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UNBOUNDED BASINS OF ATTRACTION OF LIMIT CYCLES

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Abstract. Consider a dynamical system given by a system of autonomous ordinary differential equations. In this paper we provide a sufficient local condition for an unbounded subset of the phase space to belong to the basin of attraction of a limit cycle. This condition also guarantees the existence and uniqueness of such a limit cycle, if that subset is compact. If the subset is unbounded, the positive orbits of all points of this set either are unbounded or tend to a unique limit cycle.

1. Introduction

Equilibria and periodic orbits are the simplest invariant sets in dynamical systems. While equilibria are – at least in principle – easy to determine as zeros of the right hand side of the differential equation there is no straight-forward way to find periodic orbits in general. Besides the existence and uniqueness of periodic orbits, one is also interested in their stability properties. Given an asymptotically stable periodic orbit we can define its basin of attraction consisting of all points which eventually are attracted by the periodic orbit. Our goal, in this paper, is to determine unbounded basins of attraction.

There are a number of approaches to prove existence of periodic orbits, e.g. by perturbation theory or the method of averaging (cf. [**4**], [**9**], [**16**], [**17**]). In two-dimensional systems the Poincaré-Bendixson theory can be used to show existence of periodic orbits. The Bendixson criterion for nonexistence of periodic orbits and its generalizations (cf. [**18**]) are tools to prove uniqueness.

Classical results concerning the stability are provided by linearization around the periodic orbit (cf. [**1**], [**4**], [**17**]). By the Floquet theory a necessary and sufficient condition for a periodic orbit to be exponentially asymptotically stable is that all Floquet exponents except for the trivial one have strictly negative real parts (cf. [**11**]). This can be shown using a Poincaré map. However, if we cannot determine the periodic orbit explicitly these theorems cannot be applied directly. Other criteria using the linearization are given by [**6**] and [**18**]. They are special cases of our results for two-dimensional systems. Stability of periodic orbits can also be proven by Lyapunov functions (cf. [**5**], [**13**], [**19**]).

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To determine the basin of attraction of an exponentially asymptotically stable periodic orbit one can use a Lyapunov function, too (cf. [**2**]). But even if we know the periodic orbit explicitly it is not easy to find such a Lyapunov function.

Borg [**3**] gave a sufficient condition for existence and uniqueness of a limit cycle using a certain contraction property. He showed that if this condition is valid in a bounded set, then this set belongs to the basin attraction of a unique limit cycle. Hartman and Olech [**10**] made first attemps to generalize these ideas to unbounded sets but they showed existence and uniqueness of a limit cycle only for bounded sets.

In this paper, we give sufficient conditions for an unbounded set to be part of the basin of attraction of an exponentially asymptotically stable periodic orbit. In contrast to most other approaches we do not presume the existence, uniqueness or stability of the periodic orbit. Instead, these properties are conclusions. Thus, we use the results both to prove existence and uniqueness of exponentially asymptotically stable periodic orbits and to determine a part of their basin of attraction.

Let us first briefly discuss the basic idea of the conditions. Consider the dynamical system given by the autonomous ordinary differential equation $\dot{x} = f(x)$, where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $n \geq 2$. For each point p of the phase space we define

$$\begin{aligned} L(p) &:= \max_{\|v\|=1, v \perp f(p)} L(p, v) \\ \text{with } L(p, v) &:= \langle Df(p)\, v, v \rangle. \end{aligned}$$

Here Df denotes the Jacobian of f and $\langle ., . \rangle$ the Euclidian scalar product.

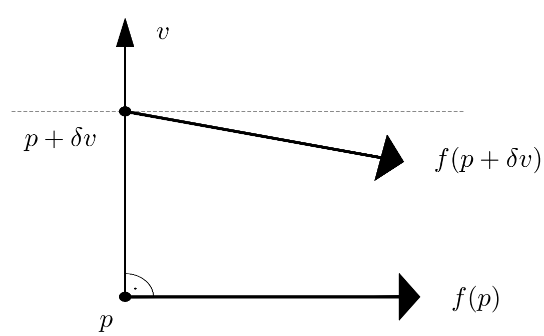

Figure 1. The meaning of the function L.

