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# A NONLINEAR EVOLUTION INCLUSION IN PERFECT PLASTICITY WITH FRICTION

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Abstract. We consider a nonlinear evolution inclusion associated with a time dependent convex set in a Hilbert space. We prove an existence and uniqueness result for the problem using classical results from the theory of evolution equations involving maximal monotone operators, a fixed point argument and a regularization method. We apply this result to a model for the quasistatic evolution process of a perfectly plastic body which is in frictional contact with a rigid foundation and obtain the existence of the unique stress field.

## 1. Introduction

In this paper we establish an existence and uniqueness theorem for an abstract Cauchy problem in a Hilbert space involving the indicator function of a time-dependent convex set. The main difficulty in studying the problem arises from this time dependence of the convex set. We deal with it by imposing sufficient regularity and compatibility on the problem data.

Problems of this type arise in models of quasistatic contact with friction of perfectly plastic materials. There the main unknown is the stress tensor and the time dependence of the convex set is a consequence of the time dependence of the volume forces and tractions.

Initial and boundary value problems for perfectly-plastic materials have been studied in [**7, 9, 14**] with classical displacements and tractions boundary conditions. An existence result for a one-dimensional problem using the Prandtl-Reuss plastic flow rule and unilateral conditions imposed on the velocity has been obtained recently in [**12**]. Moreover, the existence and uniqueness of the stress tensor and the existence of the velocity field for a quasistatic perfectly-plastic contact problem with Tresca's friction law was established in [**3**].

The aim of this paper is to extend some of the results in [**3**] to include various nonlinear boundary conditions with friction. To that end we establish an existence and uniqueness theorem for an abstract nonlinear evolution equation in a Hilbert space. Then we apply it to problems in quasistatic perfect plasticity with friction.

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The paper is structured as follows. In Section 2 we state the abstract problem, list the assumptions imposed on the data, state and then prove our main existence and uniqueness result, Theorem 2.1. The proof is based on arguments from the theory of variational inequalities, regularization and Banach's fixed point theorem. In Section 3 we present a general model for a class of problems for perfectly plastic bodies with friction and we formulate it as a variational inequality for the stress field. We apply Theorem 2.1 to this formulation and deduce the existence and uniqueness of the solution of the mechanical problem. Finally, in Section 4 we present a number of examples of friction laws to which our results apply.

## 2. An Abstract Existence Theorem

In this section we prove an abstract existence theorem for an evolution problem in a Hilbert space. We will apply it in the following sections to problems involving frictional contact of a perfectly plastic body with a rigid obstacle.

Let $H$ be a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle_H$ and the associated norm $|\cdot|_H$. Let $A\colon H \to H$ be an operator described below, let $K$ and $\Sigma_0$ be two closed convex sets in $H$ and let $\chi\colon [0,T] \to H$ be a given function. For $t \in [0,T]$, $(T > 0)$, $\Sigma(t)$ denotes the following convex set in $H$

$$\Sigma(t) = \Sigma_0 + \chi(t). \tag{2.1}$$

Let $y_0 \in H$ and denote by $\psi_{K \cap \Sigma(t)}\colon H \longrightarrow (-\infty, +\infty]$ the indicator function of the set $K \cap \Sigma(t)$.

We study the following Cauchy problem on $H$:

*Find* $y\colon [0,T] \to H$ *such that*

$$A\dot{y}(t) + \partial\psi_{K\cap\Sigma(t)}(y(t)) \ni f(t) \quad \text{a.e.} \quad t \in (0,T), \tag{2.2}$$

$$y(0) = y_0. \tag{2.3}$$

Here $f\colon [0,T] \to H$ is given, a dot above a variable represents the time derivative and $\partial\psi$ denotes the subdifferential of $\psi$.

To study problem (2.2) and (2.3) we make the following assumptions on the data: $A\colon H \longrightarrow H$ is a positive definite symmetric operator, i.e.,

$$\begin{aligned} &(a) \quad \langle Ax, x\rangle_H \geq m|x|_H^2 \quad \forall x \in H, \quad m > 0, \\ &(b) \quad \langle Ax, y\rangle_H = \langle Ay, x\rangle_H \quad \forall x, y \in H; \end{aligned} \tag{2.4}$$

$$f \in L^2(0,T;H); \tag{2.5}$$

$$\chi \in W^{1,\infty}(0,T;H); \tag{2.6}$$

$$y_0 = \chi(0) \in \Sigma(0). \tag{2.7}$$

Also, there exists a positive $\delta$ such that

$$\chi(t) + z \in K \quad \text{for all } t \in [0,T], \quad \text{for } z \in H, \quad |z|_H \leq \delta. \tag{2.8}$$

The main result of our paper is the following.

**Theorem 2.1.** *When assumptions* (2.4)–(2.8) *hold, there exists a unique solution* $y$ *of problem* (2.2) *and* (2.3), *such that*

$$y \in W^{1,2}(0,T;H).$$

The proof of the Theorem 2.1 is based on classical results from the theory of evolution equation with maximal monotone operators, fixed point arguments and a regularization method. It is carried out below in several steps. We suppose that $\eta \in L^2(0,T;H)$ and consider the following auxiliary problem:

*Find* $y_\eta : [0,T] \to H$ *such that*

$$A\dot{y}_\eta(t) + \eta(t) + \partial\psi_{\Sigma(t)}(y_\eta(t)) \ni f(t) \quad a.e.\ t \in (0,T), \tag{2.9}$$

$$y_\eta(0) = y_0. \tag{2.10}$$

**Lemma 2.2.** *Problem* (2.9) *and* (2.10) *has a unique solution* $y_\eta \in W^{1,2}(0,T;H)$.

*Proof.* We begin with a change of the dependent variable which leads to an evolution equation associated with the fixed convex set $\Sigma_0$. Let

$$\overline{y}_\eta = y_\eta - \chi. \tag{2.11}$$

It is straightforward to show that $y_\eta \in W^{1,2}(0,T;H)$ is a solution of (2.9) and (2.10) if and only if $\overline{y}_\eta \in W^{1,2}(0,T;H)$ and

$$A\dot{\overline{y}}_\eta(t) + \partial\psi_{\Sigma_0}(\overline{y}_\eta(t)) \ni f(t) - A\dot{\chi}(t) - \eta(t) \quad \text{a.e. } t \in (0,T), \tag{2.12}$$

$$\overline{y}_\eta(0) = 0. \tag{2.13}$$

We note that (2.1) and (2.7) imply that $0 \in \Sigma_0$. Moreover, $f - A\dot{\chi} - \eta \in L^2(0,T;H)$ by (2.4), (2.5) and (2.6). Therefore, by (2.4) and a classical result for evolution equations (see e.g. [**4**, p. 189]) we obtain the existence of a unique function $\overline{y}_\eta \in W^{1,2}(0,T;H)$ which solves problem (2.12) and (2.13). This concludes the proof. □

Let now $\mu > 0$ be a viscosity coefficient, and denote by $G_\mu : H \to H$ the operator

$$G_\mu(z) = \frac{1}{\mu}(z - P_K z) \qquad \forall z \in H, \tag{2.14}$$

where $P_K$ represents the projection on $K$. Below we denote by $C$ a positive constant which depends on $A$, $f$, $\chi$ and $T$, but does not depend on $\mu$, and whose value may change from line to line. We define $\Lambda_\mu : L^2(0,T;H) \longrightarrow L^2(0,T;H)$ by

$$\Lambda_\mu(\eta) = G_\mu(y_\eta) \tag{2.15}$$

where, for a given $\eta \in L^2(0,T;H)$, the function $y_\eta$ is the solution of the Cauchy problem (2.9) and (2.10) given by Lemma 2.2.

**Lemma 2.3.** *For each* $\mu > 0$ *the operator* $\Lambda_\mu$ *has a unique fixed point* $\eta_\mu$.

*Proof.* We use similar arguments to those in [**1, 2, 13**]. Let $\eta_1, \eta_2 \in L^2(0,T;H)$ and $t \in [0,T]$. Using (2.9), (2.10) and algebraic manipulations we find

$$|y_{\eta_1}(t) - y_{\eta_2}(t))|_H^2 \le C \int_0^t |\eta_1(s) - \eta_2(s)|_H^2\, ds.$$

Moreover, from (2.14) and (2.15) we obtain

$$|\Lambda_\mu(\eta_1(t)) - \Lambda_\mu(\eta_2(t))|_H \le \frac{2}{\mu} |y_{\eta_1}(t) - y_{\eta_2}(t)|_H.$$

Combining the last two inequalities yields

$$|\Lambda_\mu(\eta_1(t)) - \Lambda_\mu(\eta_2(t))|_H^2 \le \frac{C}{\mu^2} \int_0^t |\eta_1(s) - \eta_2(s)|_H^2\, ds.$$

This implies that for $p$ sufficiently large, a power $\Lambda_\mu^p$ of $\Lambda_\mu$ is a contraction on $L^2(0,T;H)$, which concludes the proof of Lemma 2.3. □

Next, we consider the following regularized problem:

*Find* $y_\mu \colon [0,T] \longrightarrow H$ *such that*

$$A\dot{y}_\mu(t) + G_\mu(y_\mu(t)) + \partial\psi_{\Sigma(t)}(y_\mu(t)) \ni f(t) \quad \textit{a.e. } t \in (0,T), \tag{2.16}$$

$$y_\mu(0) = y_0. \tag{2.17}$$

**Lemma 2.4.** *For* $\mu > 0$ *the Cauchy problem* (2.16) *and* (2.17) *has a unique solution* $y_\mu \in W^{1,2}(0,T;H)$.

*Proof.* Let $\eta_\mu$ be the unique fixed point of the operator $\Lambda_\mu$ and let $y_\mu \in W^{1,2}(0,T;H)$ be the solution of (2.9) and (2.10) for $\eta = \eta_\mu$. Then, $y_\mu$ is a solution of (2.16) and (2.17). The uniqueness of the solution is obtained from the uniqueness of the fixed point of $\Lambda_\mu$. □

We now obtain a priori estimates on the solutions $y_\mu$ of (2.16) and (2.17). Then we study the behavior of the solutions when $\mu \to 0$, which is the main ingredient of the regularization. Here and below by $\mu \to 0$ we mean any sequence $\mu_n \to 0$ for $n \to \infty$. To that end let $\mathcal{G}_\mu \colon H \to \mathbb{R}_+$ be given by

$$\mathcal{G}_\mu(z) = \frac{1}{2\mu} |z - P_K z|_H^2 \qquad \forall z \in H. \tag{2.18}$$

**Lemma 2.5.** *The sequence* $\{y_\mu\}$ *is bounded in* $L^\infty(0,T;H)$.

*Proof.* Applying (2.16) to $z - y_\mu(t)$ we obtain

$$\begin{aligned} \langle A\dot{y}_\mu(t), z - y_\mu(t)\rangle_H &+ \langle G_\mu(y_\mu(t)), z - y_\mu(t)\rangle_H \\ &\ge \langle f(t), z - y_\mu(t)\rangle_H \quad \forall z \in \Sigma(t), \end{aligned} \tag{2.19}$$

a.e. on $t \in (0,T)$. Let

$$\overline{y}_\mu = y_\mu - \chi. \tag{2.20}$$

Since $0 \in \Sigma_0$, by (2.1) we find $\chi(t) \in \Sigma(t)$ for all $t \in [0,T]$. Therefore, choosing $z = \chi(t)$ in (2.19) and using (2.20) and (2.1) we get

$$\begin{aligned} \langle A\dot{\overline{y}}_\mu(t), \overline{y}_\mu(t)\rangle_H &+ \langle G_\mu(y_\mu(t)), \overline{y}_\mu(t)\rangle_H \\ &\le \langle f(t) - A\dot{\chi}(t), \overline{y}_\mu(t)\rangle_H \quad \text{a.e.} \quad t \in (0,T). \end{aligned} \tag{2.21}$$

