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THE BLOW–UP RATE FOR A SEMILINEAR PARABOLIC EQUATION WITH A NONLINEAR BOUNDARY CONDITION

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Abstract. In this paper we obtain the blow-up rate for positive solutions of $u_t = u_{xx} - \lambda u^p$, in $(0,1)\times(0,T)$ with boundary conditions $u_x(1,t) = u^q(1,t)$, $u_x(0,t) = 0$. If $p < 2q-1$ or $p = 2q-1$, $0 < \lambda < q$, we find that the behaviour of u is given by $u(1,t) \sim (T-t)^{-\frac{1}{2(q-1)}}$ and, if $\lambda < 0$ and $p \geq 2q-1$, the blow up rate is given by $u(1,t) \sim (T-t)^{-\frac{1}{p-1}}$. We also characterize the blow-up profile in similarity variables.

1. Introduction

In this paper we consider positive solutions of the following parabolic problem,

$$
\begin{cases}
u_t = u_{xx} - \lambda u^p & \text{in } (0,1)\times[0,T),\\
u_x(1,t) = u^q(1,t) & t \in [0,T),\\
u_x(0,t) = 0 & t \in [0,T),\\
u(x,0) = u_0(x) > 0 & \text{in } (0,1),
\end{cases}
\tag{1.1}
$$

where $p, q > 1$ and $\lambda \neq 0$ are parameters.

This problem with $\lambda > 0$ was studied in [**2**] and [**12**]. Existence and regularity of solutions have been proved for initial data that satisfy a compatibility condition. In the general case one can obtain a solution in H^1 by a standard approximation procedure (see [**2**] for the details). The solution of (1.1) only exists for a finite period of time (in this case u becomes unbounded in finite time and we say that it blows up) or it is defined for all positive t (in this case we call it a global solution).

In our problem one has a nonlinear term at the boundary and a reaction term in the equation. If $\lambda > 0$, these two terms compete and the blow up phenomenon occurs if and only if $p < 2q-1$ or $p = 2q-1$ with $\lambda < q$ (see [**2**]), [**12**]). In fact there holds:

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Theorem 1.1 ([**2**, Theorems 4.1, 4.2 and 4.7] and [**12**]).

1. *Suppose that* $p < 2q - 1$ *or* $p = 2q - 1$ *with* $0 < \lambda < q$*, if* $u_0 > v$*, where* v *is any maximal stationary solution, then* u *blows up in finite time.*
2. *Suppose that* $p > 2q - 1$ *or* $p = 2q - 1$ *with* $\lambda \geq q$*, then every positive solution is global.*

Our interest is the blow-up rate, so we assume that we are dealing with a blowing up solution u, and that $p < 2q - 1$ or $p = 2q - 1$ with $\lambda < q$.

We suppose that the initial data are positive, increasing, verify a compatibility condition and $(u_0)_{xx} - \lambda u_0^p \geq \alpha > 0$ in order to guarantee $u_t \geq 0$.

The blow-up rate for solutions of (1.1) with $\lambda > 0$ was conjectured in [**2**] (see Remark 4.2 there). For the blow-up rate for the heat equation with a similar boundary condition we refer to [**5**], [**9**] and for the blow-up rate for a system to [**3**], [**4**] and [**13**].

In this paper we prove the conjecture of [**2**] and characterize the blow-up rate. We prove:

Theorem 1.2. *Let* $p, q > 1$ *and* $\lambda > 0$*. Under the above assumptions on* u_0*,*
a) If $p < 2q - 1$*, there exists positive constants* C*,* c *such that*

$$c \leq \max_{[0,1]} u(\cdot, t)(T - t)^{\frac{1}{2(q-1)}} \leq C \qquad (t \nearrow T).$$

b) If $p = 2q - 1$ *with* $\lambda < q$*, there exists positive constants* C*,* c *such that*

$$c \leq \max_{[0,1]} u(\cdot, t)(T - t)^{\frac{1}{2(q-1)}} \leq C \qquad (t \nearrow T).$$