The main condition is $L(p) < 0$ for a point p of the phase space. This condition is obviously local and guarantees that trajectories within a certain neighborhood of p approach the trajectory through p as time increases. Let us give a heuristic justification of this fact (cf. Figure 1). For example, consider a point $p + \delta v$ with $v \perp f(p)$, $\|v\| = 1$ and $\delta > 0$ small. A sufficient condition for the trajectories through p and $p+\delta v$ to move towards each other is that

$$\begin{aligned} 0 &> \langle f(p+\delta v), v\rangle \\ &\approx \langle f(p) + \delta Df(p)\, v, v\rangle \\ &= \delta \underbrace{\langle Df(p)\, v, v\rangle}_{=L(p,v)}, \text{ since } v \perp f(p). \end{aligned}$$

So if $L(p) < 0$, then the trajectories through p and $p+\delta v$ move towards each other. We will assume $L(p) < 0$ for all points p in a certain positively invariant subset of the phase space in order to assure that the trajectories through adjacent points of this subset move towards each other for all future times.

Let us now state the results, shortly discuss their implications and illustrate them with first examples. At first we give the precise definition of an exponentially asymptotically stable periodic orbit.

Definition 1.1. *Let S_t be the flow of a dynamical system given by an autonomous ordinary differential equation and let Ω be a periodic orbit. We will call Ω* exponentially asymptotically stable*, if it is orbitally stable and there are δ, $\mu > 0$ such that* $\text{dist}(q, \Omega) \leq \delta$ *implies* $\text{dist}(S_t q, \Omega)e^{\mu t} \stackrel{t\to\infty}{\longrightarrow} 0$.

In Theorem 1.2 we assume conditions for a possibly unbounded subset G of the phase space. Then one of the following two alternatives holds: either all positive orbits $\bigcup_{t\geq 0} S_t x_0$ with initial points $x_0 \in G$ are unbounded or they all approach a unique exponentially asymptotically stable periodic orbit as time increases.

Theorem 1.2. *Let $\emptyset \neq G \subset \mathbb{R}^n$ be an open and connected set. Let $\overline{G}$ be a positively invariant set, which contains no equilibrium. Moreover assume $L(p) < 0$ for all $p \in \overline{G}$, where*

$$L(p) := \max_{\|v\|=1, v\perp f(p)} L(p, v) \tag{1}$$

$$L(p, v) := \langle Df(p)\, v, v\rangle. \tag{2}$$

Then either $s(p) := \sup_{t\geq 0} \|S_t p\| = \infty$ holds for all $p \in \overline{G}$ or there exists one and only one periodic orbit $\Omega \subset \overline{G}$. Ω is exponentially asymptotically stable and its basin of attraction $A(\Omega)$ contains G.

Remark 1.3. *$L(p)$ is a continuous function with respect to p as we prove in Proposition A.2.*

Note that in Theorem 1.2 we only claim $G \subset A(\Omega)$. The points of the boundary of G can still tend to infinity, if the boundary of G is not smooth. We give an example in Section 3.4. If we have at least one point in $\overline{G}$, the positive orbit of which is bounded, then Theorem 1.2 yields the existence and uniqueness of an exponentially asymptotically stable periodic orbit in $\overline{G}$. If the positively invariant set $\overline{G}$ itself is bounded, then also the positive orbits of all points of $\overline{G}$ are bounded. Thus, we have the following corollary for compact sets K which has been shown by Borg [**3**] under slightly different assumptions. In this case the geometry of the boundary is not involved and the whole set K belongs to the basin of attraction.