Now, the operator $G_\mu$, given by (2.14), is the Gâteaux-derivative of the convex function $\mathcal{G}_\mu$ given by (2.18); therefore,

$$\mathcal{G}_\mu(z) - \mathcal{G}_\mu(y) \ge \langle G_\mu(y),\ z - y\rangle_H \qquad \forall y, z \in H. \tag{2.22}$$

Using (2.8) and (2.18) we obtain $\mathcal{G}_\mu(\chi(t)) = 0$ for all $t \in [0,T]$. It follows from (2.22) that

$$\langle G_\mu(y_\mu(t)), \overline{y}_\mu(t)\rangle_H \ge \mathcal{G}_\mu(y_\mu(t)) \qquad \forall t \in [0,T]. \tag{2.23}$$

Then, (2.21) and (2.23) imply that

$$\langle A\dot{\overline{y}}_\mu(t), \overline{y}_\mu(t)\rangle_H + \mathcal{G}_\mu(y_\mu(t)) \le \langle f(t) - A\dot{\chi}(t),\ \overline{y}_\mu(t)\rangle_H \ \text{ a.e. } t \in (0,T). \tag{2.24}$$

From (2.20), (2.17) and (2.7) we have $\overline{y}_\mu(0) = 0$, and since $\mathcal{G}_\mu(y_\mu(t)) \ge 0$ for all $t \in [0,T]$, the previous inequality and (2.4) lead to

$$|\overline{y}_\mu(s)|_H^2 \le C \int_0^s |f(t) - A\dot{\chi}(t)|_H |\overline{y}_\mu(t)|_H \, dt \qquad \forall s \in [0,T]. \tag{2.25}$$

Lemma 2.5 follows now from (2.20) and (2.25). □

**Lemma 2.6.** *The sequence $\{G_\mu(y_\mu)\}$ is bounded in $L^1(0,T;H)$.*

*Proof.* Using (2.21) and Lemma 2.5 we find

$$\int_0^T \langle G_\mu(y_\mu(t)), \overline{y}_\mu(t)\rangle_H dt \le C. \tag{2.26}$$

Let $z \in H$ such that $|z|_H \le \delta$. Using (2.8) we have $\chi(t) + z \in K$ which implies that $\mathcal{G}_\mu(\chi(t) + z) = 0$ for all $t \in [0,T]$. Therefore, we obtain from (2.20) and (2.22)

$$\langle G_\mu(y_\mu(t)), z\rangle_H \ \le \ \langle G_\mu(y_\mu(t)), \overline{y}_\mu(t)\rangle_H \qquad \forall t \in [0,T]. \tag{2.27}$$

Thus, from (2.27) we find

$$\begin{aligned} |G_\mu(y_\mu(t))|_H &= \frac{1}{\delta} \sup_{|z|_H \le \delta} \langle G_\mu(y_\mu(t)), z\rangle_H \\ &\le \frac{1}{\delta} \langle G_\mu(y_\mu(t)), \overline{y}_\mu(t)\rangle_H, \end{aligned} \tag{2.28}$$

for all $t \in [0,T]$. Lemma 2.6 follows now from (2.26) and (2.28). □

**Lemma 2.7.** *The sequence $\{\dot{y}_\mu\}$ is bounded in $L^2(0,T;H)$.*

*Proof.* Using (2.19) we find

$$\langle A\dot{y}_\mu(t) + G_\mu(y_\mu(t)) - f(t), z - y_\mu(t)\rangle_H \ \ge 0 \qquad \forall z \in \Sigma(t), \qquad \text{a.e. } t \in (0,T),$$

and by (2.1) and (2.20) we deduce

$$\langle A\dot{\overline{y}}_\mu(t) + A\dot{\chi}(t) + G_\mu(y_\mu(t)) - f(t), z - \overline{y}_\mu(t)\rangle_H \ \ge 0 \qquad \forall z \in \Sigma_0, \quad \text{a.e. } t \in (0,T).$$

This implies

$$\langle A\dot{\overline{y}}_\mu(t) + A\dot{\chi}(t) + G_\mu(y_\mu(t)) - f(t),\ \dot{\overline{y}}_\mu(t)\rangle_H = 0 \quad \text{a.e.} \quad t \in (0,T). \tag{2.29}$$

Integration of (2.29) over $[0,T]$, Lemma 2.6 and simple manipulations result in

$$|\dot{\overline{y}}_\mu|^2_{L^2(0,T;H)} + \int_0^T \langle G_\mu(y_\mu(t)), \dot{y}_\mu(t)\rangle_H \, dt \le C + C|\dot{\overline{y}}_\mu|_{L^2(0,T;H)}. \tag{2.30}$$

Then by using (2.7), (2.8) and (2.17) we obtain

$$\int_0^T \langle G_\mu(y_\mu(t)), \dot{y}_\mu(t)\rangle_H \, dt = \mathcal{G}_\mu(y_\mu(T)) - \mathcal{G}_\mu(y_\mu(0)) \ge -\mathcal{G}_\mu(y_0) = 0. \tag{2.31}$$

Lemma 2.7 follows now from (2.30) and (2.31), since $\dot{y}_\mu = \dot{\overline{y}}_\mu + \dot{\chi}$. □

We have all the ingredients needed to prove Theorem 2.1.

