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THE APPLICATION OF PICONE-TYPE IDENTITY FOR SOME NONLINEAR ELLIPTIC DIFFERENTIAL EQUATIONS

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Abstract. We established a Picone-type identity for the second order partial differential equation

$$(*) \qquad \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(r_i(x)\varphi\left(\frac{\partial u}{\partial x_i}\right)\right) + c(x)\varphi(u) = 0, \quad \varphi(u) := |u|^{p-1}u,\ p > 0.$$

Using this identity we prove the Leighton-type comparison theorem for a pair of equations of the form (*). Properties of the principal eigenvale of a certain eigenvalue problem associated with (*) are investigated as well.

1. Introduction

The recently established Picone identity [**10**] for the so-called p-Laplacian

$$(1) \qquad \Delta_p u := \operatorname{div}(||\nabla u||^{p-1}\nabla u), \quad p > 0,$$

turned out to be a very useful tool in the extension of various important properties of solutions of equations associated with the classical Laplacian (which correspond to $p = 1$ in (1)) to equations associated with p-Laplacian Δ_p. We refer to papers [**1, 2, 3, 16**] and the references given therein and also to the survey paper [**24**].

In this paper we deal with equations associated with another homogeneous operator extending the Laplace operator, namely with the operator

$$\tilde{\Delta}_p^{[r]} u := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(r_i(x)\varphi\left(\frac{\partial u}{\partial x_i}\right)\right),$$

where $\varphi(s) := |s|^{p-1}s$, $p > 0$, and $r_i(x)$, $i = 1, \dots, N$, are positive functions in a domain $\Omega \subseteq \mathbb{R}^N$. We show that the Picone identity for $\Delta_p^{[r]}$ established in [**7**] in case $r_i(x) \equiv 1$, $i = 1, \dots, N$, can be extended also to the more general operator (1), and using this identity we investigate oscillatory and spectral properties of

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the equation

$$q[u] := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(r_i(x)\varphi\left(\frac{\partial u}{\partial x_i}\right)\right) + c(x)\varphi(u) = 0. \tag{2}$$

We suppose that $r_i \in C^1(\Omega) \cap C(\bar{\Omega})$, $r_i(x) > 0$ in Ω and $c \in C(\bar{\Omega})$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain.

Throughout the paper, we use the standard notation. The scalar product in $\mathbb{R}^N$ is denoted by $\cdot$ and $||x|| = \sqrt{x \cdot x} = \sqrt{x_1^2 + \cdots + x_N^2}$ is the Euclidean norm in $\mathbb{R}^N$. For a function of N variables $u = u(x) = u(x_1, x_2 \ldots, x_N)$, $u_{x_i} = \frac{\partial u}{\partial x_i}$, $i = 1, 2, \ldots, N$, $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_N}\right)$, div $= \sum_{i=1}^{N} \frac{\partial}{\partial x_i}$. Moreover, $V(\nabla u) = (\varphi(u_{x_1}), \ldots, \varphi(u_{x_N}))^T$ is N-dimensional column vector.

In our paper we are not directly concerned with existence and regularity results, so we assume, for the sake of presentional convenience that the coefficients in the investigated equations and the boundary of the domain under consideration, unless specified otherwise, are smooth enough to guarantee the required regularity of solutions.

The paper is arranged as follows. In the next section we prove the Picone identity for a pair of differential operators of the form (2). Section 3 is devoted to the Leighton-type comparison theorem and to some of its consequences. These results are used in Section 4 to investigate Dirichlet eigenvalue problem associated with the operator $\tilde{\Delta}_p^{[r]}$.

2. Picone's identity

Together with (2) consider another equation of the same form

$$Q[u] := \operatorname{div}(R(x)V(\nabla u)) + C(x)\varphi(u) = 0, \tag{3}$$

where $R(x) = \operatorname{diag}\{R_1(x), \ldots, R_N(x)\}$ is a diagonal matrix. The functions R_i, $i = 1, \ldots, N$, and C satisfy the same assumptions as r_i and c in (2). Similarly, we denote by $r(x) = \operatorname{diag}\{r_1(x), \ldots, r_N(x)\}$ the diagonal matrix with coefficients of (2) and with this notation equation (2) can be written in the form

$$q[u] = \operatorname{div}(r(x)V(\nabla u)) + c(x)\varphi(u) = 0.$$

The classical Picone identity established by Picone [**21**] states that if x, y, px', and Py' are continuously differentiable functions on an interval I and $y(t) \neq 0$ in this interval, then

$$\begin{aligned} &\frac{d}{dt}\left[\frac{x}{y}\left(y\,px' - x\,Py'\right)\right] \\ &= (p-P)\,x'^2 + P\left(x' - \frac{x}{y}y'\right)^2 + x\,(px')' - \frac{x^2}{y}\,(Py')'. \end{aligned}$$

