partial u}{\partial n}\bigg|_s - \lambda \frac{\partial u}{\partial n}\bigg|_l = -L v_\Gamma, \tag{5}$$

$$\frac{\Delta s}{\sigma}(u - u^*) = -g(\theta)\kappa + \alpha(\theta) v_\Gamma, \tag{6}$$

$$\Omega_s(t) \mid_{t=0} = \Omega_{so}. \tag{7}$$

where ρ, c, λ are material characteristics, (assumed to be strictly positive constants), L is latent heat per unit volume, u^* is melting point, u temperature field. Discontinuity of heat flux on $\Gamma(t)$ is described by the Stefan condition (5) and the formula (6) is the Gibbs-Thompson relation on $\Gamma(t)$, where $\sigma =$ const. is the surface tension between the two phases (surface free energy). It follows from (6), that α is the coefficient of attachment kinetics, and the dimensionless function g describes anisotropy of the interface (e.g., it has the form

$$g(\theta) = 1 - \zeta \cos(m(\theta - \theta_0)),$$

where $\zeta \in \langle 0, 1)$ is strength of the m-fold anisotropy, and θ_0 is the principal direction — crystallographic orientation). The Dirichlet boundary conditions (3) are considered as an example. The conditions (4) and (7) are the initial conditions for temperature, and spatial distribution of the solid and liquid phase.

The formulation of this problem implies that the phase interface is not expressed explicitly and cannot be traced during the transition process. In the modeling of microstructure effects, the phase interface is the object of principal interest, though. Complexity of the phase interface often occurring in the microstructure evolution lead to the investigation of level-set approaches to the problem. One of them is the phase-field model presented in [**2**]

$$\begin{aligned} \rho c \frac{\partial u}{\partial t} &= \lambda \nabla^2 u + L \frac{\partial p}{\partial t}, \\ \alpha(\theta)\xi^2 \frac{\partial p}{\partial t} &= g(\theta)[\xi^2 \nabla^2 p + ap(1-p)(p-0.5)] + b\frac{\Delta s}{\sigma}(u^* - u)\xi^2 |\nabla p|_E, \end{aligned} \tag{8}$$

with initial conditions

$$u\,|_{t=0} = u_0\ ,\quad p\,|_{t=0} = p_0,$$

and with boundary conditions of Dirichlet type

$$u\,|_{\partial\Omega} = u_\Omega\ ,\quad p\,|_{\partial\Omega} = p_\Omega,$$

where a, b are positive model constants. The model is destinated to the simulation of microstructure growth in solidification. Figures 1 and 2 present examples of such simulation.

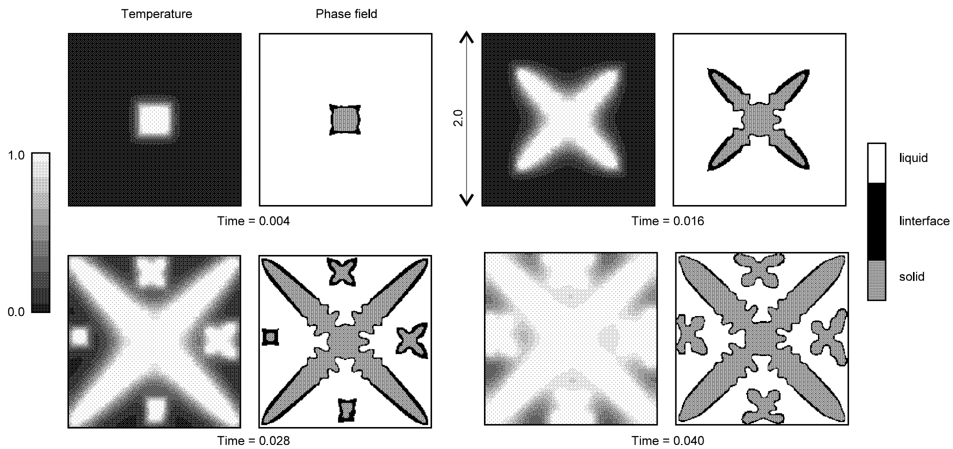

Figure 1. Anisotropic 4-fold dendritic growth with additional homogeneous nucleation, $L = 2.0$, $\beta = 900$, $\zeta = 0.2$, $\xi = 0.01$, $\alpha = 3$, $a = 4.0$, $b = 1$, $L_1 = L_2 = 2.0$, mesh $N_1 = N_2 = 200$.

The phase equation in (8) contains a modified coupling term $b\beta(u^* - u)\xi^2|\nabla p|_E$, $\beta = \frac{\Delta s}{\sigma}$, proposed as a consequence of the level-set reformulation of the condition (6) using the definition of the boundary Γ as a manifold given by a mapping $\Phi\colon \mathbb{R}^n \to \mathbb{R}$:

$$\Gamma(t) = \{\mathbf{x} \in \Omega \mid \Phi(\mathbf{x}, t) = 0\}.$$

According to the convention,

$$\Omega_s = \{\Phi(\mathbf{x}) > 0\}.$$

The mapping Φ depends on time. Then Γ also depends on time. If Φ is smooth enough and $\nabla\Phi$ is non-zero along Γ, it can be used to express the outer normal and normal interface velocity

$$\mathbf{n}_\Gamma = -\frac{\nabla\Phi}{|\nabla\Phi|_E}\quad ,\quad v_\Gamma = -\frac{\frac{\partial\Phi}{\partial t}}{|\nabla\Phi|_E}.$$

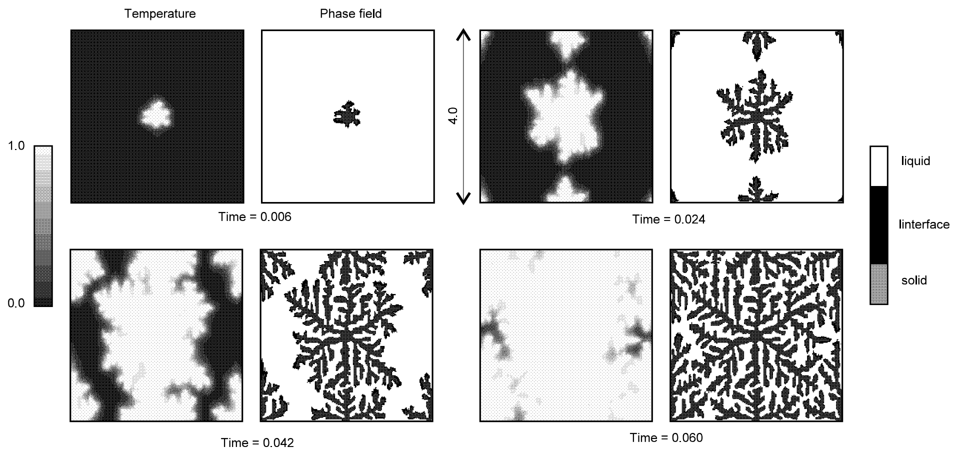

Figure 2. Large-scale growth of the 6-fold anisotropic structure. Parameters are $\xi = 0.01$, $u(0) = 0.0$, $L = 2.0$, $\beta = 900$, $a = 4.0$, $b = 1$, $D_0 = \rho c/\lambda = 1$, $\alpha = 3$, $\zeta = 0.8$ $L_1 = L_2 = 4.0$, mesh $N_1 = N_2 = 200$.

Similarly, mean curvature is expressed as

$$\kappa = \nabla \cdot \mathbf{n}_\Gamma = -\nabla \cdot \left(\frac{\nabla \Phi}{|\nabla \Phi|_E} \right) \Big|_\Gamma .$$

Substituting the previous expressions into (6), and assuming validity of the previous equation on whole Ω (increasing dimension of the problem), the level-set formulation of the Gibbs-Thompson equation is obtained:

$$\alpha \frac{\partial \Phi}{\partial t} = g |\nabla \Phi|_E \nabla \cdot \left(\frac{\nabla \Phi}{|\nabla \Phi|_E} \right) + \frac{\Delta s}{\sigma} |\nabla \Phi|_E (u^* - u). \tag{9}$$

The origin of the new coupling term in (8) (including the absolute value of the gradient of the phase function p), can be seen from (9). In spite of the phase field models presented earlier, in the theoretical part of this paper we present the analysis of the phase equation including this forcing "gradient" term in the case when $\beta(u^* - u)$, α, g are constant (isotropic case).

The results of simulation by the phase equation (in spite of theoretical part of the paper, for simulations we use anisotropic form of equation (8)) are compared to the results obtained by another method solving the sharp-interface problem (1). Angenent and Gurtin in [**1**], using the ideas of Gage and Hamilton ([**7**]), show that the anisotropic motion of the closed and strictly convex interface governed by (1) is equivalent to the following initial-boundary value problem for special PDE.

Let the initial convex curve Γ_0 be parametrized by θ, $\kappa_0(\theta)$ be its curvature and $J = [0, 2\pi]$. Assume that the function $\kappa(\theta, t)$ for all $t \in [0, T), \theta \in \mathbb{R}$ satisfies the

anisotropic curve shortening equation, i.e.

$$\partial_t \kappa = \kappa^2 \left(\frac{g\kappa - F}{\alpha} \right)_{\theta\theta} + \kappa^2 \left(\frac{g\kappa - F}{\alpha} \right), \tag{10}$$

the periodicity and initial conditions

$$\kappa(\theta, t) = \kappa(\theta + 2\pi, t), \tag{11}$$

$$\kappa(\theta, 0) = \kappa_0(\theta). \tag{12}$$

Then, the flow $\Gamma(\theta, t)$ of the curves that solves the problem (1) with the initial curve Γ_0 is given uniquely up to translation by the formula

$$\Gamma(\theta_0, t) = \Gamma(0, t) - \int_0^{\theta_0} \frac{e^{i\theta}}{\kappa(\theta, t)}\, d\theta. \tag{13}$$

In [**15**], the computational methods for solving the problem (10)–(12) were suggested as well as the convergence and error estimates of approximations are proved. Solving (10)–(12), the curvature function κ of evolving curve is obtained numericaly. Then, the real flow of the curve is reconstructed using the formula (13).

The paper is organized as follows. Section 2 presents a numerical approximation of phase field equation based on finite difference space discretization. The convergence of such scheme and relation to the sharp-interface problem are studiedin that Section, too. In Section 3, ideas of numerical solution of the anisotropic curve shortening equation (1) are discussed. In Section 4, numerical simulation by the phase equation are compared to the analytical solution if it is known, and to the numerical solution of curve-shortening equation in other cases.