Corollary 1.4. *Let $\emptyset \neq K \subset \mathbb{R}^n$ be a compact, connected and positively invariant set, which contains no equilibrium. Moreover assume $L(p) < 0$ for all $p \in K$.*

Then there exists one and only one periodic orbit $\Omega \subset K$. Ω is exponentially asymptotically stable and its basin of attraction $A(\Omega)$ contains K.

Note that the assumptions of Corollary 1.4 are sufficient, but not necessary. In order to obtain both necessary and sufficient conditions one has to allow a point-dependent Riemannian metric in (2) instead of the Euclidian metric (cf. [**7**], [**15**] and Example 3.1).

Let us discuss the statement of Theorem 1.2 in more detail. If there is an unbounded positive orbit in G, then the first alternative yields that all positive orbits of $\overline{G}$ are unbounded. If there is one bounded positive orbit in $\overline{G}$, the theorem implies that all positive orbits of G tend to a unique limit cycle. Let us consider two examples to illustrate these two alternatives.

The system

$$\begin{cases} \dot{x} &= -x \\ \dot{y} &= 1 \end{cases}$$

provides an example for the first alternative. Obviously, there is no equilibrium. Let us choose $G = \mathbb{R}^2$. To check the assumptions of Theorem 1.2, we have to calculate $L(x, y)$. Note that in two-dimensional systems there is a one-dimensional family of vectors $v \perp f(x, y)$. Since L is quadratic in v we can choose any vector $v \perp f(x, y)$ of positive length and $L(x, y; v)$ has the same sign as $L(x, y)$. Thus we define $\tilde{L}(x, y) := \left\langle Df(x, y) \begin{pmatrix} f_2(x, y) \\ -f_1(x, y) \end{pmatrix}, \begin{pmatrix} f_2(x, y) \\ -f_1(x, y) \end{pmatrix} \right\rangle$, which has the same sign as $L(x, y)$. We calculate $\tilde{L}(x, y) = -1 < 0$. Since all points on the y-axis have unbounded positive orbits, the first alternative is valid and hence all positive orbits in $\mathbb{R}^2$ are unbounded.

The following system

$$\begin{cases} \dot{x} &= x(1 - x^2 - y^2) - y \\ \dot{y} &= y(1 - x^2 - y^2) + x \end{cases}$$

provides an example for the second alternative. The only equilibrium is the origin. Choose $G = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 0.5\}$ and denote $r = \sqrt{x^2 + y^2}$. We have $\frac{d}{dt} r^2 = 2r^2(1 - r^2)$. Hence, $\overline{G}$ is positively invariant and there is a bounded positive orbit. We calculate

$$\tilde{L}(x, y) \quad = \quad r^2(2 - 6r^2 + 3r^4 - r^6).$$

Thus $\tilde{L}(x, y) < 0$ for all $(x, y) \in \overline{G}$ and G belongs to the basin of attraction of an exponentially asymptotically stable periodic orbit $\Omega \subset \overline{G}$ by Theorem 1.2. The periodic orbit in this case is given by $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

Let us describe how the paper is organized. In the second section we prove Theorem 1.2 and Corollary 1.4. In the third section we give more examples to illustrate the results, among them the FitzHugh-Nagumo equation. In an appendix we prove the continuity of the function $L(p)$ and a sufficient condition for a point to belong to a limit cycle which is needed in the proof of Theorem 1.2.