In the case $\lambda < 0$ both terms cooperate and every positive solution has finite time blow-up (see [**1**], [**14**]). For the blow up rate we observe that if $p < 2q - 1$ then the nonlinear term at the boundary determines the blow up rate while if $p > 2q - 1$ the reaction term in the equation dominates and gives the blow up rate. We prove :

Theorem 1.3. *Let* $p, q > 1$ *and* $\lambda < 0$*. Under the above assumptions on* u_0*,*
a) If $p < 2q - 1$*, there exists positive constants* C*,* c *such that*

$$c \leq \max_{[0,1]} u(\cdot, t)\ (T - t)^{\frac{1}{2(q-1)}} \leq C \qquad (t \nearrow T).$$

b) If $p = 2q - 1$*, there exists positive constants* C*,* c *such that*

$$c \leq \max_{[0,1]} u(\cdot, t)\ (T - t)^{\frac{1}{2(q-1)}} \leq C \qquad (t \nearrow T).$$

c) If $p > 2q - 1$*, there exists positive constants* C*,* c *such that*

$$c \leq \max_{[0,1]} u(\cdot, t)\ (T - t)^{\frac{1}{(p-1)}} \leq C \qquad (t \nearrow T).$$

We want to remark that if $p = 2q - 1$ then $\frac{1}{2(q-1)} = \frac{1}{p-1}$.

With this blow-up rate we can characterize the blow-up profile.

Theorem 1.4. *a) Let $p < 2q-1$. Then for any $y \geq 0$,*

$$(T-t)^{\frac{1}{2(q-1)}} u(1 - y\sqrt{T-t}, t) \to w_0(y) \qquad (t \to T)$$

where w_0 is the unique positive bounded solution of $w_{yy} - \frac{y}{2} w_y - \frac{1}{2(q-1)} w = 0$ in $(0,\infty)$ with $w_y(0) = -w^q(0)$.

b) Let $p > 2q-1$ and $\lambda < 0$. Then for any $y \geq 0$,

$$(T-t)^{\frac{1}{p-1}} u(1 - y\sqrt{T-t}, t) \to (-\frac{1}{\lambda(p-1)})^{\frac{1}{p-1}} \qquad (t \to T).$$

The convergence is uniform for $y \in [0, C]$, C being an arbitrary positive constant.

For the critical case, $p = 2q-1$, further investigation is required. Case *a*) for $\lambda > 0$ was conjectured in [**2**].

We want to remark that in the first case w_0 is not constant, while in the second the asymptotic profile is a constant. This is due to the predominance of the nonlinear term at the boundary in the first case or the nonlinear source in the second.

The paper is organized as follows, in Section 2, we prove Theorem 1.2 and Theorem 1.3, the main tool used in the proof is a scaling argument due to [**8**], [**9**]. In Section 3 we prove Theorem 1.4 using ideas from [**6**], [**7**].

2. Blow-up Rate

Let us begin by proving part *a*) of Theorems 1.2, 1.3.

Let u be a solution of (1.1) with a finite blow-up time T, and for each $0 < t^* < T$, let

$$M(t^*) = u(1, t^*) = \max_{[0,1]} u(\cdot, t^*).$$

We define

$$\varphi_\gamma(y, s) = \frac{1}{M(t^*)} u(\gamma y + 1, \gamma^2 s + t^*),$$

in $\Omega_\gamma = \{y \in R : \gamma y + 1 \in [0,1]\}$.