2. Approximation Scheme for the Phase-Field Model

This section deals with the isotropic phase-field model of mean-curvature flow in a rectangular domain $\Omega = (0, L_1) \times (0, L_2) \subset \mathbb{R}^2$. The analysis of the anisotropic model presented above still remains open despite of being used for simulation. We consider the initial-boundary value problem

$$\alpha\xi^2 \frac{\partial p}{\partial t} = g[\xi^2 \nabla^2 p + ap(1-p)(p-0.5)] + b\xi^2 F|\nabla p|_E \quad \text{in} \quad (0, T) \times \Omega, \tag{14}$$

$$\begin{aligned} & p\,|_{\partial\Omega} = p_\Omega \quad \text{on} \quad (0, T) \times \partial\Omega, \\ & p(0, x) = p_0(x) \quad \text{in} \quad \bar{\Omega}, \end{aligned} \tag{15}$$

where $\xi > 0$ is a small parameter related to the thickness of the interface layer, $\alpha, g, a, b > 0$, $F \in \mathbb{R}$, $|\cdot|_E$ denotes the Euclidean norm in $\mathbb{R}^2$. The previous equation is a level-set regularization of the mean-curvature equation

$$\alpha v_\Gamma = g\kappa - F, \tag{16}$$

for the dynamics of closed curves in $\mathbb{R}^2$ in the sense of [**3**], [**4**] as presented in [**9**]. The motion by mean curvature has been studied in [**5**], in general. Its relation to phase transitions is analysed in [**6**].

Remark. For the sake of simplicity, we consider homogeneous boundary condition. The analysis continues by weak reformulation of the problem following standard procedures. Let p is a classical solution such that $p \in \mathrm{C}^2(\langle 0,T\rangle \times \bar{\Omega})$, and let $q \in \mathcal{D}(\Omega)$. Here $\mathcal{D}(\Omega)$ means the set of test functions on Ω, $\mathcal{D}'$ the set of corresponding distributions [**12**]. Multiplying (14) by q (scalar product in $\mathrm{L}_2(\Omega)$), denoting

$$f_0(p) = ap(1-p)(p-0.5), \quad (p,q) = \int_\Omega p(\mathbf{x})q(\mathbf{x})\,d\mathbf{x}, \quad \|p\| = \sqrt{(p,p)}, \tag{17}$$

and using the Green formula, we get the following definition

Definition 2.1. Weak solution of the boundary-value problem for the phase equation is a function $p \in \mathrm{L}_1(0,T;\mathrm{H}_0^1(\Omega))$ such that

$$\begin{aligned}(\forall q \in \mathcal{D}(\Omega))\Big(\alpha\xi^2 \frac{d}{dt}(p,q) + g\xi^2(\nabla p, \nabla q) \\ = g(f_0(p),q) + b\xi^2(F|\nabla p|_E, q), \text{ a.e. in } (0,T)\Big),\end{aligned} \tag{18}$$

and

$$p(0) = p_0.$$

The previous definition has a proper sense, as follows from [**2**]. The analysis of the phase equation concerning the existence and uniqueness of the weak solution is being performed using a semi-discrete scheme based on finite differences. The following notations are introduced:

$$\mathbf{h} = (h_1,h_2)\,,\ h_1 = \frac{L_1}{N_1},\ h_2 = \frac{L_2}{N_2},\ \mathbf{x}_{ij} = [x_{ij}^1, x_{ij}^2],\ u_{ij} = u(\mathbf{x}_{ij}), \tag{19}$$

$$\omega_h = \big\{[ih_1, jh_2] \mid i = 1,\dots,N_1-1;\ j = 1,\dots,N_2-1\big\}, \tag{20}$$

$$\bar{\omega}_h = \big\{[ih_1, jh_2] \mid i = 0,\dots,N_1;\ j = 0,\dots,N_2\big\}, \tag{21}$$

$$u_{\bar{x}_1,ij} = \frac{u_{ij} - u_{i-1,j}}{h_1}, \quad u_{x_1,ij} = \frac{u_{i+1,j} - u_{ij}}{h_1}, \tag{22}$$

$$u_{\bar{x}_2,ij} = \frac{u_{ij} - u_{i,j-1}}{h_2}, \quad u_{x_2,ij} = \frac{u_{i,j+1} - u_{ij}}{h_2}, \tag{23}$$

$$u_{\bar{x}_1 x_1,ij} = \frac{1}{h_1^2}\left(u_{i+1,j} - 2u_{ij} + u_{i-1,j}\right), \tag{24}$$

and

$$\bar{\nabla}_h u = [u_{\bar{x}_1}, u_{\bar{x}_2}], \quad \nabla_h u = [u_{x_1}, u_{x_2}], \quad \Delta_h u = u_{\bar{x}_1 x_1} + u_{\bar{x}_2 x_2}, \tag{25}$$

as mappings from ω_h to $\mathbb{R}^2$ or $\mathbb{R}$, respectively. The semi-discrete scheme has the form of:

$$\begin{aligned} &\alpha\xi^2\dot{p}^h = g\xi^2\Delta_h p^h + gf_0(p^h) + b\xi^2|\bar{\nabla}_h p^h|_E F \quad \text{on} \quad \omega_h, \\ &p^h(0) = \mathcal{P}_h p_0\,. \end{aligned} \tag{26}$$

where its solution is a map $p^h\colon \bar{\omega}_h \to \mathbb{R}$. Using the discrete analogy to Sobolev inequalities, necessary a priori estimates are derived. They serve for the proof of convergence of the scheme to the weak solution showing at the same time its existence and uniqueness as follows from the main theorem:

Theorem 2.1. *Consider the problem* (18)*. If* $p_0 \in \mathrm{H}^2(\Omega)\cap \mathrm{H}^1_0(\Omega)$ *then the solutions of the semi-discrete scheme* (26) *converge in* $\mathrm{L}_2((0,T)\times\Omega)$ *to the unique solution* p *of (18) for which*

$$p \in \mathrm{L}_2(0,T;\mathrm{H}^2(\Omega)\cap\mathrm{H}^1_0(\Omega)), \quad \dot{p}\in \mathrm{L}_2(0,T;\mathrm{L}_2(\Omega)).$$

The derived a priori estimates are used to show convergence of obtained solutions of the phase equation to a step-wise constant function, if $\xi \to 0_+$. More precisely:

Theorem 2.2. *Let* p_ξ *is the solution of the problem* (18) *with the initial data satisfying* $E_\xi[p_\xi](0) < M_0$ *independently on* ξ*, where*

$$E_\xi[p] = \int_\Omega \left[g\xi\frac{1}{2}|\nabla p|_E^2 + \frac{1}{\xi}gw_0(p)\right]\,dx, \tag{27}$$

Let

$$\int_\Omega |p_\xi(0,\mathbf{x}) - v_0(\mathbf{x})|\,d\mathbf{x} \to 0,$$

as $\xi \to 0$*, for a function* $v_0 \in \mathrm{L}_1(\Omega)$*. Then for any sequence* ξ_n *tending to* 0 *there is a subsequence* $\xi_{n'}$ *such that*

$$\lim_{\xi_{n'}\to 0} p_{\xi_{n'}}(t,\mathbf{x}) = v(t,\mathbf{x}),$$

is defined a.e. in $(0,T)\times\Omega$*. The function* v *reaches values* 0 *and* 1*, and satisfies*

$$\int_\Omega |v(t_1,\mathbf{x}) - v(t_2,\mathbf{x})\,|d\mathbf{x} \le C|t_2 - t_1|^{\frac{1}{2}},$$

where $C > 0$ *is a constant, and*

$$\sup_{t\in<0,T>} \int_\Omega |\nabla v|_E\,d\mathbf{x} \le C_1,$$

in the sense of $BV(\Omega)$*, where* $C_1 > 0$ *is a constant. The initial condition is*

$$\lim_{t\to 0_+} v(t,\mathbf{x}) = v_0(\mathbf{x})$$

a.e.

In the following sections, necessary tools for proving the previous statements are presented.

2.1 Functions on a Grid

In this section, the properties and relations of grid functions are investigated. If $\mathcal{H}_h = \{f \mid f : \bar{\omega}_h \to \mathbb{R}\}$ is a set of grid functions, the following notations will be used ($f, g \in \mathcal{H}_h$):

$$\|f\|_{ph} = \left(\sum_{i,j=1}^{N_1-1,N_2-1} h_1 h_2 |f_{ij}|^p \right)^{\frac{1}{p}} \quad \text{for} \quad p > 1, \tag{28}$$

$$(f,g)_h = \sum_{i,j=1}^{N_1-1,N_2-1} h_1 h_2 f_{ij} g_{ij}, \quad \|f\|_h^2 = (f,f)_h\,, \tag{29}$$

$$(f^1, g^1\rfloor = \sum_{i=1,j=1}^{N_1,N_2-1} h_1 h_2 f_{ij}^1 g_{ij}^1, \quad \|f^1\rfloor\|^2 = (f^1, f^1\rfloor, \tag{30}$$

$$(f^2, g^2\rceil = \sum_{i=1,j=1}^{N_1-1,N_2} h_1 h_2 f_{ij}^2 g_{ij}^2, \quad \|f^2\rceil\|^2 = (f^2, f^2\rceil, \tag{31}$$

$$(\mathbf{f}, \mathbf{g}] = (f^1, g^1\rfloor + (f^2, g^2\rceil, \quad \|\mathbf{f}]\|^2 = (\mathbf{f}, \mathbf{f}], \tag{32}$$

where $\mathbf{f} = [f^1, f^2]$ and $\mathbf{g} = [g^1, g^2]$.

Referring to [**2**] we recall the following formulas

• Green formulas

$$(f, g_{\bar{x}_1 x_1})_h = -(f_{\bar{x}_1}, g_{\bar{x}_1}\rfloor + \sum_{j=1}^{N_2-1} (f g_{\bar{x}_1}\,|_{N_1,j} - f g_{x_1}\,|_{0,j}) h_2, \tag{33}$$

and

$$(f, g_{\bar{x}_2 x_2})_h = -(f_{\bar{x}_2}, g_{\bar{x}_2}\rceil + \sum_{i=1}^{N_1-1} (f g_{\bar{x}_2}\,|_{i,N_2} - f g_{x_2}\,|_{i,0}) h_1, \tag{34}$$

In a natural way, we define the space

$$l_p(\omega_h) = \{\mathcal{H}_h \mid \|\cdot\|_{ph}\}. \tag{35}$$

• Poincaré inequality. Let $u \in l_2(\omega_h)$ and $u\,|_{\gamma_h} = 0$. Then

$$\|u\|_h^2 \le C(\Omega)[\ \|u_{\bar{x}_1}\rfloor\|^2 + \|u_{\bar{x}_2}\rceil\|^2\ ]. \tag{36}$$

We continue by introducing an extension of grid functions, so that they are defined almost everywhere on Ω. Such extensions are studied by the usual technique of L_p and H^k spaces. The approach of [**17**] is adopted for the equations in question. The limiting process requires the refinement of the FDM grid $\bar{\omega}_h$, if $\mathbf{h} \to 0$. For this purpose, a proper metric should be chosen. If we intent to use the compactness technique, a mapping converting a grid function $f_h : \bar{\omega}_h \to \mathbb{R}$ into a function $f : \Omega \to \mathbb{R}$ is needed. Then, the norm of L_p space will serve as a metric for convergence of the numerical scheme.