2. Proofs of Theorem 1.2 and Corollary 1.4

The proof of Theorem 1.2 proceeds in the following steps and related propositions:

1. Define a time-dependent distance between two trajectories with nearby initial points and prove that this distance is exponentially decreasing (Propositions 2.1 to 2.3)
2. Show that the positive orbits with initial points in G are either all unbounded or all bounded (Proposition 2.4)
3. Show that in the second case the ω-limit sets of all points of G are the same (Proposition 2.5)
4. Show that this ω-limit set is an exponentially asymptotically stable periodic orbit (Proposition B.1)

In Proposition 2.1 we define a distance function

$$d(\theta) := \|S_T(p+\eta) - S_\theta p\|,$$

where $T = T_p^{p+\eta}(\theta)$ is a synchronized time. Here we assume that the initial points p and $p+\eta$ are sufficiently close, and, moreover, that $p+\eta$ lies in the hyperplane $p + f(p)^\perp$. In Proposition 2.2 we extend our results to all points q of a full neighborhood of p. Our estimates are only valid as long as the trajectory through p does not leave a certain ball $\overline{B_S(0)}$, because only restricting ourselves to this compact set we are able to derive uniform bounds. In Proposition 2.3 we assume in addition that $s(p) = \sup_{t\geq 0} \|S_t p\| \leq S$. Then the orbit stays in $\overline{B_S(0)}$ for all positive times and hence also the estimates are valid for all positive times.

Proposition 2.1. *Let the assumptions of Theorem 1.2 be satisfied. For $S > 0$ and $T_0 > 0$ there are two positive constants δ and ν such that the following holds for all $p \in \overline{G}$, for which $\|S_\theta p\| \leq S$ for all $\theta \in [0, T_0]$:*
For all $\eta \in \mathbb{R}^n$ with $\eta \perp f(p)$ and $\|\eta\| \leq \frac{\delta}{2}$, there exists a diffeomorphism $T_p^{p+\eta}\colon [0,T_0] \longrightarrow T_p^{p+\eta}([0,T_0]) \subset \mathbb{R}_0^+$ which satisfies $T_p^{p+\eta}(0) = 0$, $\frac{1}{2} \leq \dot{T}_p^{p+\eta}(\theta) \leq$ $\leq \frac{3}{2}$ and

$$\left(S_{T_p^{p+\eta}(\theta)}(p+\eta) - S_\theta p\right) \perp f(S_\theta p)$$

for all $\theta \in [0, T_0]$. Moreover, $T_p^{p+\eta}(\theta)$ depends continuously on η, and the distance function $d(\theta) = \|S_{T_p^{p+\eta}(\theta)}(p+\eta) - S_\theta p\|$ satisfies

$$d(\theta) \quad \leq \quad e^{-\frac{\nu}{4}\theta}\,\|\eta\| \text{ for all } \theta \in [0, T_0]. \tag{3}$$

Proof. Denote $G_S = \overline{G} \cap \overline{B_S(0)}$. This set is bounded and closed, and thus compact in $\mathbb{R}^n$. Hence, for the continuous function L (cf. Proposition A.2)

$\nu := -\max_{p\in G_S} L(p) > 0$ exists, so that

$$L(p) \leq -\nu < 0 \text{ for all } p \in G_S. \tag{4}$$

Df is continuous and thus uniformly continuous on G_S. Hence, there exists a $\delta_1 > 0$, so that

$$\|Df(p) - Df(p+\xi)\| \leq \frac{\nu}{2} \tag{5}$$

holds for all $p \in G_S$ and all $\xi \in \mathbb{R}^n$ with $\|\xi\| \leq \delta_1$. Since there is no equilibrium in G_S and f and Df are continuous functions on the compact sets G_S, $\overline{(G_S)_{\delta_1}} := \{q \mid \operatorname{dist}(q, G_S) \leq \delta_1\}$ respectively, there are positive constants ϵ_1 and ϵ_2, such that the following inequalities hold:

$$0 < \epsilon_1 \leq \|f(p)\| \leq \epsilon_2 \text{ for all } p \in G_S \tag{6}$$

$$\|Df(q)\| \leq \epsilon_2 \text{ for all } q \in \overline{(G_S)_{\delta_1}}. \tag{7}$$

We set

$$\delta := \min\left(\delta_1, \frac{\epsilon_1^2}{5\epsilon_2^2}\right). \tag{8}$$