This identity (or its various modifications) is a powerful tool in the oscillation theory of second order linear ordinary differential equations ([**6, 14, 17**]), of second order nonlinear differential equations ([**15**]) and of fourth order nonlinear differential equations ([**18**]). The Picone's identity has been also extended to partial differential equations, see [**1, 2, 3, 10, 16, 20, 23**].

Let Ω be a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ has a piecewise continuous unit normal. Recall that we suppose that $r_i, R_i \in C^1(\Omega)\cap C(\bar{\Omega})$, $i = 1, \ldots, N$, and $c, C \in C(\bar{\Omega})$. The domain $\mathcal{D}_Q$ of Q is defined as the set of all real-valued functions $v \in C^1(\Omega)$ such that all derivatives involved in $Q[v]$ exist and are continuous at every point in Ω. A solution of equation (3) is a real valued function $v \in \mathcal{D}_Q$ satisfying (3) at every point in Ω. A similar definition is applied to the domain $\mathcal{D}_q$ of q and solutions of $q[u] = 0$.

Theorem 1. *If $u \in \mathcal{D}_q$, $v \in \mathcal{D}_Q$ and $v \neq 0$ in Ω, then*

$$
\begin{aligned}
&\operatorname{div}\left\{\frac{u}{\varphi(v)}\left[\varphi(v)\, r(x)\, V(\nabla u) - \varphi(u)\, R(x)\, V(\nabla v)\right]\right\} \\
&= \nabla u\, \left[r(x)\, V(\nabla u) - R(x)\, V(\nabla u)\right] + \left[C(x) - c(x)\right] u\, \varphi(u) \qquad (4)\\
&\quad + \frac{u}{\varphi(v)}\left[\varphi(v)\, q[u] - \varphi(u)\, Q[v]\right] \\
&+ \left[\nabla u\cdot R(x)\, V(\nabla u) + p\frac{u}{v}\varphi\left(\frac{u}{v}\right)\nabla v\cdot R(x)\, V(\nabla v) - (p+1)\varphi\left(\frac{u}{v}\right)\nabla u\cdot R(x)\, V(\nabla v)\right]
\end{aligned}
$$

Proof. Multiplying (2) by u we get

$$
\begin{aligned}
u\, q[u] &= u\operatorname{div}\left(r(x)\, V(\nabla u)\right) + c(x)\, u\, \varphi(u) \\
&= \operatorname{div}\left[u\, r(x)\, V(\nabla u)\right] - \nabla u\cdot r(x)\, V(\nabla u) + c(x)\, u\, \varphi(u)
\end{aligned}
$$

and hence

$$
\operatorname{div}\left[u\, r(x)\, V(\nabla u)\right] = \nabla u\cdot r(x)\, V(\nabla u) - c(x)\, u\, \varphi(u) + \frac{u}{\varphi(v)}\left[\varphi(v)\ q[u]\right]. \qquad (5)
$$

Similarly

$$
\begin{aligned}
&\operatorname{div}\left[\frac{u}{\varphi(v)}\left(\varphi(u)\, R(x)\, V(\nabla v)\right)\right] = \operatorname{div}\left[u\, \varphi(u)\right]\cdot\frac{R(x)\, V(\nabla v)}{\varphi(v)} \\
&\quad + u\, \varphi(u)\operatorname{div}\left(\frac{R(x)\, V(\nabla v)}{\varphi(v)}\right) \\
&= \nabla u\cdot\varphi(u)\,\frac{R(x)\, V(\nabla v)}{\varphi(v)} + u\,\frac{\varphi'(u)}{\varphi(v)}\,\nabla u\cdot R(x)\, V(\nabla v) \\
&\quad - u\, \varphi(u)\,\frac{\varphi'(v)}{\varphi^2(v)}\,\nabla v\cdot R(x)\, V(\nabla v) + u\,\frac{\varphi(u)}{\varphi(v)}\operatorname{div}\left(R(x)\, V(\nabla v)\right).
\end{aligned}
$$