Definition 2.2. Be $\bar{\omega}_h$ an uniform rectangular grid imposed on a domain $\Omega \subset \mathbb{R}^2$. Let $\mathbf{h} = [h_1, h_2]$ is the mesh size. Then, the dual grid is a set

$$\bar{\omega}_h^* = \left\{ \Sigma_{ij} \subset \bar{\Omega} \;\middle|\; \Sigma_{ij} = \left(x_i^1 - \frac{h_1}{2}, x_i^1 + \frac{h_1}{2} \right) \times \left(x_j^2 - \frac{h_2}{2}, x_j^2 + \frac{h_2}{2} \right) \cap \bar{\Omega} \text{ for } [x_i^1, x_j^2] \in \bar{\omega}_h \right\}.$$

The dual simplicial grid is a set

$$\bar{\omega}_h^{*s} = \bar{\omega}_h^{*\triangleleft} \cup \bar{\omega}_h^{*\triangleright}, \tag{37}$$

with

$$\bar{\omega}_h^{*\triangleleft} = \left\{ \Sigma_{ij}^{\triangleleft} \subset \bar{\Omega} \mid \Sigma_{ij}^{\triangleleft} = [x_{i,j}, x_{i-1,j}, x_{i,j-1}]_{\varkappa} \cap \bar{\Omega} \text{ for } [x_i^1, x_j^2] \in \bar{\omega}_h \right\},$$

$$\bar{\omega}_h^{*\triangleright} = \left\{ \Sigma_{ij}^{\triangleright} \subset \bar{\Omega} \mid \Sigma_{ij}^{\triangleright} = [x_{i-1,j-1}, x_{i-1,j}, x_{i,j-1}]_{\varkappa} \cap \bar{\Omega} \text{ for } [x_i^1, x_j^2] \in \bar{\omega}_h \right\},$$

where $[\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3]_{\varkappa} = \{ \mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \sum_{j=1}^3 \gamma_j \mathbf{z}_j \wedge \sum_{j=1}^3 \gamma_j = 1 \wedge \gamma_{1,2,3} \geq 0 \}$ is the convex hull of the set $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$.

Remark. Consequently, $\bigcup_{\Sigma \in \bar{\omega}_h^*} \Sigma = \bar{\Omega}$ - the system $\bar{\omega}_h^*$ covers the domain Ω. Each (rectangular) set $\Sigma \in \bar{\omega}_h^*$ has the point $[x_i^1, x_j^2]$ in its center. Similarly, the system $\bar{\omega}_h^{*s}$ also covers $\bar{\Omega}$.

Definition 2.3. Let $\mathcal{H}_h$ is a set of grid functions on $\bar{\omega}_h$. Define the following mappings:

- $\mathcal{Q}_h \colon \mathcal{H}_h \to \mathcal{C}(\bar{\Omega})$ such that for each $u \in \mathcal{H}_h$

  $$(\mathcal{Q}_h u)(x^1, x^2) = u_{i-1,j-1} + \nabla_h u_{h,i-1,j-1} \cdot [x^1 - x^1_{i-1,j-1}, x^2 - x^2_{i-1,j-1}],$$

  if $[x^1, x^2] \in \Sigma_{ij}^{\triangleright}$, $\Sigma_{ij}^{\triangleright} \in \bar{\omega}_h^{*\triangleright}$;

  $$(\mathcal{Q}_h u)(x^1, x^2) = u_{ij} + \bar{\nabla}_h u_{ij} \cdot [x^1 - x^1_{ij}, x^2 - x^2_{ij}],$$

  if $[x^1, x^2] \in \Sigma_{ij}^{\triangleleft}$, $\Sigma_{ij}^{\triangleleft} \in \bar{\omega}_h^{*\triangleleft}$.
- $\mathcal{S}_h \colon \mathcal{H}_h \to \mathrm{L}_1(\Omega)$ such that for each $u \in \mathcal{H}_h$

  $$(\mathcal{S}_h u)(x^1, x^2) = u_{ij},$$

  if $[x^1, x^2] \in \Sigma_{ij}$, $\Sigma_{ij} \in \bar{\omega}_h^*$;
- $\mathcal{P}_h \colon \mathcal{C}(\bar{\Omega}) \to \mathcal{H}_h$ such that for each $u \in \mathcal{C}(\bar{\Omega})$

  $$(\mathcal{P}_h u)_{ij} = u(\mathbf{x}_{ij}),$$

  if $\mathbf{x}_{ij} \in \bar{\omega}_h$.

Remark. The operator $\mathcal{P}_h$ is linear and continuous from $\mathcal{C}(\bar{\Omega})$ to $\mathcal{H}_h$, and can be extended to $\mathrm{H}^1(\Omega)$ via density argument. $\mathcal{Q}_h u$ is a continuous piecewise linear function, $\nabla(\mathcal{Q}_h u)$ exists a.e. in Ω. We proceed by determining basic properties of the above defined maps as implied by [**2**]

1. If $u, v \mid_{\gamma_h}= 0$, the scalar product coincides with the scalar product in $l_2(\omega_h)$

$$\int_\Omega \mathcal{S}_h u \mathcal{S}_h v \, dx = (u, v)_h. \tag{38}$$

2. Let ω_h be a grid on the domain Ω with the mesh $\mathbf{h}$, let $u, v \in \mathcal{H}_h$ is such that $u, v \mid_{\gamma_h}= 0$. Then

$$(\nabla(\mathcal{Q}_h u), \nabla(\mathcal{Q}_h v)) = (\bar{\nabla}_h u, \bar{\nabla}_h v]. \tag{39}$$

3. Let ω_h be a grid on the domain Ω with the mesh $\mathbf{h}$, let $u \in \mathcal{H}_h$. Then

$$\|\mathcal{Q}_h u\|_{\mathrm{L}_2(\Omega)} \leq \|\mathcal{S}_h u\|_{\mathrm{L}_2(\Omega)}. \tag{40}$$

4. Let ω_h be a grid on the domain Ω with the mesh $\mathbf{h}$, let $u \in \mathcal{H}_h$, $u \mid_{\gamma_h}= 0$. Then

$$\int_\Omega |\mathcal{Q}_h u - \mathcal{S}_h u|^2 \, d\mathbf{x} \leq \frac{|\mathbf{h}|^2}{6} \|\bar{\nabla}_h u]|^2, \tag{41}$$

if $u \mid_{\gamma_h}= 0$.

5. Let ω_h be a grid on the domain Ω with the mesh $\mathbf{h}$, let $u \in \mathcal{H}_h$, $u \mid_{\gamma_h}= 0$. Then there is a constant $C_5 > 0$ such that

$$\int_\Omega |\nabla \mathcal{Q}_h u - \mathcal{S}_h \nabla_h u|^2 \, d\mathbf{x} \leq C_5 |\mathbf{h}|^2 \|\Delta_h u\|_h^2, \tag{42}$$

6. Let $p \in \mathcal{C}^{0,\nu}(\Omega)$, $\nu \in (0, 1)$. Then,

$$\mathcal{S}_h(\mathcal{P}_h p) \to p \quad \text{in } \mathrm{L}_s(\Omega), \text{ if } \mathbf{h} \to 0, \tag{43}$$

for $s > 1$.

7. Let $u \in \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega)$. Then

$$\mathcal{Q}_h(\mathcal{P}_h u) \to u \tag{44}$$

in $\mathrm{H}^1(\Omega)$, if $\mathbf{h} \to 0$.

8. Let $p \in \mathcal{C}^2(\Omega)$ and $p \mid_{\partial\Omega}= 0$. Then

$$\nabla(\mathcal{Q}_h(\mathcal{P}_h p)) \to \nabla p, \tag{45}$$

in $\mathrm{L}_2(\Omega)$, if $\mathbf{h} \to 0$.

2.2 Proof of the Main Theorem

Proof of Theorem 2.1. Consider the semi-discrete scheme (26). The theory of ordinary differential equations implies existence and uniqueness of the maximal solution p^h of (26) on a time interval $\langle 0, T_h)$, if $\bar{\omega}_h$ remains fixed. We will derive an a-priori estimate allowing to show independence of T_h on $\mathbf{h}$ as well as convergence of p^h if $\mathbf{h} \to 0$.

Multiplying (26) by $\dot{p}^h$ and summing over ω_h, the following equality takes place:

$$\alpha\xi^2 \parallel \dot{p}^h \parallel_h^2 = g\xi^2(\Delta_h p^h, \dot{p}^h)_h + g(f_0(p^h), \dot{p}^h)_h + b\xi^2(F|\bar{\nabla}_h p^h|_E, \dot{p}^h)_h. \tag{46}$$

Taking into account that

$$f_0(p^h)\dot{p}^h = -\frac{d}{dt} w_0(p^h), \text{ where } w_0(s) = \frac{a}{64}[1-(1-2s)^2]^2,$$

The previous relation takes the form of

$$\alpha\xi^2 \parallel \dot{p}^h \parallel_h^2 + g\xi^2(\bar{\nabla}_h p^h, \bar{\nabla}_h \dot{p}^h] = -g\frac{d}{dt}(w_0(p^h), 1)_h + b\xi^2(F|\bar{\nabla}_h p^h|_E, \dot{p}^h)_h. \tag{47}$$

The Schwarz inequality allows to obtain ($\epsilon > 0$)

$$\begin{aligned} &\alpha\xi^2 \parallel \dot{p}^h \parallel_h^2 + g\xi^2 \frac{1}{2}\frac{d}{dt} \parallel \bar{\nabla}_h p^h]|^2 \\ &\quad \le -g\frac{d}{dt}(w_0(p^h),1)_h + \epsilon\frac{b^2|F|^2}{2\alpha}\xi^2 \parallel \bar{\nabla}_h p^h]|^2 + \frac{1}{2\epsilon}\alpha\xi^2 \parallel \dot{p}^h \parallel_h^2 . \end{aligned} \tag{48}$$

If setting $\epsilon = 0.5$, we get

$$\begin{aligned} &\Big\{g\xi^2\frac{1}{2} \parallel \bar{\nabla}_h p^h]|^2 + g(w_0(p^h),1)_h\Big\}(t) \\ &\quad \le \Big\{g\xi^2\frac{1}{2} \parallel \bar{\nabla}_h p^h]|^2 + g(w_0(p^h),1)_h\Big\}(0)\exp\Big(\frac{b^2|F|^2}{2\alpha g}t\Big), \end{aligned} \tag{49}$$

and integrating the previous estimate over $(0, T)$, we have

$$\begin{aligned} &\int_0^T \Big\{g\xi^2\frac{1}{2} \parallel \bar{\nabla}_h p^h]|^2 + g(w_0(p^h),1)_h\Big\}(t)\, dt \\ &\quad \le \Big\{g\xi^2\frac{1}{2} \parallel \bar{\nabla}_h p^h]|^2 + g(w_0(p^h),1)_h\Big\}(0)\frac{2\alpha g}{b^2|F|^2}\Big\{\exp\Big(\frac{b^2|F|^2}{2\alpha g}\Big)T - 1\Big\}, \end{aligned} \tag{50}$$

which means that (according to (44) applied to the initial condition),

$$\begin{aligned} \bar{\nabla}_h p^h &\in \mathrm{L}_\infty(0,T;\mathrm{l}_2(\omega_h)), \qquad & \bar{\nabla}_h p^h &\in \mathrm{L}_2(0,T;\mathrm{l}_2(\omega_h)), \\ p^h &\in \mathrm{L}_\infty(0,T;\mathrm{l}_4(\omega_h)), & p^h &\in \mathrm{L}_4(0,T;\mathrm{l}_4(\omega_h)), \end{aligned}$$

are bounded independently on $\mathbf{h}$.