This function φ_γ, satisfies $0 \leq \varphi_\gamma \leq 1$, $\varphi_\gamma(0,0) = 1$, $\frac{\partial \varphi_\gamma}{\partial s} \geq 0$ and

$$\begin{cases} (\varphi_\gamma)_s = (\varphi_\gamma)_{yy} - \lambda \gamma^2 M^{p-1} (\varphi_\gamma)^p, \\ (\varphi_\gamma)_y(0, s) = \gamma M^{q-1} (\varphi_\gamma)^q(0, s), \\ (\varphi_\gamma)_y(-1/\gamma, s) = 0. \end{cases} \tag{2.1}$$

Now we choose $\gamma = \frac{1}{M^{q-1}}$ and observe that γ goes to zero as t^* goes to T. We define $K_\gamma = \lambda \gamma^2 M^{p-1} = \lambda M^{p-2q+1}$ and observe that K_γ goes to 0 as t^* goes to T because $p < 2q-1$.

We claim that there exists a constant C such that for every γ small

$$\frac{\partial \varphi_\gamma}{\partial s}(0,0) \geq C.$$

To prove this claim, suppose the contrary. Then there exists a sequence $\gamma_j \to 0$ such that

$$\frac{\partial \varphi_{\gamma_j}}{\partial s}(0,0) \to 0.$$

As φ_{γ_j} is uniformly bounded in $C^{2+\alpha,1+\alpha/2}$ (see [**10**], [**11**]), passing to a subsequence if necessary, we obtain a positive function φ, such that $\varphi_{\gamma_j} \to \varphi$ in $C^{2+\beta,1+\beta/2}$, (for some $\beta < \alpha$) and verify $0 \leq \varphi \leq 1$, $\varphi(0,0) = 1$, $\frac{\partial \varphi}{\partial s} \geq 0$ and

$$\begin{cases} \varphi_s = \varphi_{yy}, \\ \varphi_y(0,s) = \varphi^q(0,s) \end{cases} \tag{2.2}$$

in $\{y < 0\} \times (-\infty, 0]$. We set $w = \varphi_s$ and as w satisfies the heat equation, a boundary condition of the type $w_y(0,s) \geq 0$ and $w(0,0) = 0$, we conclude, using the Hopf lemma that $w \equiv 0$, that is φ does not depend on s and then, using that $0 \leq \varphi \leq 1$, $\varphi(0,0) = 1$, and that φ verifies (2.2) we obtain a contradiction.

So we have proved that

$$\frac{\partial \varphi_\gamma}{\partial s}(0,0) \geq C.$$

In terms of u, that is $\frac{\gamma^2 u_t(1,t^*)}{M(t^*)} \geq C$. As $M(t^*) = u(1,t^*)$ and $\gamma = \frac{1}{M^{q-1}}$, this implies

$$M^{1-2q}(t^*) M'(t^*) \geq C.$$

Now we integrate between t and T and obtain (using that $q > 1$)

$$C(T-t) \leq \int_t^T M^{1-2q}(t^*) M'(t^*)\, dt^* \leq \int_{M(t)}^{+\infty} s^{1-2q}\, ds = \frac{C}{M(t)^{2(q-1)}}.$$

And then,

$$M(t) \leq \frac{C}{(T-t)^{\frac{1}{2(q-1)}}}.$$

To prove the other inequality we observe that φ_γ is uniformly bounded in $C^{2+\alpha,1+\alpha/2}$ and then there exists a constant C such that,

$$\frac{\partial \varphi_\gamma}{\partial s}(0,0) \leq C.$$

In terms of u, that is $\frac{\gamma^2 u_t(1,t^*)}{M(t^*)} \leq C$. Using again that $M(t^*) = u(1,t^*)$ and that $\gamma = \frac{1}{M^{q-1}}$, we have,

$$M^{1-2q}(t^*) M'(t^*) \leq C.$$

Again, we integrate between t and T and obtain (using that $q > 1$)

$$c(T-t) \geq \int_t^T M^{1-2q}(t^*)M'(t^*)\ dt = \int_{M(t)}^{+\infty} s^{1-2q}\ ds = \frac{C}{M(t)^{2(q-1)}}.$$