Applying (5) and the connection $\varphi'(v) = p\,\frac{\varphi(v)}{v}$, we have

$$
\begin{aligned}
\operatorname{div}\left[\frac{u\varphi(u)}{\varphi(v)}\left(R(x)\, V(\nabla v)\right)\right] &= \nabla u\cdot R(x)\, V(\nabla u) - \Phi\left[u, v\right] - u\, \varphi(u)\, C(x) \qquad (6)\\
&\quad + u\, \varphi\left(\frac{u}{v}\right) Q\left[v\right],
\end{aligned}
$$

where

$$\Phi\,[u,v] \;=\; \nabla u\cdot R(x)\,V(\nabla u)+p\,\frac{u}{v}\,\varphi\left(\frac{u}{v}\right)\,\nabla v\cdot R(x)\,V(\nabla v) \qquad (7)$$
$$-(p+1)\,\varphi\left(\frac{u}{v}\right)\,\nabla u\cdot R(x)\,V(\nabla v).$$

Combining (5) and (6) we obtain

$$\begin{aligned}&\operatorname{div}\left\{\frac{u}{\varphi(v)}\,[\varphi(v)\,R(x)\,V(\nabla u)-\varphi(u)\,R(x)\,V(\nabla v)]\right\}\\ =\;&\nabla u\;[(r(x)-R(x))\,V(\nabla u)]+[C(x)-c(x)]\;u\,\varphi(u)\\ &+\frac{u}{\varphi(v)}\,[\varphi(v)\,q\,[u]-\varphi(u)\,Q\,[v]]+\Phi\,[u,v]\end{aligned}$$

what has to be proved. □

Remark 1. In the previous theorem the matrices r, R are diagonal with positive diagonal entries r_i, R_i, $i=1,\dots,N$. A closer examination of the proof of Theorem 1 reveals that its statement, identity (4), remains valid if we replace diagonal matrices r, R by any $N\times N$ matrices. However, in the later application of this identity, the crucial is played by the inequality

$$x\cdot R\,V(x)-(p+1)\,x\cdot R\,V(y)+p\,y\cdot R\,V(y)\geq 0, \qquad (8)$$

where $V(x)=(\varphi(x_1),\dots,\varphi(x_N))^T$, $V(y)=(\varphi(y_1),\dots,\varphi(y_N))^T$, which we were able to prove in the next section only for diagonal R with positive diagonal entries R_i. Therefore, if one would be able to prove (8) also for general positive definite matrix R, our result of the next section could be extended also to (3) with general positive definite matrix $R(x)$. Note also that in the linear case $p=1$ (8) holds for any symmetric positive definite matrix R and this is also the reason why in the linear case comparison theorems, eigenvalue problems, etc. are treated for elliptic equations with the operator $\operatorname{div}(R(x)\nabla u)$ with general symmetric positive definite matrix R.

3. Comparison theorems

The classical Leighton comparison theorem [**19**] concerns the pair of Sturm-Liouville differential equations of the form $(r(t)x')'+c(t)x=0$. Here we prove a similar statement for a pair of partial differential equations (2), (3).

Theorem 2. *If there exists a nontrivial solution u of $q\,[u]=0$ such that $u=0$ on $\partial\Omega$ and*

$$M\,[u]=\int_\Omega\left\{\nabla u\cdot[(r(x)-R(x))\,V(\nabla u)]+(C(x)-c(x))\,|u|^{p+1}\right\}dx\geq 0, \qquad (9)$$

then every solution v of $Q\,[v]=0$ has to vanish at some point of Ω except in the case $u=kv$, where k is a real constant.

Proof. Since $u \in \mathcal{D}_q$ and $u = 0$ on $\partial\Omega$, there exists a sequence $u_n \in C_0^\infty(\Omega)$ such that $\|u_n - u\| \to 0$ as $n \to \infty$ in the $W_0^{1,p+1}(\Omega)$ norm

$$\|w\| = \left(\int_\Omega \left[\sum_{i=1}^N |w_{x_i}|^{p+1} + |w|^{p+1} \right] dx \right)^{\frac{1}{p+1}}. \tag{10}$$