*Proof of Theorem* 2.1.

**Existence.** Using Lemmas 2.5 and 2.7 we deduce that there exists an element $y \in W^{1,2}(0,T;H)$ such that, for a subsequence which is still denoted by $\{y_\mu\}$, when $\mu \to 0$ we have

$$y_\mu \longrightarrow y \quad \text{weak}^\star \text{ in } L^\infty(0,T;H), \tag{2.32}$$

$$\dot{y}_\mu \longrightarrow \dot{y} \quad \text{weakly in } L^2(0,T;H). \tag{2.33}$$

We show that $y$ is the solution of problem (2.2) and (2.3). First, we note that (2.32) and (2.33) imply

$$y_\mu(t) \longrightarrow y(t) \quad \text{weakly in } H \quad \forall t \in [0,T]. \tag{2.34}$$

It follows from (2.16) and (2.34) that $y(t) \in \Sigma(t)$ for all $t \in [0,T]$. Moreover, we have from (2.22)

$$\mathcal{G}_\mu(y(t)) \le \mathcal{G}_\mu(y_\mu(t)) - \langle G_\mu(y(t)), y_\mu(t) - y(t)\rangle_H$$

for all $t \in [0,T]$ and, using (2.14), we obtain

$$\mu \int_0^T \mathcal{G}_\mu(y(t))\, dt \le \mu \int_0^T \mathcal{G}_\mu(y_\mu(t))\, dt - \int_0^T \langle y(t) - P_K y(t), y_\mu(t) - y(t)\rangle_H \, dt. \tag{2.35}$$

Using (2.24) and Lemma 2.5 yields

$$\int_0^T \mathcal{G}_\mu(y_\mu(t))\, dt \le C. \tag{2.36}$$

It now follows from (2.18), (2.32), (2.35) and (2.36) that

$$\int_0^T |y(t) - P_K y(t)|^2_H \, dt = 0,$$

which implies that

$$y(t) \in K \quad \forall t \in \ [0,T]. \tag{2.37}$$

Now, using (2.19) and (2.22) we have

$$\begin{aligned}\langle A\dot{y}_\mu(t), z - y_\mu(t)\rangle_H &+ \mathcal{G}_\mu(z) - \mathcal{G}_\mu(y_\mu(t)) \\ &\geq \langle f(t), z - y_\mu(t)\rangle_H \quad \forall z \in \Sigma(t), \text{ a.e. } t \in (0,T).\end{aligned}$$

Therefore, since $\mathcal{G}_\mu(y_\mu(t)) \geq 0$, we find

$$\langle A\dot{y}_\mu(t), z - y_\mu(t)\rangle_H \geq \langle f(t), z - y_\mu(t)\rangle_H \qquad \forall z \in K \cap \Sigma(t), \quad \text{a.e.} \quad t \in (0,T).$$

Choosing $z \in L^2(0,T;H)$, such that $z \in K \cap \Sigma(t)$ a.e. $t \in (0,T)$ in the previous inequality and integrating over $[0,s]$ we get

$$\int_0^s \langle A\dot{y}_\mu, z\rangle_H \, dt \geq \int_0^s \langle A\dot{y}_\mu, y_\mu\rangle_H \, dt + \int_0^s \langle f, z - y_\mu\rangle_H \, dt \qquad \forall s \in [0,T].$$

Using this inequality, (2.32), (2.33) and the lower-semicontinuity of the norm we deduce

$$\int_0^s \langle A\dot{y}, z\rangle_H \, dt \geq \int_0^s \langle A\dot{y}, y\rangle_H \, dt + \int_0^s \langle f, z - y\rangle_H \, dt \qquad \forall s \in [0,T]. \tag{2.38}$$

We now apply in (2.38) the Lebesgue point argument for an $L^1$ function, and obtain

$$\begin{aligned}\langle A\dot{y}(t), z(t)\rangle_H \; &\geq \langle A\dot{y}(t), y(t)\rangle_H \; + \langle f(t), z(t) - y(t)\rangle_H \\ &\forall z \in K \cap \Sigma(t), \text{ a.e. } \; t \in (0,T).\end{aligned} \tag{2.39}$$

Inequality (2.2) follows now from $y(t) \in \Sigma(t)$ for all $t \in [0,T]$, (2.37) and (2.39). Finally, (2.3) is a consequence of (2.17) and (2.34). We conclude that $y \in W^{1,2}(0,T;H)$ is a solution of the Cauchy problem (2.2) and (2.3).

**Uniqueness.** To prove uniqueness, let $\{y_i\}$ for $i = 1,2$, be two solutions of (2.2) and (2.3), then

$$\langle A\dot{y}_1(t), z - y_1(t)\rangle_H \; \geq \langle f(t), z - y_1(t)\rangle_H,$$

$$\langle A\dot{y}_2(t), z - y_2(t)\rangle_H \; \geq \langle f(t), z - y_2(t)\rangle_H,$$

for all $z \in K \cap \Sigma(t)$, a.e. $t \in (0,T)$. It is straightforward to show that

$$\langle A\dot{y}_1(t) - A\dot{y}_2(t), \; y_1(t) - y_2(t)\rangle_H \leq 0 \quad \text{a.e.} \quad t \in (0,T). \tag{2.40}$$

Integration of (2.40) over $[0,s]$ and (2.3) yield

$$|y_1(s) - y_2(s)|_H^2 \leq 0 \qquad \forall s \in [0,T]. \tag{2.41}$$

The uniqueness of the solution is now a consequence of (2.41). □

## 3. Stress Formulation in Perfect Plasticity

In this section we consider a quasistatic problem which describes the frictional contact between an elastic perfectly plastic body and a rigid foundation. We obtain a variational formulation for the model and, by applying Theorem 2.1, we establish the existence and uniqueness of the solution.