Setting $\epsilon = 1$, we get

$$\frac{1}{2}\alpha\xi^2 \parallel \dot{p}^h \parallel_h^2 + g\xi^2 \frac{1}{2}\frac{d}{dt} \parallel \bar{\nabla}_h p^h]|^2 + g\frac{d}{dt}(w_0(p^h),1)_h \le \frac{b^2|F|^2}{2\alpha}\xi^2 \parallel \bar{\nabla}_h p^h]|^2. \tag{51}$$

Integrating over $(0,T)$,

$$\frac{1}{2}\alpha\xi^2 \int_0^T \parallel \dot{p}^h \parallel_h^2 \, dt + \Big\{ g\xi^2 \frac{1}{2} \parallel \bar{\nabla}_h p^h]|^2 + g(w_0(p^h),1)_h \Big\}(T)$$
$$\le \Big\{ g\xi^2 \frac{1}{2} \parallel \bar{\nabla}_h p^h]|^2 + g(w_0(p^h),1)_h \Big\}(0) + \frac{b^2|F|^2}{2\alpha}\xi^2 \int_0^T \parallel \bar{\nabla}_h p^h]|^2 \, dt.$$

Again, (44) together with the assumption $p_0 \in \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega)$ imply that $\dot{p}^h \in \mathrm{L}_2(0,T;\mathrm{l}_2(\omega_h))$ is uniformly bounded with respect to $\mathbf{h}$ (see (45)). The time interval is therefore independent on $\mathbf{h}$.

Introduce the extension operators $\mathcal{S}_h$ and $\mathcal{Q}_h$. Let $\mathbf{h}_n$ is a sequence of mesh sizes converging towards 0. It produces a sequence p^{h_n} of semi-discrete solutions to (26) which are defined on $\langle 0,T\rangle$. Then, $\bar{\nabla}_{h_n}\mathcal{P}_{h_n}p_0 \overset{n\to\infty}{\to} \nabla p_0$ in $\mathrm{L}_2(\Omega)$ by (44), and $\mathcal{S}_{h_n}\mathcal{P}_{h_n}p_0 \overset{n\to\infty}{\to} p_0$ in $\mathrm{L}_4(\Omega)$ by the (43). Following (38), (39), and (40), we find that after a multi-step selection, there is a subsequence n' for which

- $\{\mathcal{S}_{h_{n'}}p^{h_{n'}}\}_{n'=1}^{\infty}$ converges weakly in $\mathrm{L}_2(0,T;\mathrm{H}_0^1(\Omega))$;
- $\{\mathcal{S}_{h_{n'}}\dot{p}^{h_{n'}}\}_{n'=1}^{\infty}$ converges weakly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$;
- $\{\mathcal{Q}_{h_{n'}}p^{h_{n'}}\}_{n'=1}^{\infty}$ converges weakly in $\mathrm{L}_2(0,T;\mathrm{H}_0^1(\Omega))$;
- $\{\mathcal{Q}_{h_{n'}}\dot{p}^{h_{n'}}\}_{n'=1}^{\infty}$ converges weakly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$;

We replace the notation n' back by n. The non-linear terms in the equation (26) require stronger convergence result. Using the lemma on the compact imbedding, we conclude that $\mathcal{Q}_{h_n}p^{h_n}$ converges strongly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$. Relation (41) implies the same result for $\mathcal{S}_{h_n}p^{h_n}$. Denote their common limit as p and the weak limit of $\mathcal{S}_{h_n}\dot{p}^{h_n}$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ as q_1. The estimates

$$\|\mathcal{S}_h(|\nabla_h p^h|)\| \le \|\nabla_h p^h]| = \|\nabla(\mathcal{Q}_h p^h)\|,$$

and

$$\begin{aligned} &\|f_0(\mathcal{S}_h p^h)\|_{\mathrm{L}_{4/3}(\Omega)} \\ &\quad\le a\Big[\frac{1}{2}\|\mathcal{S}_h p^h\|_{\mathrm{L}_{4/3}(\Omega)} + \frac{3}{2}\|(\mathcal{S}_h p^h)^2\|_{\mathrm{L}_{4/3}(\Omega)} + \|(\mathcal{S}_h p^h)^3\|_{\mathrm{L}_{4/3}(\Omega)}\Big] \\ &\quad= a\Big[\frac{1}{2}\|\mathcal{S}_h p^h\|_{\mathrm{L}_{4/3}(\Omega)} + \frac{3}{2}\|\mathcal{S}_h p^h\|^2_{\mathrm{L}_{8/3}(\Omega)} + \|\mathcal{S}_h p^h\|^3_{\mathrm{L}_4(\Omega)}\Big], \end{aligned} \tag{52}$$

justify the existence of weak limit of $f_0(\mathcal{S}_{h_n}p_{h_n})$ in $\mathrm{L}_2(0,T;\mathrm{L}_{4/3}(\Omega))$ denoted by q_2 (dual space), and of $\mathcal{S}_{h_n}(|\nabla_{h_n}p_{h_n}|_E)$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ denoted by q_3. These limits exist as a consequence of the a priori estimate and of (41), (40). We prove that $q_1 = \dot{p}$, $q_2 = f_0(p)$, and $q_3 = |\nabla p|_E$. First relation is implied by the uniquenes of the limit in $\mathcal{D}'(0,T)$, as

$$\int_0^T (\mathcal{S}_{h_n}\dot{p}^{h_n} - \mathcal{Q}_{h_n}\dot{p}^{h_n}, q)\psi(t)\,dt = -\int_0^T (\mathcal{S}_{h_n}p^{h_n} - \mathcal{Q}_{h_n}p^{h_n}, q)\dot{\psi}(t)\,dt,$$

where $q \in \mathcal{D}(\Omega)$, $\psi \in \mathcal{D}(0,T)$. The remaining equalities are proven in the following lemmas.

Lemma 2.1. *If p denotes the weak limit of $\mathcal{S}_{h_n}p^{h_n}$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$, then*

$$f_0(\mathcal{S}_{h_n}p^{h_n}) \to f_0(p) \text{ weakly in } \mathrm{L}_{\frac{4}{3}}(0,T;\mathrm{L}_{\frac{4}{3}}(\Omega)).$$

Proof. According to the compact imbedding, we have that $\mathcal{S}_{h_n}p^{h_n}$ converges strongly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ and it can be considered to converge a.e. in this space (see [**12**]). Furthermore, we observe that as $\mathcal{S}_{h_n}p^{h_n}$ was bounded in $\mathrm{L}_\infty(0,T;\mathrm{L}_4(\Omega))$ (see (52), $f_0(\mathcal{S}_{h_n}p^{h_n})$ is bounded in $\mathrm{L}_\infty(0,T;\mathrm{L}_{\frac{4}{3}}(\Omega))$. These two facts together with the Aubin lemma [**2**] give the final result. □

Before proceeding in the proof, we show more about regularity of p.

Lemma 2.2. *Under the assumptions of the theorem, the function p belongs to $\mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega)$.*

Proof. Multiply (26) by a function $\mathcal{P}_{h_n}q$, where $q \in \mathcal{D}(\Omega)$.

$$\begin{aligned}\alpha\xi^2(\dot{p}^{h_n}, \mathcal{P}_{h_n}q)_h + g\xi^2(\bar{\nabla}_{h_n}p^{h_n}, \bar{\nabla}_{h_n}\mathcal{P}_{h_n}q] \\ = g(f_0(p^{h_n}), \mathcal{P}_{h_n}q)_h + b\xi^2(F|\bar{\nabla}_{h_n}p^{h_n}|_E, \mathcal{P}_{h_n}q)_h.\end{aligned} \tag{53}$$

In terms of $\mathrm{L}_2(\Omega)$, this means that

$$\begin{aligned}\alpha\xi^2(\mathcal{S}_{h_n}\dot{p}^{h_n}, \mathcal{S}_{h_n}(\mathcal{P}_{h_n}q)) + g\xi^2(\nabla(\mathcal{Q}_{h_n}p^{h_n}), \nabla\mathcal{Q}_{h_n}(\mathcal{P}_{h_n}q)) \\ = g(f_0(\mathcal{S}_{h_n}p^{h_n}), \mathcal{S}_{h_n}(\mathcal{P}_{h_n}q)) + b\xi^2(F\mathcal{S}_{h_n}|\bar{\nabla}_{h_n}p^{h_n}|_E, \mathcal{S}_{h_n}(\mathcal{P}_{h_n}q)).\end{aligned} \tag{54}$$

According to (45), we realize that $\mathcal{Q}_{h_n}(\mathcal{P}_{h_n}q) \overset{n\to\infty}{\to} q$ in $\mathrm{H}_0^1(\Omega)$, and $\mathcal{S}_{h_n}(\mathcal{P}_{h_n}q) \overset{n\to\infty}{\to} q$ in $\mathrm{L}_2(\Omega)$ (see (43)). We can pass to the limit in the sense of $\mathcal{D}'(0,T)$ obtaining

$$\alpha\xi^2(\dot{p}, q) + g\xi^2(\nabla p, \nabla q) = g(q_2, q) + b\xi^2(Fq_3, q). \tag{55}$$

Consequently, the function p is continuous from $\langle 0,T\rangle$ into $\mathrm{L}_2(\Omega)$. We rewrite the previous equality in the sense of $\mathcal{D}'(\Omega)$,

$$\alpha\xi^2\dot{p} = g\xi^2\Delta p + gq_2 + b\xi^2Fq_3. \tag{56}$$

Note that $q_2 = f_0(p)$ and $p \in \mathrm{L}_\infty(0,T,\mathrm{L}_s(\Omega))$ for any $s > 1$. Consequently, $q_2 \in \mathrm{L}_2(0,T,\mathrm{L}_2(\Omega))$. As $\dot{p}$, q_2, q_3 belong to $\mathrm{L}_2(\Omega)$, this means that $\Delta p \in \mathrm{L}_2(\Omega)$ for each $t \in (0,T)$. After several simple implications, we find that p must be in the domain of Δ – see [**2**]:

$$p(t) \in D(\Delta) = \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega) \text{ for } t \in (0,T).$$ □

Next statement investigates the convergence of the norm of gradient.