If $v \neq 0$ throughout Ω, we can use the Picone-type identity (4) in the form

$$\begin{aligned} &\nabla \left\{ \frac{u}{\varphi(v)} \left[\varphi(v)\, r(x)\, V(u) - \varphi(u)\, R(x)\, V(v) \right] \right\} \\ &= F[u] + \frac{u}{\varphi(v)} \left[\varphi(v)\, q[u] - \varphi(u)\, Q[v] \right] + \Phi[u,v] \\ &= F[u] + \Phi[u,v]. \end{aligned} \tag{11}$$

where

$$F[u] = \nabla u \cdot [(r(x) - R(x)) V(\nabla u)] + (C(c) - c(x))|u|^{p+1},$$

i.e.

$$M[u] = \int_\Omega F[u]\, dx.$$

Using the Young inequality ([**13**, Theorem 41]) which holds for any $X, Y \in \mathbb{R}$ and $p > 0$

$$|X|^{p+1} + p\,|Y|^{p+1} - (p+1) X\,|Y|^{p-1}\, Y \geq 0 \tag{12}$$

with equality if and only if $X = Y$, we have got for $\Phi[u,v]$ that
(13)

$$\Phi[u,v] = \sum_{i=1}^N R_i(x) \left[|u_{x_i}|^{p+1} + p \left| \frac{u}{v} v_{x_i} \right|^{p+1} - (p+1) \left| \frac{u}{v} v_{x_i} \right|^{p-1} \frac{u}{v} v_{x_i}\, u_{x_i} \right] \geq 0$$

since by (12) with $X = u_{x_i}$, $Y = \frac{u}{v} v_{x_i}$

$$|u_{x_i}|^{p+1} + p \left| \frac{u}{v} v_{x_i} \right|^{p+1} - (p+1) \left| \frac{u}{v} v_{x_i} \right|^{p-1} \frac{u}{v} v_{x_i}\, u_{x_i} \geq 0$$

and $R_i(x) > 0$ for all $i = 1, \dots, N$. Equality holds in (13) if and only if

$$\frac{u}{v} v_{x_i} = u_{x_i}, \quad i = 1, \dots, N, \tag{14}$$

i.e. $v \left(\frac{u}{v} \right)_{x_i} = 0$ which implies that $\frac{u}{v} = k$ in Ω. As $Q[v] = 0$ in Ω, $u_n = 0$ on $\partial\Omega$, and (9) holds, it follows upon integration of (11) over Ω (and using the Gauss theorem) that

$$M[u_n] = \int_\Omega F[u_n]\, dx = -\int_\Omega \Phi[u_n, v]\, dx \leq 0. \tag{15}$$

In view of our hypotheses on the functions r_i, R_i, c and C, these functions are bounded in Ω. Hence there exists a constant $\alpha > 0$ independent of n such that

$$|M[u_n] - M[u]| \le \alpha \left[\int\limits_\Omega \sum_{i=1}^N \left| |(u_n)_{x_i}|^{p+1} - |u_{x_i}|^{p+1} \right| dx + \int\limits_\Omega \left| |u_n|^{p+1} - |u|^{p+1} \right| dx \right]$$

Then an application of the Young's inequality and the inequality

$$\left| |a|^{p+1} - |b|^{p+1} \right| \le (p+1)(|a|+|b|)^p \, |a-b| \tag{16}$$

(which follows from the Lagrange mean value theorem) for a, $b \in \mathbb{R}$ and $p > 0$, we have the estimate

$$\begin{aligned} &\sum_{i=1}^N \int\limits_\Omega \left| |(u_n)_{x_i}|^{p+1} - |u_{x_i}|^{p+1} \right| dx \\ &\le (p+1) \sum_{i=1}^N \int\limits_\Omega (|(u_n)_{x_i}| + |u_{x_i}|)^p \, |(u_n)_{x_i} - u_{x_i}| \, dx \\ &\le (p+1) \sum_{i=1}^N \left[\int\limits_\Omega (|u_{x_i}| + |u_{x_i}|)^{p+1} dx \right]^{\frac{p}{p+1}} \left[\int\limits_\Omega (|(u_n)_{x_i} - u_{x_i}|)^{p+1} dx \right]^{\frac{1}{p+1}} \\ &\le N(p+1) \left(\|u_n\| + \|u\| \right)^p \|u_n - u\| . \end{aligned}$$