Hence,

$$M(t) \geq \frac{c}{(T-t)^{\frac{1}{2(q-1)}}}$$

as we wanted to prove. □

To prove part b) of Theorems 1.2, 1.3 we proceed as before but in this case we obtain that φ_γ verifies $0 \leq \varphi_\gamma \leq 1$, $\varphi_\gamma(0,0) = 1$, $\frac{\partial \varphi_\gamma}{\partial s} \geq 0$ and (2.1) with $K_\gamma = \lambda\gamma^2 M^{p-1} = \lambda$. As before we claim that there exists a constant C such that

$$\frac{\partial \varphi_\gamma}{\partial s}(0,0) \geq C. \tag{2.3}$$

If not, passing to a subsequence and using Hopf Lemma, we obtain a nontrivial solution of

$$\begin{cases} 0 = \varphi_{yy} - \lambda\varphi^p, & y < 0, \\ \varphi_y(0) = \varphi^q(0) = 1 \end{cases}$$

with $0 \leq \varphi \leq 1$, which is a contradiction (this solution can not exist). We remark that in this case, $p = 2q - 1$, we are using that $q > \lambda$.

From inequality (2.3) and the reverse one (that follows by $C^{2+\alpha,1+\alpha/2}$ regularity, see [**11**]) it follows that

$$c \leq u(1,t)(T-t)^{\frac{1}{p-1}} = M(t)(T-t)^{\frac{1}{2(q-1)}} \leq C. \qquad \square$$

To prove part c) of Theorem 1.3 we proceed as in the previous case but this time we choose $\gamma^2 = M^{-(p-1)}$ and hence φ_γ verifies $0 \leq \varphi_\gamma \leq 1$, $\varphi_\gamma(0,0) = 1$, $\frac{\partial \varphi_\gamma}{\partial s} \geq 0$ and (2.1) with $\gamma M^{q-1} = M^{q-1/2-p/2}$, that goes to zero as t^* goes to T because $p > 2q - 1$. As before we claim that there exists a constant C such that

$$\frac{\partial \varphi_\gamma}{\partial s}(0,0) \geq C.$$

If not, passing to a subsequence and using Hopf Lemma, we find a nontrivial solution of $0 \leq \varphi \leq 1$, $\varphi(0) = 1$, and

$$\begin{cases} 0 = \varphi_{yy} - \lambda\varphi^p, \\ \varphi_y(0) = 0 \end{cases}$$

which is a contradiction (λ is negative).

As in the previous cases, from this inequality and the reverse one (that follows by $C^{2+\alpha,1+\alpha/2}$ regularity) it follows that

$$c \leq u(1,t)(T-t)^{\frac{1}{p-1}} = M(t)(T-t)^{\frac{1}{p-1}} \leq C. \qquad \square$$

3. Blow-up Profile

We begin by part a) so we are dealing with $p < 2q-1$. Let us introduce the similarity variables,

$$w(y,s) = (T-t)^{\frac{1}{2(q-1)}} u(x,t), \qquad y = \frac{1-x}{\sqrt{T-t}}, \qquad s = -\ln(T-t).$$

Then w satisfies the following equation and boundary conditions

$$\begin{cases} w_s = w_{yy} - \frac{y}{2} w_y - \frac{1}{2(q-1)} w - \lambda e^{-sk} w^p, \\ w_y(0,s) = -w^q(0,s), \\ w_y(e^{s/2}, s) = 0, \\ w(y, -\ln T) = T^{\frac{1}{2(q-1)}} u_0(1-y\sqrt{T}). \end{cases} \tag{3.1}$$

in a domain of the form $\Omega = \{(y,s); 0 < y < e^{s/2},\ s > -\ln T\}$, here $k = \frac{2q-1-p}{2(q-1)} > 0$ because $p < 2q-1$.

The corresponding stationary problem was studied in [**5**].

Lemma 3.1 ([**5**, Lemma 3.1]). *There is a unique positive bounded solution, w_0, of $0 = w_{yy} - \frac{y}{2} w_y - \frac{1}{2(q-1)} w$, $(y > 0)$ with the boundary condition $w_y(0) = -w^q(0)$ (for an explicit formula for w_0 see* [**5**]*).*

With this Lemma we can find the blow-up profile.