The mechanical state of a perfectly plastic body which occupies the domain $\Omega \subset \mathbb{R}^M$ $(M = 2, 3)$ evolves over the time interval $[0, T]$, for $T > 0$, due to body forces and surface tractions which act on it. The surface $\Gamma = \partial\Omega$, which is assumed to be Lipschitz continuous, is decomposed into three disjoint (measurable) parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$, such that $meas\,\Gamma_D > 0$. The body is clamped on $\Gamma_D \times (0, T)$, surface tractions $\mathbf{f}_N$ act on $\Gamma_N \times (0, T)$ and volume forces of density $\mathbf{f}_A$ act in $\Omega \times (0, T)$. We assume slow variation of the external forces and so the accelerations may be neglected, leading to a quasistatic approximation of the process. The body is in frictional contact with a rigid obstacle on $\Gamma_C \times (0, T)$, which we model by a subdifferential type of inequality. The mechanical problem of frictional contact may be formulated classically as follows:

*Find a displacement field* $u\colon \Omega \times [0, T] \longrightarrow \mathbb{R}^M$ *and a stress field* $\sigma\colon \Omega \times [0, T] \longrightarrow S_M$ *such that*

$$\mathcal{A}\dot{\sigma} + \partial\psi_K(\sigma) \ni \varepsilon(\dot{u}) \qquad \text{in} \quad \Omega \times (0, T), \tag{3.1}$$

$$\operatorname{Div}\sigma + \mathbf{f}_A = 0 \qquad \text{in} \quad \Omega \times (0, T), \tag{3.2}$$

$$u = 0 \qquad \text{on} \quad \Gamma_D \times (0, T), \tag{3.3}$$

$$\sigma\nu = \mathbf{f}_N \qquad \text{on} \quad \Gamma_N \times (0, T), \tag{3.4}$$

$$u \in U, \quad \sigma\nu \cdot (v - \dot{u}) \geq \varphi(\dot{u}) - \varphi(v) \quad \forall v \in U \qquad \text{on} \quad \Gamma_C \times (0, T), \tag{3.5}$$

$$u(0) = u_0, \qquad \sigma(0) = \sigma_0 \qquad \text{in} \quad \Omega. \tag{3.6}$$

Here, $S_M$ represents the space of second order symmetric tensors on $\mathbb{R}^M$, (3.1) is the elastic-perfectly plastic constitutive law in which $K$ is the set of elastic stresses, $\mathcal{A}$ is the tensor of elastic compliance, $\partial\psi_K$ denotes the subdifferential of the indicator function $\psi_K\colon S_M \longrightarrow (-\infty, +\infty]$ and $\nu$ denotes the outward unit normal to $\Omega$ on $\Gamma$. The inequality (3.5) models the frictional contact conditions, $U$ represents the set of admissible test functions, $\sigma\nu$ denotes the Cauchy stress vector and $\varphi$ is a given function. In the next section we shall present a number of concrete examples of friction laws which may be cast in the form (3.5) by appropriate choice of $\varphi$ and $U$.

To obtain a weak formulation of (3.1)-(3.6) we introduce the following notation and present preliminary material. For further details, we refer to [**5, 6, 11**]. We denote by "$\cdot$" and $|\cdot|$ the inner product and the Euclidean norm on $S_M$ and $\mathbb{R}^M$, respectively. We also use the spaces

$$H = \{\ u = (u_i) \mid u_i \in L^2(\Omega)\ \}, \quad \mathcal{H} = \{\ \sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\ \},$$

$$H_1 = \{\ u = (u_i) \mid u_i \in H^1(\Omega)\ \}, \quad \mathcal{H}_1 = \{\ \sigma \in \mathcal{H} \mid \sigma_{ij,j} \in H\ \}.$$

Here and below $i, j = 1, \dots, M$, summation over repeated indices is implied and an index that follows a comma indicates a partial derivative. $H$, $\mathcal{H}$, $H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with their canonical inner products given by

$$\langle u, v\rangle_H = \int_\Omega u_i v_i \, dx, \quad \langle \sigma, \tau\rangle_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\, dx,$$

$$\langle u, v\rangle_{H_1} = \langle u, v\rangle_H + \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}, \quad \langle \sigma, \tau\rangle_{\mathcal{H}_1} = \langle \sigma, \tau\rangle_{\mathcal{H}} + \langle \mathrm{Div}\ \sigma, \mathrm{Div}\ \tau\rangle_H,$$

respectively, where $\varepsilon\colon H_1 \to \mathcal{H}$ and $\mathrm{Div}\colon \mathcal{H}_1 \to H$ are the **deformation** and the **divergence** operators, respectively, defined by

$$\varepsilon(v) = (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad \mathrm{Div}\ \sigma = (\sigma_{ij,j}).$$

The associated norms on the spaces $H$, $\mathcal{H}$, $H_1$ and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$, respectively.