Now fix $p \in G_S$ and $\eta \in \mathbb{R}^n$ with $\eta \perp f(p)$ and $\|\eta\| \leq \frac{\delta}{2}$. We synchronize the time of the trajectories through p and $p+\eta$ while we define $T_p^{p+\eta}(\theta)$ implicitly by

$$Q(T, \theta, \eta) := \langle S_T(p+\eta) - S_\theta p, f(S_\theta p)\rangle = 0. \tag{9}$$

$Q(0, 0, \eta) = 0$ implies $T_p^{p+\eta}(0) = 0$. Since $\partial_T Q(0, 0, \eta) \neq 0$, as we show later, $T_p^{p+\eta}(\theta)$ is defined by (9) locally near $\theta = 0$ and depends continuously on η by the implicit function theorem. We will later show by a prolongation argument that, in fact, $T_p^{p+\eta}$ is defined for all times $\theta \in [0, T_0]$. We write now $T = T_p^{p+\eta}$. As long as $T(\theta)$ is defined, we set

$$d\colon \begin{cases} \mathbb{R}_0^+ & \longrightarrow \quad \mathbb{R}_0^+ \\ \theta & \longmapsto \quad \|S_{T(\theta)}(p+\eta) - S_\theta p\| \end{cases} \tag{10}$$

$d(0) \neq 0$ implies $d(\theta) \neq 0$ for all $\theta \in [0, T_0]$. In this case we set $v(\theta) := \frac{S_{T(\theta)}(p+\eta) - S_\theta p}{d(\theta)}$. $v(\theta)$ is a vector of length one, and it is perpendicular to $f(S_\theta p)$ for each θ by (9). Note that the following equation holds

$$S_{T(\theta)}(p+\eta) - S_\theta p = d(\theta) v(\theta).$$

We calculate the derivative $\dot{T}(\theta)$ using the implicit function theorem.

$$\begin{aligned}
\dot{T}(\theta) &= -\frac{\partial_\theta Q(T,\theta,\eta)}{\partial_T Q(T,\theta,\eta)} \\
&= \frac{\|f(S_\theta p)\|^2 - \langle S_T(p+\eta) - S_\theta p, Df(S_\theta p) f(S_\theta p)\rangle}{\langle f(S_T(p+\eta)), f(S_\theta p)\rangle} \\
&= \frac{\|f(S_\theta p)\|^2 - d(\theta)\langle v(\theta), Df(S_\theta p) f(S_\theta p)\rangle}{\langle f(S_\theta p + d(\theta)v(\theta)), f(S_\theta p)\rangle} \\
&= \frac{\|f(S_\theta p)\|^2 - d(\theta)\langle v(\theta), Df(S_\theta p) f(S_\theta p)\rangle}{\|f(S_\theta p)\|^2 + d(\theta)\langle \int_0^1 Df(S_\theta p + \lambda d(\theta)v(\theta))\, d\lambda\ v(\theta), f(S_\theta p)\rangle}.
\end{aligned}$$

The last equation follows from the mean value theorem. As $d(0) = \|\eta\| \leq \frac{\delta}{2}$, the continuous function d satisfies $d(\theta) \leq \delta$ for θ small enough. We will show later that, however, this inequality holds for all $\theta \in [0, T_0]$.

Since $\overline{G}$ is positively invariant and $\|S_\theta p\| \leq S$ for all $\theta \in [0, T_0]$, we have $S_\theta p \in G_S$ for all $\theta \in [0, T_0]$, and therefore $S_\theta p + \lambda d(\theta) v(\theta) \in \overline{(G_S)_\delta}$, supposed that $d(\theta) \leq \delta$ and $\lambda \in [0,1]$. Using (7) we can conclude $\| \int_0^1 Df(S_\theta p + \lambda d(\theta)v(\theta))\, d\lambda\| \leq$ $\leq \epsilon_2$. Equations (6), (7) and (8) imply