To prove Theorem 1.4 we have to prove that $w(y,s) \to w_0(y)$ as $s \to \infty$. We use ideas from [**6**].

Part a) of Theorem 1.2 and Theorem 1.3 implies that w is bounded and that $0 < c \le w(0,s) \le C$. Also from the proof of the blow-up rate and the maximum principle we obtain $u_t(\cdot,t) \le C(T-t)^{-\frac{2q-1}{2(q-1)}}$, $u_x(\cdot,t) \le C(T-t)^{-\frac{q}{2(q-1)}}$ and $u_{xx}(\cdot,t) \le C(T-t)^{-\frac{2q-1}{2(q-1)}}$. Hence w_{yy}, $\frac{y}{2} w_y$ and w_s are bounded.

Now we adapt arguments from Propositions 6 and 7 in [**6**]. Let s_j be a sequence tending to ∞ and let $w_j(y,s) = w(y, s+s_j)$. By the previous estimates there is a subsequence (that we still denote by w_j) such that $w_j(y,s) \to w_\infty(y,s)$ uniformly on compact sets and $(w_j)_y(y,m) \to (w_\infty)_y(y,m)$ pointwise in $\{y > 0\}$ for each integer m.

We have the identity

$$\int_0^R \rho(y) w_s (w_s + \lambda e^{-sk} w^p)(y,s)\,dy - \rho(R) w_s(R,s) w_y(R,s) = -\frac{d}{ds} E_R(w)(s)$$

where $\rho(y) = e^{-y^2/4}$ and

$$E_R(w)(s) = \frac{1}{2} \int_0^R \rho w_y^2\,dy + \frac{1}{4(q-1)} \int_0^R \rho w^2\,dy - \frac{1}{q+1} w^{q+1}(0,s).$$

Taking $R(s)=s$ we obtain

$$-\frac{d}{ds}E_s(w)(s)=\int_0^s w_s^2(y,s)\rho(y)dy-G(s)$$

where

$$G(s)=\rho(s)(\frac{1}{2}w_y^2(s,s)+\frac{1}{4(q-1)}w^2(s,s)+w_yw_s(s,s))-\int_0^s\lambda e^{-sk}w^p$$

Integration in time gives an identity which enables us to proceed exactly as in Proposition 6 of [**6**] to prove that w_∞ is independent of s, we leave the details to the reader. Also, in the same way as in [**6**] we obtain that w_∞ is a weak (hence strong) stationary solution of (3.1) and that $E_\infty(w_\infty)$ is independent of s_j. To verify that w_∞ is independent of s_j we use the fact that

$$E_\infty(w_0)>E_\infty(0).$$

Hence $w_\infty=w_0$ or $w_\infty=0$. To rule out the last possibility we only have to remark that $w(0,s)\to w_0(0)$ by the blow-up rate that we have proved in Section 2. □

To prove part b) of Theorem 1.4 we proceed just as before, but in this case the similarity variables are

$$w(y,s)=(T-t)^{\frac{1}{p-1}}u(x,t),\qquad y=\frac{1-x}{\sqrt{T-t}},\qquad s=-\ln(T-t),$$

and w satisfies

$$\text{(3.2)}\qquad\begin{cases} w_s=w_{yy}-\frac{y}{2}w_y-\frac{1}{p-1}w-\lambda w^p,\\ w_y(0,s)=-e^{sk}w^q(0,s),\\ w_y(e^{s/2},s)=0,\\ w(y,-\ln T)=T^{\frac{1}{p-1}}u_0(1-y\sqrt{T}).\end{cases}$$

in a domain of the form $\Omega=\{(y,s);0<y<e^{s/2},\ s>-\ln T\}$, in this case $k<0$ $(p>2q-1)$.