For every element $v \in H_1$ we also use the notation $v$ to denote the trace of $v$ on $\Gamma$ and we denote by $v_\nu$ and $v_\tau$ the **normal** and the **tangential** components of $v$ on $\Gamma$ given by

$$v_\nu = v \cdot \nu, \qquad v_\tau = v - v_\nu \nu. \tag{3.7}$$

We also denote by $\sigma_\nu$ and $\sigma_\tau$ the **normal** and the **tangential** traces of $\sigma \in \mathcal{H}_1$ on $\Gamma_C$ (see, e.g., [**11**]), and note that when $\sigma$ is a regular function,

$$\sigma_\nu = \sigma\nu \cdot \nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu, \tag{3.8}$$

and the Green formula holds:

$$\langle \sigma, \varepsilon(v)\rangle_{\mathcal{H}} + \langle \mathrm{Div}\ \sigma, v\rangle_H = \int_\Gamma \sigma\nu \cdot v \, da \qquad \forall v \in H_1. \tag{3.9}$$

Let $V$ be a closed subspace of $H_1$ given by

$$V = \{\ v \in H_1 \mid v = 0 \ \text{ on } \ \Gamma_1\ \}.$$

By assumption $meas\,\Gamma_D > 0$, and thus the Korn inequality holds, i.e., there exists $C > 0$ depending only on $\Omega$ and $\Gamma_D$ such that

$$|\varepsilon(u)|_{\mathcal{H}} \geq C|u|_{H_1} \qquad \forall u \in V \tag{3.10}$$

(see, e.g., [**10**, p. 79]). Next, the inner product $\langle\cdot,\cdot\rangle_V$ on $V$ is given by

$$\langle u, v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}, \tag{3.11}$$

then $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on $V$. Therefore, $(V, |\cdot|_V)$ is a real Hilbert space.

We make the following assumptions on the data of problem (3.1)–(3.6).

The operator $\mathcal{A}\colon \Omega \times S_M \longrightarrow S_M$ is a symmetric positive definite tensor, i.e.,

$$\begin{aligned} &(a)\mathcal{A}_{ijkh} \in L^\infty(\Omega)\ i, j, k, h = 1, \dots, M; \\ &(b)\mathcal{A}\sigma \cdot \tau = \sigma \cdot \mathcal{A}\tau \quad \sigma,\ \tau \in S_M,\ \text{ a.e. in } \Omega; \\ &(c) \text{ there exists } \alpha > 0 \text{ such that } \mathcal{A}\sigma\cdot\sigma \geq \alpha|\sigma|^2,\ \sigma \in S_M,\ \text{ a.e. in } \Omega. \end{aligned} \tag{3.12}$$

$$K \subset S_M \text{ is a closed convex set such that } 0 \in K. \tag{3.13}$$

The forces and the tractions satisfy

$$\mathbf{f}_A \in W^{1,\infty}(0,T;H), \qquad \mathbf{f}_N \in W^{1,\infty}(0,T;L^2(\Gamma_N)^M). \tag{3.14}$$

Let $\mathcal{K}$ denote the set

$$\mathcal{K} = \{\ \tau \in \mathcal{H} \mid \tau(x) \in K \ \text{ a.e. } \ x \in \Omega\ \} \tag{3.15}$$

and, for all $t \in [0,T]$, let $\mathbf{F}(t)$ be the element of $V$ given by

$$\langle \mathbf{F}(t), v\rangle_V = \int_\Omega \mathbf{f}_A(t)\cdot v\,dx + \int_{\Gamma_N} \mathbf{f}_N(t)\cdot v\,da \qquad \forall v \in V. \tag{3.16}$$

We suppose that $U \subset H_1$, $\varphi\colon \Gamma_C \times \mathbb{R}^M \longrightarrow \mathbb{R}$, and define the functional $j\colon H_1 \to (-\infty, +\infty]$ by

$$j(v) = \begin{cases} \displaystyle\int_{\Gamma_C} \varphi(v)\,da & \text{if } \varphi(v) \in L^1(\Gamma_C), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.17}$$

We also denote by $D(j)$ the effective domain of $j$ in $V \cap U$, i.e.,

$$D(j) = \{\ v \in V \cap U \mid j(v) < +\infty\ \},$$

and, for all $t \in [0,T]$, let $\Sigma(t)$ be given by

$$\Sigma(t) = \{\ \tau \in \mathcal{H} \mid \langle \tau, \varepsilon(v)\rangle_{\mathcal{H}} + j(v) \geq \langle \mathbf{F}(t), v\rangle_V \ \forall v \in D(j)\ \}. \tag{3.18}$$

In the sequel we denote by $\chi(t) = \varepsilon(\mathbf{F}(t))$ and assume that there exists $\delta > 0$ such that

$$\chi(t) + \tau \in \mathcal{K} \quad \text{for all } t \in [0,T], \quad \text{for } \tau \in \mathcal{H}, \quad |\tau|_{\mathcal{H}} \leq \delta, \tag{3.19}$$

$$D(j) = V \cap U \quad \text{and} \quad V \cap U \text{ is a subspace of } H_1, \tag{3.20}$$

$$j(v) \geq 0 \quad \text{and} \quad j(\alpha v) = \alpha j(v) \quad \forall v \in D(j),\ \alpha \geq 0, \tag{3.21}$$

$$\sigma_0 = \chi(0). \tag{3.22}$$

We note that assumptions (3.19) and (3.22) represent a compatibility condition on the problem data similar to the one used in [**7, 9, 14**]. Moreover, $j$ is a positively homogeneous functional on $V \cap U$.

Next, we obtain a variational formulation of the mechanical problem (3.1)–(3.6). To this end, we assume that $\{u, \sigma\}$ are regular functions satisfying (3.1)–(3.6) and let $v \in D(j)$, $t \in [0,T]$. Using (3.9) and (3.2) we have

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} = \langle \mathbf{f}_A(t), v - \dot{u}(t)\rangle_H + \int_\Gamma \sigma(t)\nu \cdot (v - \dot{u}(t))\,da,$$

and, using (3.3), (3.4) and (3.16), we find

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} = \langle \mathbf{F}(t), v - \dot{u}(t)\rangle_V + \int_{\Gamma_C} \sigma(t)\nu \cdot (v - \dot{u}(t))\,da. \tag{3.23}$$

Moreover, it follows from (3.5) and (3.17) that

$$\int_{\Gamma_C} \sigma(t)\nu \cdot (v - \dot{u}(t))\, da \geq j(\dot{u}(t)) - j(v). \tag{3.24}$$