Moreover

$$\begin{aligned} &\int\limits_\Omega \left| |u_n|^{p+1} - |u|^{p+1} \right| dx \le (p+1) \int\limits_\Omega (|u_n| + |u|)^p \, |u_n - u| \, dx \\ &\le (p+1) \left[\int\limits_\Omega (|u_n| + |u|)^{p+1} dx \right]^{\frac{p}{p+1}} \left[\int\limits_\Omega (|u_n - u|)^{p+1} dx \right]^{\frac{1}{p+1}} \\ &\le (p+1) \left(\|u_n\| + \|u\| \right)^p \|u_n - u\| . \end{aligned}$$

Therefore

$$|M[u_n] - M[u]| \le \beta \left(\|u_n\| + \|u\| \right)^p \|u_n - u\| , \tag{17}$$

where the constant β depends on α, N and p. From (17) it follows that

$$\lim_{n\to\infty} M[u_n] = M[u] ,$$

and we get from (15) that $M[u] \le 0$ which together with (9) implies that $M[u] = 0$. Let S be an arbitrary domain with $\overline{S} \subset \Omega$. Then (15) implies that

$$\int\limits_S F[u_n] \, dx = - \int\limits_S \Phi[u_n, v] \, dx \le 0. \tag{18}$$

Next we show that

$$\int\limits_S \Phi[u_n, v] \, dx \to \int\limits_S \Phi[u, v] \, dx \quad \text{as } n \to \infty. \tag{19}$$

Applying Young's inequality and (16), since the functions R_i, $i = 1, \dots, N$, are bounded in Ω, we obtain

$$\left| \int_S \Phi\,[u_n, v]\, dx - \int_S \Phi\,[u, v]\, dx \right|$$

$$= \left| \int_S \sum_{i=1}^{N} R_i(x) \left\{ \left[\left| u_{n_{x_i}} \right|^{p+1} - |u_{x_i}|^{p+1} \right] + p \left[\left| \frac{u_n}{v} v_{x_i} \right|^{p+1} - \left| \frac{u}{v} v_{x_i} \right|^{p+1} \right] \right. \right.$$

$$\left. \left. -(p+1) \left[\left| \frac{u_n}{v} v_{x_i} \right|^{p-1} u_n\, u_{n_{x_i}} - \left| \frac{u}{v} v_{x_i} \right|^{p-1} u\, u_{x_i} \right] \frac{v_{x_i}}{v} \right\} dx \right|$$

$$\leq \gamma \left(\left\| \frac{u_n}{v} \right\| + \left\| \frac{u}{v} \right\| \right)^p \left\| \frac{u_n - u}{v} \right\|,$$

where γ is a constant independent of n.

From the last computation we get that (19) holds as $\|u_n - u\| \to 0$. Therefore from (19) and (18) we get that

$$\int_S \Phi\,[u, v]\, dx = 0,$$

and hence $\Phi[u, v] = 0$ in S. Since S is arbitrary, $\Phi[u, v] = 0$ throughout Ω. This implies that (14) holds, i.e. v is a constant multiple of u what we needed to prove. □

A generalization of the Sturm-Picone theorem can be formulated in the following version:

Corollary 1. *The conclusion of Theorem 2 holds if* $r_i(x) \geq R_i(x)$ $i = 1, 2, ..., N$, *and* $c(x) \leq C(x)$ *on* $x \in \Omega$.

Let us denote $U(\Omega) := \{u \in \mathcal{D}_Q : u = 0 \text{ on } \partial\Omega\}$ and let us define the functional $J : U(\Omega) \to \mathbb{R}$ by

$$J\,[u] := \int_\Omega \left(\nabla u\, R(x)\, V(u) - C(x)\, |u|^{p+1} \right) dx.$$

Then we have the following corollary (Leighton-Swanson generalized variational theorem).

Corollary 2. *If there exists a nontrivial function* $u \in \mathcal{D}_Q$ *such that* $u = 0$ *on the boundary* $\partial\Omega$ *and*

$$J\,[u] \leq 0,$$

then every solution $v \in \mathcal{D}_Q$ *of* $Q\,[v] = 0$ *has to vanish at some point of* Ω, *unless* v *is a constant multiple of* u.

Proof. The existence of a positive (negative) solution v in Ω of $Q[v] = 0$ leads via Picone's identity, using the same argument as in the proof of Theorem 2, to the identity $\Phi[u, v] = 0$ in Ω which implies that v is a constant multiple of u. □

Now we can formulate the following version of Wirtinger's inequality:

Corollary 3. *If there exists a solution v of $Q[v] = 0$ such that $v \neq 0$ in Ω, then for all $u \in U(\Omega)$*

$$J[u] \geq 0,$$

where equality holds if and only if u is a constant multiple of v.