Lemma 2.3. *The sequence $\nabla\mathcal{Q}_{h_n}p^{h_n}$ converges strongly to ∇p in $L_2((0,T)\times\Omega)$ and $\mathcal{S}_{h_n}|\bar\nabla_{h_n}p^{h_n}|_E$ converges weakly to $|\nabla p|_E$ in $L_2((0,T)\times\Omega)$.*

Proof. Following the technique of [**15**], the statement of the lemma is shown. Multiply the equation (26) by $p^{h_n}-\mathcal{P}_{h_n}p$ and sum over ω_h.

$$\begin{aligned}\alpha\xi^2(\dot p^{h_n},p^{h_n}-\mathcal{P}_{h_n}p)_h+g\xi^2(\bar\nabla_{h_n}p^{h_n},\bar\nabla_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p)]\\ =g(f_0(p^{h_n}),p^{h_n}-\mathcal{P}_{h_n}p)_h+b\xi^2(F|\bar\nabla_{h_n}p^{h_n}|_E,p^{h_n}-\mathcal{P}_{h_n}p)_h.\end{aligned}\tag{57}$$

Rewrite this equality in terms of $\mathrm{L}_2(\Omega)$, and integrate over $(0,T)$.

$$\begin{aligned}\alpha\xi^2&\int_0^T(\mathcal{S}_{h_n}\dot p^{h_n},\mathcal{S}_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p))\,dt\\ &+g\xi^2\int_0^T(\nabla(\mathcal{Q}_{h_n}p^{h_n}),\nabla\mathcal{Q}_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p))\,dt\\ =\;&g\int_0^T(f_0(\mathcal{S}_{h_n}p^{h_n}),\mathcal{S}_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p))\,dt\\ &+b\xi^2\int_0^T(F\mathcal{S}_{h_n}|\bar\nabla_{h_n}p^{h_n}|_E,\mathcal{S}_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p))\,dt.\end{aligned}$$

As we have shown that $p\in\mathrm{L}_2(0,T;\mathrm{H}^2(\Omega))$ satisfies (55), it means that $p(t)\in\mathcal{C}^{0,1}(\Omega)$, $t\in(0,T)$, and consequently, $\mathcal{S}_{h_n}(\mathcal{P}_{h_n}p)\to p$, and $\nabla\mathcal{Q}_{h_n}(\mathcal{P}_{h_n}p)\to\nabla p$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ (see (43), (44)). Therefore, by triangular inequaltity, $\mathcal{S}_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p)\to 0$, and $\nabla\mathcal{Q}_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p)\to 0$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$. We add and subtract a term

$$g\xi^2\int_0^T(\nabla(\mathcal{Q}_{h_n}(\mathcal{P}_{h_n}p)),\nabla\mathcal{Q}_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p))\,dt$$

to the equality (57) knowing that it tends to 0 as

$$\nabla\mathcal{Q}_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p)\to 0,$$

weakly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$, if $n\to\infty$. Then, we have

$$\begin{aligned}g\xi^2&\int_0^T(\nabla(\mathcal{Q}_{h_n}p^{h_n}-\mathcal{P}_{h_n}p),\nabla\mathcal{Q}_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p))\,dt\\ =&-\alpha\xi^2\int_0^T(\mathcal{S}_{h_n}\dot p^{h_n},\mathcal{S}_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p))\,dt\\ &+g\int_0^T(f_0(\mathcal{S}_{h_n}p^{h_n}),\mathcal{S}_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p))\,dt\\ &+b\xi^2\int_0^T(F\mathcal{S}_{h_n}|\bar\nabla_{h_n}p^{h_n}|_E,\mathcal{S}_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p))\,dt\\ &+g\xi^2\int_0^T(\nabla(\mathcal{Q}_{h_n}(\mathcal{P}_{h_n}p)),\nabla\mathcal{Q}_{h_n}(p^{h_n}-\mathcal{P}_{h_n}p))\,dt.\end{aligned}$$

As all terms in the right hand side tend to 0 if $n \to \infty$, we see that $\nabla(\mathcal{Q}_{h_n}(p^{h_n} - \mathcal{P}_{h_n} p)) \to 0$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$, which together with (42) and (45) gives the desired result. The previous lemma together with the Lebesgue-dominated convergence theorem, and the technique of the Lemma 2.1 gives, that

$$\mathcal{S}_{h_n}|\nabla_{h_n} p^{h_n}|_E \rightharpoonup |\nabla p|_E,$$

in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$. □

Passage to the limit. Take the equation (26) into the consideration, multiply it by a test function $\mathcal{P}_{h_n} w$, where $w \in \mathcal{D}(\Omega)$. Integrate it over ω_h. Then, we have, in terms of $\mathrm{L}_2(\Omega)$,

$$\begin{aligned}(58)\qquad &\alpha\xi^2(\mathcal{S}_{h_n}\dot{p}^{h_n}, \mathcal{S}_{h_n}\mathcal{P}_{h_n} w) + g\xi^2(\nabla \mathcal{Q}_{h_n} p^{h_n}, \nabla \mathcal{Q}_{h_n}\mathcal{P}_{h_n} w)\\ &\quad = g(f_0(\mathcal{S}_{h_n} p^{h_n}), \mathcal{S}_{h_n}\mathcal{P}_{h_n} w) + b\xi^2(F\mathcal{S}_{h_n}|\bar{\nabla}_{h_n} p^{h_n}|_E, \mathcal{S}_{h_n}\mathcal{P}_{h_n} w).\end{aligned}$$

Knowing that

1. $\mathcal{S}_{h_n}\dot{p}^{h_n}$ converges weakly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ to $\dot{p}$;
2. $\nabla \mathcal{Q}_{h_n} p^{h_n}$ converges strongly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ to ∇p;
3. $\mathcal{S}_{h_n}|\bar{\nabla}_{h_n} p^{h_n}|_E$ converges weakly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ to $|\nabla p|_E$;
4. $\mathcal{S}_{h_n}\mathcal{P}_{h_n} p_0$ converges strongly to p_0 in $\mathrm{H}^1_0(\Omega)$,

multiply (58) by a scalar function $\psi(t) \in \mathcal{C}^1\langle 0,T\rangle$, for which $\psi(T) = 0$. We integrate by parts. Taking into account all previous results, the fact that $\mathcal{S}_{h_n} p^{h_n}(0) = \mathcal{S}_{h_n}\mathcal{P}_{h_n} p_0$, and the Lebesgue theorem, we are able to pass to the limit.

$$\begin{aligned}(59)\qquad &\alpha\xi^2(p_0, w)\psi(0) - \int_0^T \alpha\xi^2(p,w)\dot{\psi}\,dt\\ &\quad + \int_0^T \psi\big[g\xi^2(\nabla p, \nabla w) + - g(f_0(p), w) - b\xi^2(F|\nabla p|_E, w)\big]\,dt = 0.\end{aligned}$$

If $\psi \in \mathcal{D}(0,T)$, we have

$$(60)\qquad \alpha\xi^2 \frac{d}{dt}(p,w) + g\xi^2(\nabla p, \nabla w) = g(f_0(p), w) + b\xi^2(F|\nabla p|_E, w).$$

It remains to show that the weak solution satisfies the initial condition. Multiplying (18) by a scalar function $\psi(t) \in \mathcal{C}^1\langle 0,T\rangle$, for which $\psi(T) = 0$, and integrating by parts, we obtain

$$\begin{aligned}(61)\qquad &\alpha\xi^2(p(0), w)\psi(0) - \int_0^T \alpha\xi^2(p,w)\dot{\psi}\,dt\\ &\quad + \int_0^T \psi\big[g\xi^2(\nabla w, \nabla w) - g(f_0(p), w) - b\xi^2(F|\nabla p|_E, w)\big]\,dt = 0.\end{aligned}$$

Subtracting this equation from (59), we get

$$(p_0 - p(0), w)\psi(0) = 0 \quad \forall w \in \mathcal{D}((\Omega)).$$

From this we see that $p(0) = p_0$ in $\mathrm{L}_2(\Omega)$.

Uniqueness. Suppose, there are two solutions to the weak problem (18) denoted as p, q. Subtracting corresponding weak identities and denoting $w = p - q$, we have

$$\alpha\xi^2 \frac{d}{dt}(w, v) + g\xi^2(\nabla w, \nabla v) = g(f_0(p) - f_0(q), v) + b\xi^2(F[|\nabla p|_E - |\nabla q|_E], v),$$

$$w(0) = 0. \tag{62}$$

As $f_0(p) = -ap^3 + \frac{3}{2}ap^2 - \frac{1}{2}ap$, we also find

$$f_0(p) - f_0(q) = w\Psi(p, q),$$

where $\Psi(p, q) = -\frac{1}{2}a + \frac{3}{2}a(p + q) - a(p^2 + pq + q^2)$.