Using now (3.23) and (3.24) we obtain

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} + j(v) - j(\dot{u}(t)) \geq \langle \mathbf{F}(t), v - \dot{u}(t)\rangle_V. \tag{3.25}$$

Choosing now $v = 2\dot{u}(t)$ and $v = 0$ in (3.25) and using (3.21), we deduce

$$\langle \sigma(t), \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} + j(\dot{u}(t)) = \langle \mathbf{F}(t), \dot{u}(t)\rangle_V. \tag{3.26}$$

Thus, from (3.25) and (3.26) it follows that $\sigma(t) \in \Sigma(t)$ and since by (3.1) and (3.15) $\sigma(t) \in \mathcal{K}$, we obtain

$$\sigma(t) \in \mathcal{K} \cap \Sigma(t). \tag{3.27}$$

Using now (3.18) and (3.26) we find

$$\langle \tau - \sigma(t), \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} \geq 0 \qquad \forall \tau \in \Sigma(t), \tag{3.28}$$

and from (3.1) it follows that

$$\langle \mathcal{A}\dot{\sigma}(t), \tau - \sigma(t)\rangle_{\mathcal{H}} \geq \langle \tau - \sigma(t), \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} \qquad \forall \tau \in \mathcal{K}. \tag{3.29}$$

We conclude from (3.28) and (3.29) that

$$\langle \mathcal{A}\dot{\sigma}(t), \tau - \sigma(t)\rangle_{\mathcal{H}} \geq 0 \qquad \forall \tau \in \mathcal{K} \cap \Sigma(t). \tag{3.30}$$

Therefore, by (3.6), (3.27) and (3.30) we obtain the following variational formulation in terms of the stress field of the problem (3.1)–(3.6):

*Find* $\sigma\colon [0,T] \longrightarrow \mathcal{H}_1$ *such that*

$$\mathcal{A}\dot{\sigma}(t) + \partial\psi_{\mathcal{K}\cap\Sigma(t)}(\sigma(t)) \ni 0 \qquad \text{a.e.} \quad t \in (0,T), \tag{3.31}$$

$$\sigma(0) = \sigma_0. \tag{3.32}$$

Here, $\partial\psi_{\mathcal{K}\cap\Sigma(t)}$ denotes the subdifferential of the indicator function $\psi_{\mathcal{K}\cap\Sigma(t)}\colon \mathcal{H} \longrightarrow (-\infty, +\infty]$, for all $t \in [0,T]$.

The main result in this section is the following.

**Theorem 3.1.** *Under the assumptions* (3.12)–(3.14), (3.19)–(3.22), *there exists a unique solution* $\sigma$ *of the Cauchy problem* (3.31) *and* (3.32) *such that* $\sigma \in W^{1,2}(0,T;\mathcal{H})$. *Moreover, if* $\mathcal{D}(\Omega)^M \subset U$, *then* Div $\sigma \in W^{1,\infty}(0,T;H)$.

We conclude from Theorem 3.1 the existence of a unique weak solution, in terms of stress, to the mechanical problem (3.1)–(3.6).

*Proof.* Let $\Sigma_0 = \{\, \tau \in \mathcal{H} \mid \langle \tau, \varepsilon(v)\rangle_{\mathcal{H}} + j(v) \geq 0 \ \forall v \in D(j) \,\}$. Then $\Sigma_0$ is a closed convex set in $\mathcal{H}$ and, using (3.11), (3.18) and keeping in mind that $\chi = \varepsilon(\mathbf{F})$, we find

$$\Sigma(t) = \Sigma_0 + \chi(t) \qquad \forall t \in [0,T]. \tag{3.33}$$

Moreover, it follows from (3.15) that $\mathcal{K}$ is a closed convex set in $\mathcal{H}$ and from (3.14) and (3.16) we deduce $\mathbf{F} \in W^{1,\infty}(0,T;V)$ which implies

$$\chi \in W^{1,\infty}(0,T;\mathcal{H}). \tag{3.34}$$

Since $j(v) \geq 0\ \forall v \in D(j)$ (see (3.21)), it follows that $0 \in \Sigma_0$ and from (3.33) and (3.22) we find

$$\sigma_0 = \chi(0) \in \Sigma(0). \tag{3.35}$$

Using now (3.33), (3.34), (3.12) (3.19) and (3.35), we may apply Theorem 2.1 with the space $\mathcal{H}$ and $f = 0$. Thus, we obtain the existence and uniqueness of $\sigma \in W^{1,2}(0,T;\mathcal{H})$ which satisfies (3.31) and (3.32). Since $\sigma(t) \in \mathcal{K} \cap \Sigma(t)$ for all $t \in [0,T]$, it follows from (3.18) that

$$\langle \sigma(t), \varepsilon(v)\rangle_{\mathcal{H}} + j(v) \geq \langle \mathbf{F}(t), v\rangle_V \quad \forall v \in D(j),\ t \in [0,T].$$

Next, we remark that if $\mathcal{D}(\Omega)^M \subset U$ then $\mathcal{D}(\Omega)^M \subset V \cap U = D(j)$ (see (3.20)). Therefore, taking $v = \pm\varphi \in \mathcal{D}(\Omega)^M$ in the previous inequality and using (3.16) and (3.17) we deduce

$$\mathrm{Div}\ \sigma(t) + \mathbf{f}_A(t) = 0 \ \text{ in } \ \Omega, \quad \forall t \in [0,T]. \tag{3.36}$$

The fact that Div $\sigma \in W^{1,\infty}(0,T;H)$ is now a consequence of (3.36) and (3.14). □

## 4. Examples of Friction Laws

In this section we present a number of friction laws which lead to inequalities of the form (3.5) and for which conditions (3.20) and (3.21) hold. We conclude from Theorem 3.1 the existence of a unique solution for each one of the quasistatic contact problems with these frictional boundary conditions.