There is a unique positive bounded solution of the associated stationary problem $0=w_{yy}-\frac{y}{2}w_y-\frac{1}{p-1}w-\lambda w^p$, $(y>0)$ with the boundary condition $w_y(0)=0$ (this solution is the constant $(-\frac{1}{\lambda(p-1)})^{\frac{1}{p-1}}$, see [**6**]).

Now part c) of Theorem 1.3 implies that w is bounded and that $0<c\le w(0,s)\le C$. Also from Section 2 and the maximum principle we obtain that w_{yy}, $\frac{y}{2}w_y$ and w_s are bounded.

Hence we can proceed just as before with the identity

$$\int_0^R \rho(y)w_s^2(y,s)dy - \rho(R)w_s(R,s)w_y(R,s) + e^{sk}w_s w^q(0,s) = -\frac{d}{ds}E_R(w)(s)$$

where $\rho(y) = e^{-y^2/4}$ and

$$E_R(w)(s) = \frac{1}{2}\int_0^R \rho w_y^2 dy + \frac{1}{2(p-1)}\int_0^R \rho w^2 dy + \frac{\lambda}{p+1}\int_0^R \rho w^{p+1} ds.$$

We obtain a limit $w(y, s+s_j) \to w_\infty(y)$ that has to be independent of s and then w_∞ is a weak (hence strong) stationary solution of (3.2) and that $E_\infty(w_\infty)$ is independent of s_j. To verify that w_∞ is independent of s_j we use the fact that

$$E_\infty(w_0) > E_\infty(0).$$

We only have to observe that w_∞ is positive because by part c) of Theorem 1.3, $w(0,s) \geq c > 0$. □

References

1. Acosta G. and Rossi J. D., *Blow-up vs. global existence for quasilinear parabolic systems with a nonlinear boundary condition*, Z. Angew. Math. Phys. **48**(5) (1997), 711–724.
2. Chipot M., Fila M. and Quittner P., *Stationary solutions, blow up and convergence to stationary solutions for semilinear parabolic equations with nonlinear boundary conditions*, Acta Math. Univ. Comenianae **LX**(1) (1991), 35–103.
3. Deng K., *Blow-up rates for parabolic systems*, Z. Angew. Math. Phys. **46** (1995), 110–118.
4. Deng K., Fila M. and Levine H. A., *On critical exponents for a system of heat equations coupled in the boundary conditions*, Acta Math. Univ Comenianae **LXIII** (1994), 169–192.
5. Fila M. and Quittner P., *The blow-up rate for the heat equation with a nonlinear boundary condition*, Math. Meth. Appl. Sci. **14** (1991), 197–205.
6. Giga Y. and Kohn R. V., *Asymptotically self-similar blow-up of semilinear heat equations*, Comm. Pure Appl. Math. **38** (1985), 297–319.
7. ______, *Characterizing blow-up using similarity variables*, Indiana Univ. Math. J. **36** (1987), 1–40.
8. Gidas B. and Spruck J., *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations **6** (1981), 883–901.
9. Hu B. and Yin H. M., *The profile near blow-up time for the solution of the heat equation with a nonlinear boundary condition*, Trans. Amer. Math. Soc. **346**(1) (1995), 117–135.
10. Lieberman G. M., *Hölder continuity of the gradient of solutions of uniformly parabolic equations with conormal boundary conditions*, Ann. di Math. Pura ed Appl. **148** (1987), 77–99.
11. ______, *Second Order Parabolic Differential Equations*, World Sci, 1996.
12. Lopez Gomez J., Marquez V. and Wolanski N., *Dynamic behavior of positive solutions to reaction-diffusion problems with nonlinear absorption through the boundary*, Rev. Unión Matemática Argentina **38** (1993), 196–209.
13. Rossi J. D., *The blow-up rate for a system of heat equations with non-trivial coupling at the boundary*, Math. Meth. Appl. Sci. **20** (1997), 1–11.
14. Rossi J. D. and Wolanski N., *Global existence and nonexistence for a parabolic system with nonlinear boundary conditions*, Diff. Int. Eq. **11** (1998), 179–190.

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