The a-priori estimate guarantees that $\|p\|_{\mathrm{L}_2(\Omega)}, \|p\|_{\mathrm{L}_4(\Omega)}, \|q\|_{\mathrm{L}_2(\Omega)}, \|q\|_{\mathrm{L}_4(\Omega)}$ are uniformly bounded in $(0, T)$. Due to the inequality

$$\begin{aligned} \|\Psi(p, q)\|_{\mathrm{L}_2(\Omega)} \le \frac{1}{2}a|\Omega| &+ \frac{3}{2}a\big(\|p\|_{\mathrm{L}_2(\Omega)} + \|q\|_{\mathrm{L}_2(\Omega)}\big) \\ &+ a\big(\|p\|^2_{\mathrm{L}_4(\Omega)} + \|p\|_{\mathrm{L}_4(\Omega)}\|q\|_{\mathrm{L}_4(\Omega)} + \|q\|^2_{\mathrm{L}_4(\Omega)}\big), \end{aligned} \tag{63}$$

there is a constant C' such that

$$|(f_0(p) - f_0(q), v)| \le C'\|w\|_{\mathrm{L}_4(\Omega)}\|v\|_{\mathrm{L}_4(\Omega)}.$$

The triangular inequality yields

$$||\nabla p|_E - |\nabla q|_E| \le |\nabla w|_E.$$

Putting $v = w$ into (62) (in the sense of the technique of the proof of existence), we obtain

$$\alpha\xi^2 \frac{1}{2}\frac{d}{dt}\|w\|^2 + g\xi^2\|\nabla w\|^2 \le gC'\|w\|^2_{\mathrm{L}_4(\Omega)} + b\xi^2|F|\|\nabla w]\|\|w\|. \tag{64}$$

At this point, we use the fact that

$$\|v\|_{\mathrm{L}_4(\Omega)} \le 2^{\frac{1}{4}}\|\nabla v\|^{\frac{1}{2}}\|v\|^{\frac{1}{2}} \quad \forall v \in \mathrm{H}^1_0(\Omega), \tag{65}$$

which is valid for a domain $\Omega \subset \mathbb{R}^2$. Then,

$$\alpha\xi^2 \frac{1}{2}\frac{d}{dt}\|w\|^2 + g\xi^2\|\nabla w\|^2 \leq (gC'\sqrt{2} + b\xi^2|F|)\|\nabla w\|\|w\| \leq g\xi^2\|\nabla w\|^2 + \frac{1}{4}\frac{(gC'\sqrt{2} + b\xi^2|F|)^2}{g\xi^2}\|w\|^2. \tag{66}$$

We conclude that

$$\|w(t)\|^2 \leq \|w(0)\|^2 e^{\frac{1}{2}\frac{(gC'\sqrt{2}+b\xi^2|F|)^2}{\alpha g\xi^4}t}.$$

As the initial condition was $w(0) = 0$, we have $p(t) = q(t)$ for all $t \in (0, T)$. A direct consequence of the above statement is that the entire sequence of FDM approximate solutions converges to the weak solution. Note that the proof was based on the key estimate (65) of the L_4-norm being valid in 2D. For the 3D case, a similar estimate is valid. □

Remark. The imbedding

$$H^2(\Omega) \subset\!> \mathcal{C}^{0,\lambda}(\bar{\Omega}) \quad \lambda \in \langle 0, 1)\,,$$

implies that we finally have

$$p \in \mathcal{C}(\langle 0, T\rangle \times \bar{\Omega}).$$

2.3 Convergence towards the Sharp Interface Model

This section deals with the relation of the phase equation to the sharp-interface description of the mean-curvature flow. It uses estimates derived above to show certain compactness results leading to the existence of a step function defining the position of the solid subdomain Ω_s in time. The objective of the investigation will be the dependence of the solution on ξ. The a priori estimate gave the result:

$$\alpha\xi^2 \parallel \dot{p} \parallel^2 + g\xi^2 \frac{1}{2}\frac{d}{dt} \parallel \nabla p \parallel^2 \leq -g\frac{d}{dt}(w_0(p), 1) + \epsilon\frac{b^2|F|^2}{2\alpha}\xi^2 \parallel \nabla p \parallel^2 + \frac{1}{2\epsilon}\alpha\xi^2 \parallel \dot{p} \parallel^2, \tag{67}$$

(for each $\epsilon > 0$), which bounded the expression:

$$E_\xi[p] = \int_\Omega \Big[g\xi\frac{1}{2}|\nabla p|_E^2 + \frac{1}{\xi}gw_0(p)\Big]\,dx,$$

as follows

$$E_\xi[p](t) \leq E_\xi[p](0)\exp\Big\{\frac{b^2|F|^2}{2\alpha g}t\Big\} \quad t \in (0, T). \tag{68}$$

Additionally, there is a constant C_T such that

$$\frac{1}{2}\alpha\xi\int_0^T \parallel \dot{p} \parallel^2 \, dt + E_\xi[p](T) \le C_T E_\xi[p](0). \tag{69}$$

These estimates allow to use the method proposed in [**3**]. Define the following monotone function

$$G(s) = \int_0^s |1 - (1-2r)^2| \, dr. \tag{70}$$

The function G allows to demonstrate asymptotic behaviour of the solution p_ξ, if $\xi \to 0$. Next lemma is valid (see [**2**]):

Lemma 2.4. *Be p_ξ the solution of* (18) *where* $E_\xi[p_\xi](0) \le M_0$ *independently on ξ. Then there are constants $M > 0$ and $M_1 > 0$ such that*

$$\sup\left\{ \int_\Omega |\nabla G(p_\xi)| \, dx \mid t \in \langle 0, T\rangle \right\} \le M \tag{71}$$

and, for $0 \le t_1 < t_2$,

$$\int_{t_1}^{t_2} \int_\Omega |\partial_t G(p_\xi)| \, dx \, dt \le M_1 (t_2 - t_1)^{0.5}. \tag{72}$$

The previous statement leads to the existence of a step function as expected and allows to use the proof of the Theorem 2.2 presented in [**3**].

Remark. In the previous analysis, we did not recover any sharp-interface relation for the function v. However, this question exceeds the scope of this work. It has been partially studied elsewhere, e.g. in [**5**].

3. Approximation Schemes for Anisotropic Curve Shortening Flow

This section presents numerical schemes destinated to the solution of the anisotropic curve shortening equation. We briefly sketch the results from [**15**]. Provided

$$b(v) = -\frac{1}{v}, \quad A = D = \frac{g}{\alpha}, \quad B = \left(\frac{g}{\alpha}\right)_x, \quad G = -\frac{F}{\alpha} - \left(\frac{F}{\alpha}\right)_{xx}, \tag{73}$$

the problem (10) may be transformed into the form

$$\begin{aligned} \partial_t b(v) &= (Av_x)_x + (Bv)_x + Dv + G, \\ v(x,t) &= v(x + 2\pi, t), \\ v(x,0) &= v_0(x), \end{aligned} \tag{74}$$

for the unknown function $v(x,t)$, $x \in \mathbb{R}$ and $t \in [0,T)$, corresponding to the curvature of the evolving convex curve (by a convention, it is negative) with v_0 representing smooth initial condition. Thus $b(s)$ is smooth function, defined in interval $(-\infty,0)$, satisfying $b'(s) > 0$ and

$$b(s) \to 0 \text{ and } b'(s) \to 0 \text{ for } s \to -\infty, \tag{75}$$

$$b(s) \to \infty \text{ and } b'(s) \to \infty \text{ for } s \to 0^-. \tag{76}$$

So, the nonlinearity in the problem (74) is represented by the increasing function $b(s)$ admitting asymtotical **degeneracy of two types** expressed by the properties (75)–(76). In this sense the model covers locally both slow and fast diffusions. Due to the special form of the problem, the solution also may **blow up** in a finite time. The blow up of the curvature corresponds to the shrinking of the curve.

Denote $I = (0,T)$, $J = (0,2\pi)$, $Q_T = J \times I$. Let $V = \{w \in W_2^1(0,2\pi) : w(0) = w(2\pi)\}$ and assume that $v_0(x) \in V$.

Definition 3.1. The function $v \in L_2(I,V)$ with $\partial_t b_R(v) \in L_2(I,V^*)$ is called a weak solution of (74), iff $v(x,0) = v_0(x)$ and the identity

$$\langle \partial_t b(v), \varphi\rangle + (Av_x, \varphi_x) + (Bv, \varphi_x) - (Dv, \varphi) = (G, \varphi), \tag{77}$$

holds for all $\varphi \in V$ and for a.e. $t \in I$.

In [**15**] the existence and uniqueness of the weak solution of (74) is proved and two approximation schemes for solving degenerate parabolic problems of the type (74) have been suggested. First, the problem (74) is regularized by considering a globally Lipschitz continuous function b_R instead of b, for which $\Gamma \geq b'_R(s) \geq \gamma > 0$. The regularization is choosen so that b_R and b are the same for arguments in the interval $(-R, -\frac{1}{R})$, where R is a (large) regularization parameter. Let $v(x,0) \in L_\infty(J) \cap V$ and

$$-R + \delta < v(x,0) < -\frac{1}{R} - \delta$$

where δ is positive real number. Then the L_∞-estimates for v and $b_R(v)$ are derived, where v is the solution of the regularized problem, which imply the existence of time T_1, $0 < T_1 < \infty$, such that the inequalities

$$-R \leq v(x,t) \leq -\frac{1}{R}$$

hold for every $t \in (0,T_1)$, a.e. $x \in J$ (see [**15**, Theorem 2.2]). The functions b_R and b are the same for the arguments satisfying the previous inequality, so the solutions of regularized and unregularized problems are the same in time interval

$(0, T_1)$. Choosing R sufficiently large, T_1 can be near to the blow-up time of the curvature. In the sequel, we identify T_1 with T and thus replace b_R by b.

In [**15**], the following two approximation schemes for solving (74) have been suggested and we use them also for the computations of sharp interface evolution presented in this paper.

First approximation scheme:

Let $n \in \mathbb{N}$, $\tau = \frac{T}{n}$, $t_i = i\tau$ for $i = 0, \ldots, n$, $u_0 = v_0(x)$. For every $i = 1, \ldots, n$, let $u_i \in V$ be the function such that $\forall \varphi \in V$

$$\left(b'(u_{i-1})\Big(\frac{u_i - u_{i-1}}{\tau}\Big), \varphi\right) + (Au_{ix}, \varphi_x) + (Bu_i, \varphi_x) - (Du_i, \varphi) = (G, \varphi). \tag{78}$$

From these functions we construct Rothe's functions

$$\begin{aligned} u^{(n)}(t) &= u_{i-1} + (t - t_{i-1})\frac{u_i - u_{i-1}}{\tau}, \quad \text{for } t_{i-1} \le t \le t_i, i = 1, \ldots, n \\ \overline{u}^{(n)}(t) &= u_i, \quad \text{for } t_{i-1} < t \le t_i, i = 1, \ldots, n, \quad \overline{u}^{(n)}(0) = u_0, \end{aligned} \tag{79}$$

for which one can prove that

$$u^{(n)} \to u \quad \text{in } C(I, L_2),$$

where u is the unique weak solution of (74) (see [**15**, Theorem 2.1]).