**Example 4.1.** *Bilateral contact with Tresca's friction law.*

We consider, following, e.g., [**1, 3, 5, 8**], the boundary condition when contact is maintained and the friction bound is prescribed on $\Gamma_C \times (0,T)$. Thus,

$$\begin{aligned} &u_\nu = 0, && |\sigma_\tau| \leq g, \\ &|\sigma_\tau| < g && \Rightarrow \dot{u}_\tau = 0, \\ &|\sigma_\tau| = g && \Rightarrow \dot{u}_\tau = -\lambda\sigma_\tau, \quad \text{for some } \ \lambda \geq 0. \end{aligned} \tag{4.1}$$

Here $g$ represents the friction bound, i.e., the magnitude of the limiting friction traction at which slip begins. We assume $g \in L^\infty(\Gamma_C)$ and $g \geq 0$ a.e. on $\Gamma_C$. The contact described with (4.1) is bilateral, i.e., there is no loss of contact during the process. The set of admissible test functions $U$ is given by

$$U = \{\ v \in H_1 \mid v_\nu = 0 \quad \text{on} \quad \Gamma_C\},$$

and, using (3.7) and (3.8), it is straightforward to show that if $\{u, \sigma\}$ is a pair of regular functions satisfying (4.1) then

$$\sigma\nu \cdot (v - \dot{u}) \geq g|\dot{u}_\tau| - g|v_\tau| \qquad \forall v \in U,$$

a.e. on $\Gamma_C \times (0,T)$. Thus, (3.5) holds with the choice $\varphi(v) = g|v_\tau|$ and we obtain from (3.17) that

$$j(v) = \int_{\Gamma_C} g|v_\tau|da \qquad \forall v \in H_1.$$

The assumptions (3.20) and (3.21) hold in this case. Therefore, Theorem 3.1 yields the existence and uniqueness of a function $\sigma$ which represents a weak solution in terms of the stress for the mechanical problem (3.1)–(3.4), (3.6), (4.1). The existence of a weak solution in terms of the velocity for the same problem has been proved in [**3**], using the properties of the spaces $BD(\Omega)$.

**Example 4.2.** *Frictionless contact problem with damped normal response.*

We consider now a contact problem similar to that proposed in [**5**, p. 147]. It consists of the following boundary conditions on $\Gamma_C \times (0,T)$:

$$\begin{array}{ll} g_1 \le \sigma_\nu \le g_2, & \\ g_1 < \sigma_\nu < g_2 & \Rightarrow \dot{u}_\nu = 0, \\ \sigma_\nu = g_1 & \Rightarrow \dot{u}_\nu \ge 0, \\ \sigma_\nu = g_2 & \Rightarrow \dot{u}_\nu \le 0, \\ \sigma_\tau = 0. & \end{array} \tag{4.2}$$

Here $g_1$ and $g_2$ are prescribed functions in $L^\infty(\Gamma_C)$ which satisfy $g_1 \le 0 \le g_2$, a.e. on $\Gamma_C$. It is straightforward to show that if $\{u, \sigma\}$ is a pair of regular functions satisfying (4.2), then (3.5) holds with

$$U = H_1, \qquad \varphi(v) = g_2 v_\nu^- - g_1 v_\nu^+.$$

Here, $r^+$ and $r^-$ represent the positive and the negative part of $r$, respectively, given by $r^+ = \max\{r, 0\}$, $r^- = \max\{-r, 0\}$. Then it follows from (3.17) that

$$j(v) = \int_{\Gamma_C} (g_2 v_\nu^- - g_1 v_\nu^+)da \qquad \forall v \in H_1.$$

The assumptions (3.20) and (3.21) are satisfied; hence, we may apply Theorem 3.1 to the mechanical problem (3.1)–(3.4), (3.6), (4.2) and conclude that it has a unique weak solution, in terms of the stress.

**Example 4.3.** *Damped normal response and Tresca's friction law.*

This is a combination of (4.1) and (4.2) above. The boundary conditions on $\Gamma_C \times (0,T)$ are the following:

$$\begin{array}{lll} g_1 \le \sigma_\nu \le g_2, & & \\ g_1 < \sigma_\nu < g_2 & \Rightarrow \dot{u}_\nu = 0, & \\ \sigma_\nu = g_1 & \Rightarrow \dot{u}_\nu \ge 0, & \\ \sigma_\nu = g_2 & \Rightarrow \dot{u}_\nu \le 0, & \\ |\sigma_\tau| \le g, & & \\ |\sigma_\tau| < g & \Rightarrow \dot{u}_\tau = 0, & \\ |\sigma_\tau| = g & \Rightarrow \dot{u}_\tau = -\lambda\sigma_\tau & \text{for some } \lambda \ge 0. \end{array} \tag{4.3}$$

Here $g_1$, $g_2$ and $g$ are prescribed functions lying in $L^\infty(\Gamma_C)$ which satisfy $g \ge 0$, $g_1 \le 0 \le g_2$, a.e. on $\Gamma_C$. The boundary conditions (4.3) imply (3.5) with

$$U = H_1, \qquad \varphi(v) = g_2 v_\nu^- - g_1 v_\nu^+ + g|v_\tau|,$$

where $r^+$ and $r^-$ are defined as in example 4.2. Using (3.17) we obtain

$$j(v) = \int_{\Gamma_C} (g_2 v_\nu^- - g_1 v_\nu^+ + g|v_\tau|)\, da \qquad \forall v \in H_1.$$

We conclude that the mechanical problem (3.1)–(3.4), (3.6) and (4.3) has a unique weak solution, in terms of the stress, by Theorem 3.1, since the assumptions (3.20) and (3.21) are satisfied.

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