Remark 2. In Remark 1 we observed that in the proof of Theorem 2 we needed inequality (8) which we were able to prove only for diagonal matrix R. However, in the proof of Theorem 2 diagonality of the matrix r has been never used and we actually formulated the result for diagonal r just only because of symmetry reason (symmetry between operators Q and q). Therefore, the statement of Theorem 2 remains valid if we suppose that $r \in C^1(\Omega) \cap C(\bar{\Omega})$ is any symmetric $N \times N$ matrix for which (9) holds.

4. Eigenvalue problem

Consider the eigenvalue problem

$$\begin{cases} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(|u_{x_i}|^{p-1} u_{x_i} \right) + \lambda g(x) |u|^{p-1} u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{20}$$

where Ω is an open simply connected bounded domain and $g \in L^{N/p+1}(\Omega) \cap L^{\infty}_{loc}(\Omega)$ is a weight function which is positive in Ω. The following additional notation will be used

$$\Phi_1[u, v] \; = \; \sum_{i=1}^{N} \left[|u_{x_i}|^{p+1} + p \left| \frac{u}{v} v_{x_i} \right|^{p+1} - (p+1) \left| \frac{u}{v} v_{x_i} \right|^{p-1} \frac{u}{v} u_{x_i} v_{x_i} \right],$$

i.e., $\Phi_1[u, v] = \Phi[u, v]$, with Φ defined by (7) and $R(x) = I$, i.e., $R_i(x) = 1$, $i = 1, \dots, N$.

In this section we present the application of the Picone-type identity (4) for proving some properties of the eigenfunctions and eigenvalues of problem (20). In the case $p = 1$, the eigenvalue problem (20) is equivalent to the well-known problem

$$\begin{cases} \Delta u + \lambda g(x) u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{21}$$

Eigenvalue problem (21) has been studied extensively in the literature, see e.g. [**5**], and many results have been extended to p-Laplacian Δ_p in [**4, 8, 9**] and the references given therein.

Here we consider only weak solutions; a nontrivial function u is said to be a (week) *eigenfunction* of (20) and λ is corresponding *eigenvalue* if $u \in W_0^{1,p+1}(\Omega)$ and

$$\int_{\Omega} \left\{ \sum_{i=1}^{N} |u_{x_i}|^{p-1} u_{x_i} v_{x_i} \right\} dx = \lambda \int_{\Omega} g(x) |u|^{p-1} u v \, dx$$

for all $v \in W_0^{1,p+1}(\Omega)$.

The assumption on the function g guaranties that solution of our eigenvale problem is actually of the class $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$, see [**25**]. Moreover, from the Harnack type inequality of [**26**] it follows that if $u \geq 0$ in Ω, then actually $u(x) > 0$ for all $x \in \Omega$ and every eigenvalue λ is positive. The first (smallest) eigenvalue is of special importance. It is defined as the minimum of the quotient

$$\lambda_1 := \frac{\int\limits_\Omega \sum_{i=1}^N |u_{x_i}|^{p+1}}{\int\limits_\Omega g(x)|u|^{p+1}\,dx}.$$

If the weight function g is supposed to be essentially bounded in Ω, the eigenfunction u_1 associated with the first eigenvalue λ_1 does not change its sign, see [**11**].

We show the simplicity of the eigenvalue λ_1 associated to the eigenfunction $u_1 > 0$ (or $u_1 < 0$) in Ω, i.e. the positive eigenfunction corresponding to λ_1 is unique up to a constant multiple.

Theorem 3. *Let $v \in C^1(\Omega)$, $v(x) > 0$ for $x \in \Omega$ and*

$$\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(|v_{x_i}|^{p-1}\, v_{x_i}\right) + \lambda g(x)|v|^{p-1}u \leq 0, \quad x \in \Omega, \tag{22}$$

for some $\lambda > 0$. Then for every $u \in W_0^{1,p+1}(\Omega)$ with $u(x) \geq 0$ in Ω we have

$$\int_\Omega \sum_{i=1}^N |u_{x_i}|^{p+1} \geq \lambda \int\limits_\Omega g(x)|u|^{p+1}\,dx \tag{23}$$

and $\lambda \leq \lambda_1$. The equality in (23) *holds if and only if $\lambda = \lambda_1$, u is a constant multiple of v, and v is a constant multiple of u_1. Moreover, the first eigenvalue of* (20) *is simple.*