Second approximation scheme is based on works [**13**] and [**11**] concerning numerical solution of the slow diffusion and Stefan-like problems and mainly on [**10**] describing solution of elliptic-parabolic problems. This scheme is based on a time discretization using relaxation function by means of which the original nonlinear problem is reduced to an iterative solution of linear elliptic problems:

Let $n \in \mathbb{N}$, $\tau = \frac{T}{n}$, $t_i = i\tau$ for $i = 0, \ldots, n$, $v_0 = v_0(x), u_0 = v_0$. For $i = 1, \ldots, n$ we look for functions $u_i \in V$ and $\mu_i \in L_\infty(J)$ such that $\forall \varphi \in V$

$$(\mu_i(u_i - v_{i-1}), \varphi) + \tau(Au_{ix}, \varphi_x) + \tau(Bu_i, \varphi_x) - \tau(Du_i, \varphi) = \tau(G, \varphi), \tag{80}$$

satisfying the "convergence condition"

$$\alpha\frac{\gamma}{2} \le \mu_i \le \frac{b(v_{i-1} + \alpha(u_i - v_{i-1})) - b(v_{i-1})}{u_i - v_{i-1}}, \tag{81}$$

where $0 < \alpha < 1$. If $u_i = v_{i-1}$ then we replace the difference quotient by $\alpha b'(v_{i-1})$. The function v_i is obtained by the algebraic correction

$$b(v_i) := b(v_{i-1}) + \mu_i(u_i - v_{i-1}). \tag{82}$$

There are many possibilities to determine u_i, μ_i satisfying (80)–(81). If we choose $\alpha\frac{\gamma}{2} \leq \mu_i \leq \alpha\gamma$, then (80)–(81) are satisfied. However, it is more interesting (from numerical point of view) to choose μ_i very close to the difference quotient in (81). This can be done iteratively in the following way

$$(\mu_{i,k-1}(u_{i,k} - v_{i-1}), \varphi) + \tau(Au_{i,kx}, \varphi_x) + \tau(Bu_{i,k}, \varphi_x) - \tau(Du_{i,k}, \varphi) = \tau(G, \varphi),$$

$$\overline{\mu}_{i,k} = \frac{b(v_{i-1} + \alpha(u_{i,k} - v_{i-1})) - b(v_{i-1})}{u_{i,k} - v_{i-1}}, \tag{83}$$

$$\mu_{i,k} := \overline{\mu}_{i,k}, \quad \text{for } 1 \leq k \leq k_0, \quad \mu_{i,k} := \min\{\overline{\mu}_{i,k}, \mu_{i,k-1}\}, \quad \text{for } k = k_0 + 1, \ldots$$

starting with

$$\mu_{i,0} = b'(v_{i-1}).$$

From u_i, v_i, obtained in each time step of (80)–(82), the Rothe functions $\overline{u}^{(n)}, \overline{v}^{(n)}$ are constructed as in (79). Then one can prove that

$$\overline{v}^{(n)} \to u \ \text{ in } L_2(Q_T), \quad \overline{u}^{(n)} \to u \ \text{ in } L_2(I, V), \tag{84}$$

where u is a weak solution of (74) (see [**15**, Theorems 2.3 and 2.4]).

In practical implementations, k_0 can be choosen in accordance with the shape of $b(s)$. In our numerical realization (for the case (1.9) of anisotropic curve shortening flow) we have observed that $\mu_{i,k} \equiv \overline{\mu}_{i,k}$ are convergent and hence we have used k_0 sufficiently large. In practice, the iterations very quickly converge to functions satisfying (80)–(81). Practical realization of "algebraic correction" in our case of anisotropic curve shortening is simple, because the inverse function $b^{-1}(s)$ can be determined explicitly. After time discretization, in each iteration, the scheme requires to solve the linear convection-diffusion equation (with convective term only in the presence of anisotropy). For this purpose, we use the so called "power-law scheme" described in [**16**], which respects the "up-wind" principle.

In the numerical experiments documented in the Sections 4, we use the both approximation schemes. Their results are comparable. Let us note that the second approximation scheme is more precise near the degeneracy $b'(s) = \infty$ (strong anisotropy). In general, the time step τ was choosen as 0.001 and we have computed the numerical solution until the numerical blow up of the solution occurs i.e. the curvature is of the order of 10^6.

4. Computational Results

The Gibbs-Thompson condition (6) itself belongs to the class of equations governing motion of hyperplanes. If $n = 2$, we investigate motion of curves in plane in the form of (1) (see [**9**], [**15**], [**14**]). If the initial curve is a circle of radius $r(0)$, $g \equiv 1$, $F \equiv 0$, then the radius decay is given by

$$r(t) = \sqrt{r(0)^2 - 2t}, \tag{85}$$

and the blow-up time (when the circle shrinks to a point) is

$$T_0 = \frac{S(0)}{2\pi}\,. \tag{86}$$

Following [**15**], circle satisfies the condition

$$\mathrm{Iso}\,(t) = \frac{L(t)^2}{4\pi S(t)} = 1\,, \tag{87}$$

where Iso is isoperimetric ratio (in general greater than or equal to 1, for closed curves). In other cases, it is difficult to find an analytical solution, however, the flow can be investigated numerically (see [**15**]). The corresponding phase-field model of such mean-curvature flow has the form

$$\alpha(\theta)\xi^2\frac{\partial p}{\partial t} = g(\theta)\big[\xi^2\nabla^2 p + ap(1-p)(p-0.5)\big] + b\xi^2 F|\nabla p|_E, \tag{88}$$

$$p\,|_{\partial\Omega} = 1, \tag{89}$$

$$p\,|_{t=0} = 1-\chi_{B_0}, \tag{90}$$

where χ_{B_0} is characteristic function of the initial shape $B_0 \subset \mathbb{R}^2$. In the presented results, the function $\alpha \equiv 1$, $b = 1$ and the anisotropy was set up by $g(\theta) = 1-\zeta\cos(\theta)$ with the anisotropy strength $\zeta > 0$ and θ being the angle between ∇p and the axis $\mathbf{x}^1$). We compare the solution of (74) to the solution of (88) in order to investigate their agreement and behaviour in particular situations of curve dynamics.

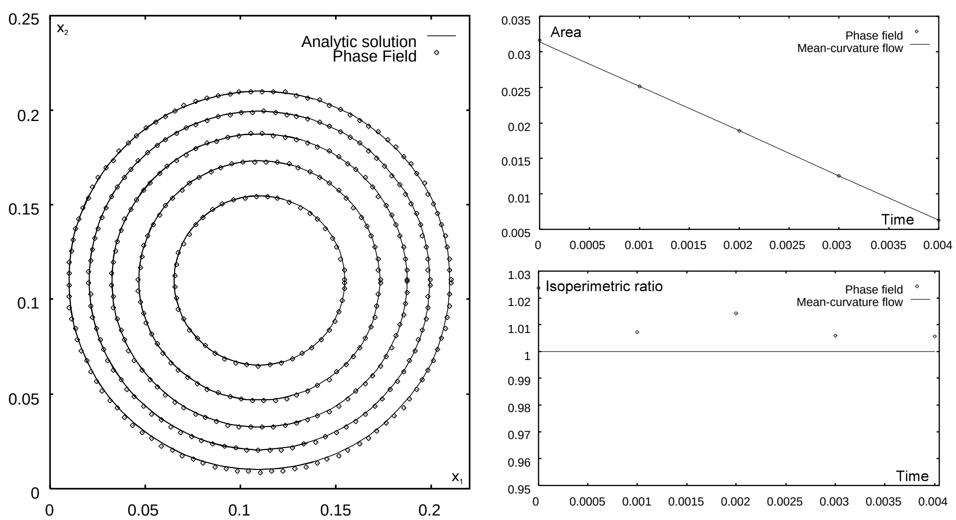

Figure 3. Isotropic mean-curvature flow compared to phase-field model ($\zeta = 0.0$, $F = 0.0$, $a = 4.0$, $\xi = 0.005$, $\alpha = 1$, $L_1 = L_2 = 0.22$, mesh $N_1 = N_2 = 220$).

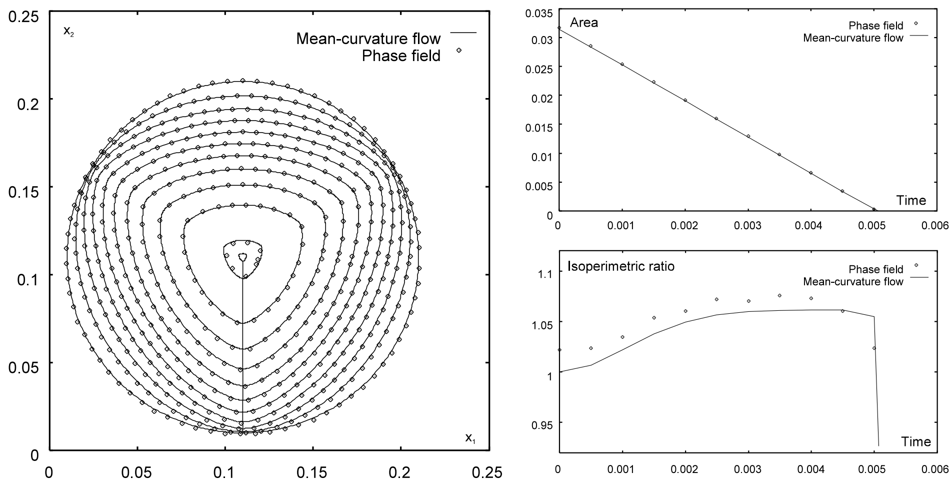

Figure 4. Anisotropic 3-fold mean-curvature flow compared to phase-field model ($\zeta = 0.88889$, $F = 0.0$, $a = 4.5$, $\xi = 0.0025$, $\alpha = 1$, $L_1 = L_2 = 0.22$, mesh $N_1 = N_2 = 390$).

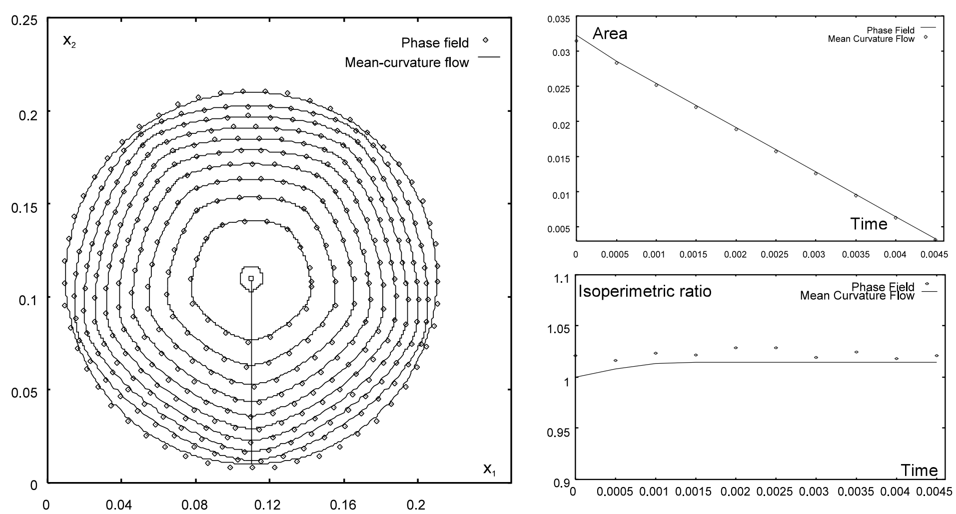

Figure 5. Anisotropic 5-fold mean-curvature flow compared to phase-field model ($\zeta = 0.7$, $F = 0.0$, $a = 4.0$, $\xi = 0.005$, $\alpha = 1$, $L_1 = L_2 = 0.22$, mesh $N_1 = N_2 = 220$).