Proof. Let $u \in W_0^{1,p+1}(\Omega)$, $u \geq 0$ in Ω, be arbitrary and let $\Omega_S \subset \Omega$ be such that $\bar{\Omega}_S \subset \Omega$. Let $s_n \in C_0^\infty(\Omega)$, $s_n(x) \geq 0$ in Ω_S, $s_n \to u$ in $W_0^{1,p+1}(\Omega)$ as $n \to \infty$. We apply the identity

$$\frac{\partial}{\partial x_i}\left(\frac{|s_n|^{p+1}}{|v|^{p-1}v}\right)|v_{x_i}|^{p-1}\,v_{x_i} = (p+1)\left|\frac{s_n v}{v_{x_i}}\right|^{p-1}\frac{s_n}{v}v_{x_i}(s_n)_{x_i} - p\left|\frac{s_n}{v}v_{x_i}\right|^{p+1}$$

Note that the fact that $s_n \in C_0^\infty(\Omega)$ eliminates possible irregularities of the boundary $\partial\Omega$. We have

$$\begin{aligned} 0 \leq \Phi_1[s_n, v] \quad &= \quad \int\limits_{\Omega_S} \sum_{i=1}^N \left[|(s_n)_{x_i}|^{p+1} + p\left|\frac{s_n}{v}v_{x_i}\right|^{p+1} - \right. \\ &\left. -(p+1)\left|\frac{s_n}{v}v_{x_i}\right|^{p-1}\frac{s_n}{v}v_{x_i}(s_n)_{x_i}\right]dx \leq \end{aligned}$$

$$\leq \int_\Omega \sum_{i=1}^N \left[|(s_n)_{x_i}|^{p+1} - \frac{\partial}{\partial x_i}\left(\frac{|s_n|^{p+1}}{|v|^{p-1}v}\right) |v_{x_i}|^{p-1} v_{x_i}\right] dx$$

$$\leq \int_\Omega \sum_{i=1}^N |(s_n)_{x_i}|^{p+1}\, dx + \int_\Omega \frac{|s_n|^{p+1}}{|v|^{p-1}v} \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(|v_{x_i}|^{p-1} v_{x_i}\right) dx$$

and hence by (22)

$$\Phi_1[s_n, v] \leq \int_\Omega \sum_{i=1}^N |(s_n)_{x_i}|^{p+1}\ dx - \int_\Omega \lambda g(x)|s_n|^{p+1}\, dx.$$

As $s_n \to u$ in $W_0^{1,p+1}(\Omega)$, then using the continuity of the mapping

$$u \longmapsto \int_\Omega \left\{ \sum_{i=1}^N |u_{x_i}|^{p+1} - g(x)|u|^{p+1} \right\} dx$$

which can be proved using the same arguments as the proof of continuity of the mapping $u \longmapsto M[u]$ in Theorem 2, we have

$$\int_\Omega \left\{ \sum_{i=1}^N |(s_n)_{x_i}|^{p+1} - \lambda g(x)|s_n|^{p+1} \right\} dx \to \int_\Omega \left\{ \sum_{i=1}^N |u_{x_i}|^{p+1} - \lambda g(x)|u|^{p+1} \right\} dx,$$

as $n \to \infty$, hence

$$0 \leq \Phi[u, v] \leq \int_\Omega \left\{ \sum_{i=1}^N |u_{x_i}|^{p+1} - \lambda g(x)|u|^{p+1} \right\} dx$$

which is the same as (23). Now suppose that in (23) equality holds for some $u \in W_0^{1,p+1}(\Omega)$ which is nonnegative in Ω, i.e.

$$\int_\Omega \sum_{i=1}^N |u_{x_i}|^{p+1}\ dx = \lambda_1 \int_\Omega g(x)|u|^{p+1}. \tag{24}$$

Then from the previous computation follows

$$0 \leq \int_\Omega \Phi_1[u, v]\, dx \leq \int_\Omega \left\{ \sum_{i=1}^N |u_{x_i}|^{p+1} - \lambda g(x)|u|^{p+1} \right\} dx = 0,$$

hence $\Phi_1[u, v] = 0$ in Ω_S and since $\Omega_S \subset \Omega$ was arbitrary, $\Phi_1[u, v] = 0$ in Ω which means that $u = kv$ for some $k \in \mathbb{R}$ in view of (14). Finally, since (24) holds for $u = u_1$, u_1 being the eigenfunction corresponding to λ_1, (23) implies that $\lambda \leq \lambda_1$. Moreover $v \in W_0^{1,p+1}(\Omega)$ and $v > 0$ in Ω, hence substituting v for u and u_1 for v in the previous considerations, we get $\Phi[v, u_1] = 0$ in Ω which means that $v = ku_1$ and this also implies simplicity of the principal eigenvalue λ_1. □

Now we show the monotonical dependence of $\lambda_1(\Omega)$ with respect to Ω.

Theorem 4. *Let $\Omega_0 \subset \Omega$ and $\Omega_0 \neq \Omega$, and denote by $\lambda_1(\Omega_0)$ the first eigenvalue of* (20) *with Ω_0 instead of Ω. Then*

$$\lambda_1(\Omega_0) > \lambda_1(\Omega).$$

Proof. Let us denote by u_1^0 the positive eigenfunction of (20) associated with $\lambda_1(\Omega_0)$ and let $s_n \in C_0^\infty(\Omega)$, $s_n \to u_1^0$ in $W_0^{1,p+1}(\Omega)$. Then similarly as in the previous proof

$$0 \leq \int\limits_\Omega \Phi_1[s_n, u_1]\,dx \leq \int\limits_{\Omega_0} \sum_{i=1}^N |s_{n_{x_i}}|^{p+1}\,dx - \int\limits_{\Omega_0} \lambda_1(\Omega)|s_n|^{p+1}\,dx$$

and letting $n \to \infty$ we get

$$\begin{aligned} 0 \quad &\leq \quad \Phi[u_1^0, u_1] \leq \int\limits_{\Omega_0} \left\{ \sum_{i=1}^N \left|(u_1^0)_{x_i}\right| - \lambda_1(\Omega) g(x)|u_1^0|^{p+1} \right\} dx \\ &= \quad [\lambda_1(\Omega_0) - \lambda_1(\Omega)] \int\limits_{\Omega_0} g(x)|u_1^0|^{p+1}\,dx. \end{aligned}$$

This implies that $\lambda_1(\Omega_0) - \lambda_1(\Omega) \geq 0$. If $\lambda_1(\Omega_0) = \lambda_1(\Omega)$, then $\Phi_1[u_1^0, u_1] = 0$, i.e., u_1^0 is a constant multiple of u_1, which is impossible in the domain $\Omega_0 \subset \Omega$ and $\Omega_0 \neq \Omega$. □

The first eigenfunction u_1 has a special property; it is the only positive (or only negative) eigenfunction in Ω. Any eigenfunction u of (20) associated to the eigenvalue $\lambda \neq \lambda_1$ changes its sign in Ω.

Theorem 5. *Let $\lambda > \lambda_1$ be an eigenvalue of the eigenvalue problem* (20) *and u be the associated eigenfunction in Ω. Then any eigenfunction u has to vanish at some point of Ω.*

Proof. Let us assume that $u > 0$ in Ω, then

$$\begin{aligned} 0 \quad &\leq \quad \Phi_1[u_1, u] \leq \int\limits_\Omega \sum_{i=1}^N |\frac{\partial}{\partial x_i} u_1|^{p+1}\,dx - \lambda \int\limits_\Omega g(x)|u_1|^{p+1}\,dx \\ &\leq \quad (\lambda_1 - \lambda) \int\limits_\Omega g(x)|u_1|^{p+1}\,dx \leq 0. \end{aligned}$$

From this follows that $\Phi_1[u_1, u] = 0$, i.e., u_1 is a constant multiple of u, This is a contradiction. Therefore, every solution u of (20) has to vanish at some point of Ω. □

Remark 3. In our treatment we suppose, for the sake of simplicity, that the weight function g is positive in Ω. The eigenvalue problem for p-Laplacian Δ_p with indefinite weight has been investigated in several recent papers [**1, 2, 3**]. A subject of the present investigation is the modification of methods used in these papers to be applicable to our operator $\tilde{\Delta}_p u = \sum_{i=1}^N \frac{\partial}{\partial x_i}\varphi(u_{x_i})$. We concentrate our attention also to some other problems associated with the operator $\tilde{\Delta}_p$ like description of higher eigenvalues and nodal domains of corresponding eigenfunctions, Fučík spectrum, boundary value problems, etc.

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