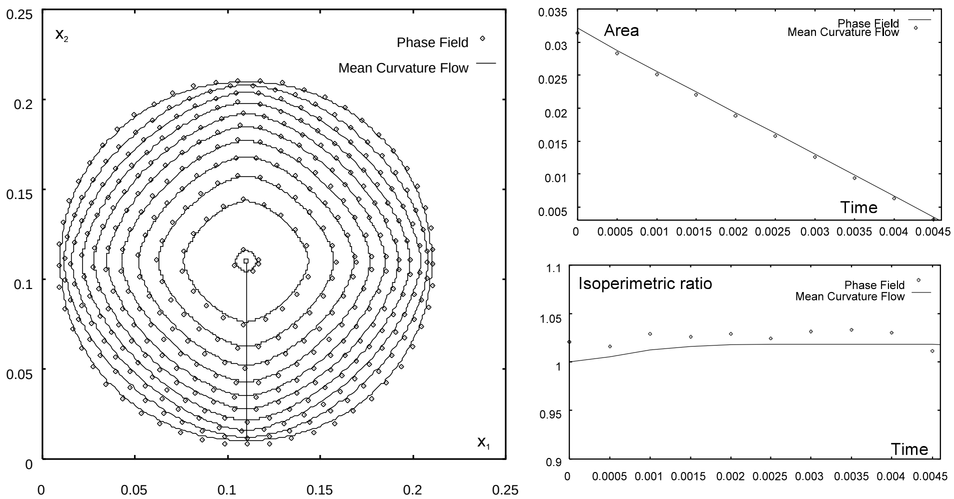

Figure 6. Anisotropic 4-fold mean-curvature flow compared to phase-field model ($\zeta = 0.7$, $F = 0.0$, $a = 4.0$, $\xi = 0.005$, $\alpha = 1$, $L_1 = L_2 = 0.22$, mesh $N_1 = N_2 = 220$).

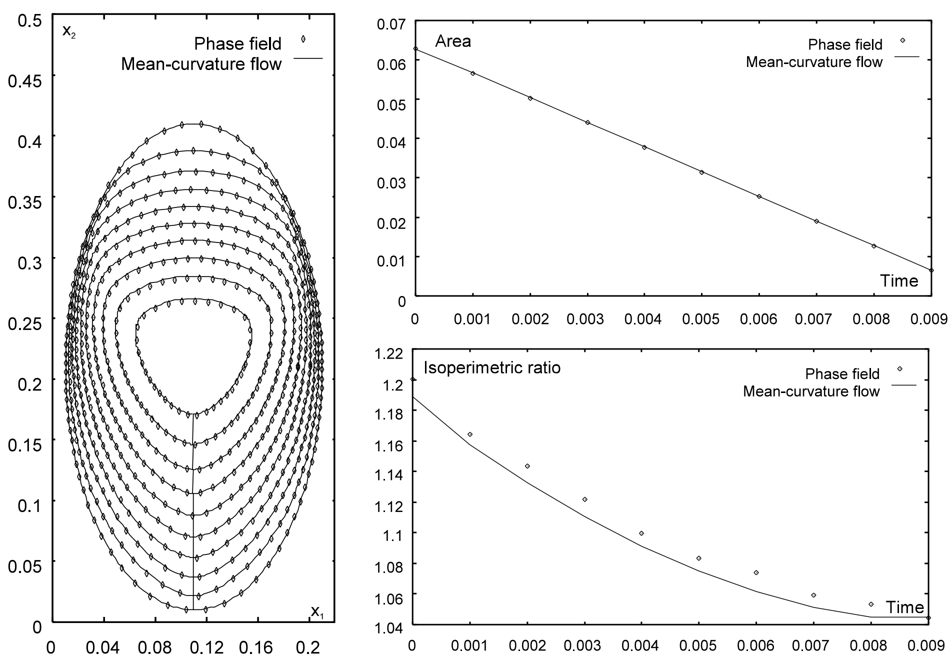

Figure 7. Anisotropic 3-fold mean-curvature flow compared to phase-field model ($\zeta = 0.7$, $F = 0.0$, $a = 4.0$, $\xi = 0.0025$, $\alpha = 1$, $L_1 = L_2 = 0.22$, mesh $N_1 = N_2 = 440$).

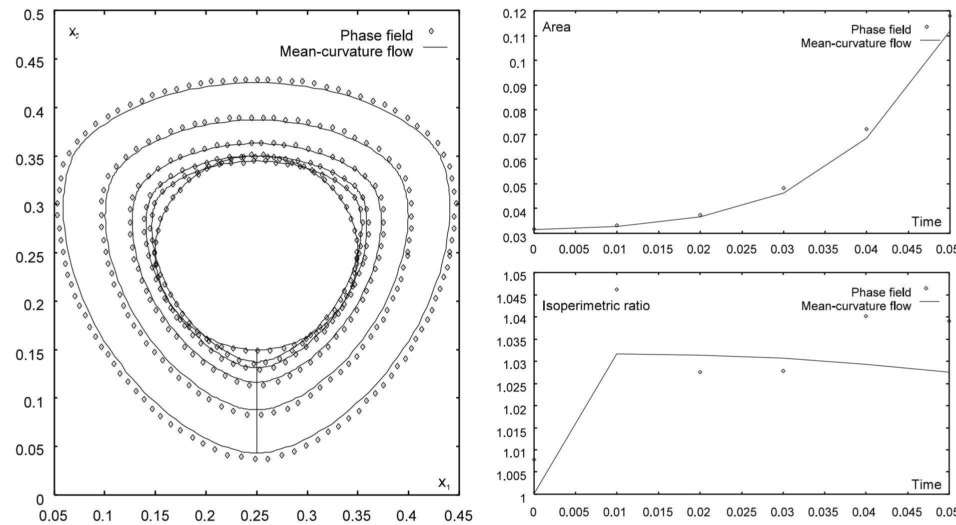

Figure 8. Anisotropic 3-fold mean-curvature flow with force compared to phase-field model ($\zeta = 0.7$, $F = -10.0$, $a = 4.0$, $b = 1$, $\xi = 0.005$, $\alpha = 1$, $L_1 = L_2 = 0.5$, mesh $N_1 = N_2 = 250$).

Figure 3 shows shrinking of a circle by isotropic mean-curvature flow (comparison of the phase-field and analytical solution). Figures 5 and 6 demostrate the shortening of a circle towards a 5-fold or 4-fold anisotropic shape. Figure 7 shows shrinking of an ellipse by the 3-fold anisotropic motion (comparison of phase-field solution with the numerical solution of the curve-shortening equation (see [**15**])). Figure 8 shows an expansion of a circle by mean-curvature flow with forcing term.

Remark. From the presented studies, it follows that for the values $\xi \sim 0.005$ and less, a good agreement has been achieved (the mesh size about one half of ξ). Regarding the conditions for the motion in a larger scale simulating the dendritic growth, such a choice of ξ, and $\mathbf{h}$ was difficult to use due to the increase of CPU time needed for the simulation. This would require an improvement of the simulation algorithm.

5. Conclusion

The presented results of comparison between the two mentioned different approaches to the curve dynamics in the plane show the consistency of both of them. They also serve as an information about the behaviour of the phase field model in such very special cases and indicate the degree of approximation to the sharp interface. The performed studies also stimulated new ideas on how the phase field model and algorithms for the numerical solution can be improved.

Acknowledgement. First author was partially supported by the Grant No. 3097486 of the Czech Technical University. Second author was supported by grants 1/3034/96 and M63G of the Grant Agency for Science of the Slovak Republic.

References

1. Angenent S. B. and Gurtin M. E., *Multiphase thermomechanics with an interfacial structure 2.evolution of an isothermal interface*, Arch. Rat. Mech. Anal. **108** (1989), 323–391.

2. Beneš M., *Phase-Field Model of Microstructure Growth in Solidification of Pure Substances*, PhD thesis, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, 1997.

3. Bronsard L. and Kohn R., *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics*, J. Differential Equations **90** (1991), 211–237.

4. Caginalp G., *The dynamics of a conserved phase field system: Stefan-like, Hele-Shaw, and Cahn-Hilliard models as asymptotic limits*, IMA J. Appl. Math. **44** (1990), 77–94.

5. Chen Y.-G., Giga Y. and Goto S., *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Diff. Geom. **33** (1991), 749–786.

6. Evans L. C., Soner H. M. and Sougadinis P. E., *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math. **45** (1992), 1097–1123.

7. Gage M. and Hamilton R. S., *The heat equation shrinking convex plane curves*, J. Diff. Geom. **23** (1986), 285–314.

8. Gurtin M., *On the two-phase Stefan problem with interfacial energy and entropy*, Arch. Rational Mech. Anal. **96** (1986), 200–240.

9. ______, *Thermomechanics of Evolving Phase Boundaries in the Plane*, Clarendon Press, Oxford, 1993.

10. Jäger W. and Kačur J., *Approximation of degenerate elliptic-parabolic problems by nondegenerate elliptic and parabolic problems*, Technical report, University of Heidelberg, 1991.

11. ______, *Solution of porous medium type systems by linear approximation schemes*, Num. Math. **60** (1991), 407–427.

12. Kufner A., John O. and Fučík S., *Function Spaces*, Academia, Prague, 1977.

13. Magenes E., Nochetto R. H. and Verdi C., *Energy error estimates for a linear scheme to approximate nonlinear parabolic problems*, Math. Mod. Num. Anal. **21** (1987), 655–678.

14. Mikula K., *Solution of nonlinear flow governed by curvature of plane convex curves*, Applied Numerical Mathematics **23** (1997), 347–360.

15. Mikula K. and Kačur J., *Evolution of convex plane curves describing anisotropic motions of phase interfaces*, SIAM J. Scientific Computing **17** (1996), 1302–1327.

16. Patankar S., *Numerical heat transfer and fluid flow*, Hemisphere Publ. Comp., 1980.

17. Raviart P. A., *Sur la resolution de certaines equations paraboliques non lineaires.*, J. Func. Anal. **5** (1970), 299–328.

M. Beneš, Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University of Prague, Trojanova 13, 120 00 Prague, Czech Republic, *e-mail:* benes@km1.fjfi.cvut.cz

K. Mikula, Department of Mathematics and Descriptive Geometry, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia, *e-mail:* mikula@ops.svf.stuba.sk