$$\begin{aligned}
\dot{T}(\theta) &\leq \frac{\|f(S_\theta p)\|^2 + \delta\epsilon_2^2}{\|f(S_\theta p)\|^2 - \delta\epsilon_2^2} \\
&\leq \frac{\|f(S_\theta p)\|^2 + \frac{\epsilon_1^2}{5}}{\|f(S_\theta p)\|^2 - \frac{\epsilon_1^2}{5}} \\
&\leq 1 + \frac{\frac{2}{5}\epsilon_1^2}{\|f(S_\theta p)\|^2 - \frac{\epsilon_1^2}{5}} \\
&\leq 1 + \frac{2\epsilon_1^2}{5\epsilon_1^2 - \epsilon_1^2} = \frac{3}{2}.
\end{aligned}$$

Similarly we can conclude $\dot{T}(\theta) \geq \frac{1}{2}$. In particular we have shown $\partial_T Q(0,0,\eta) \neq 0$. $\dot{T}(\theta) \geq \frac{1}{2}$ shows that $T(\theta)$ is a strictly increasing function. The inverse map $\theta(T)$ satisfies $\frac{2}{3} \leq \dot{\theta}(T) \leq 2$. As long as $d(\theta) \leq \delta$ and $S_\theta p \in G_S$ hold, we can thus define $T(\theta)$ by a prolongation argument.

Next we show that $d(\theta)$ tends to zero exponentially. That will imply that we can define $T(\theta)$ for all $\theta \in [0, T_0]$. We calculate the time derivative of $d^2(\theta)$ with respect to θ (cf. (10)) and use $v(\theta) \perp f(S_\theta p)$.

$$\begin{aligned}
\frac{d}{d\theta} d^2(\theta) &= 2\left\langle f(S_{T(\theta)}(p+\eta)) \frac{dT}{d\theta}(\theta) - f(S_\theta p), S_{T(\theta)}(p+\eta) - S_\theta p \right\rangle \\
&= 2d(\theta)\langle f(S_\theta p + d(\theta)v(\theta))\dot{T}(\theta) - f(S_\theta p), v(\theta)\rangle \\
&= 2d(\theta)\langle f(S_\theta p + d(\theta)v(\theta)), v(\theta)\rangle \dot{T}(\theta). \qquad (11)
\end{aligned}$$

As $\|\lambda d(\theta)v(\theta)\| \leq \delta$ provided that $\lambda \in [0,1]$ and $d(\theta) \leq \delta$, which holds for small θ, (5) implies $\|Df(S_\theta p + \lambda d(\theta)v(\theta)) - Df(S_\theta p)\| \leq \frac{\nu}{2}$. The mean value theorem yields with $v(\theta) \perp f(S_\theta p)$, (4) and (5)

$$
\begin{aligned}
&\langle f(S_\theta p + d(\theta)v(\theta)), v(\theta)\rangle \\
&\qquad = \quad d(\theta) \left\langle \int_0^1 Df(S_\theta p + \lambda d(\theta)v(\theta))\, d\lambda \ v(\theta), v(\theta) \right\rangle \\
&\qquad = \quad d(\theta) \left\langle \int_0^1 [Df(S_\theta p + \lambda d(\theta)v(\theta)) - Df(S_\theta p)]\, d\lambda \ v(\theta), v(\theta) \right\rangle \\
&\qquad\quad\ + d(\theta) \underbrace{\langle Df(S_\theta p)v(\theta), v(\theta)\rangle}_{\leq L(S_\theta p) \leq -\nu} \\
&\qquad \leq \quad -\frac{d(\theta)\nu}{2}.
\end{aligned}
$$

Plugging this into (11) we conclude

$$
\frac{d}{d\theta} d^2(\theta) \quad \leq \quad 2\, d(\theta) \left(-\frac{d(\theta)\nu}{2} \right) \dot{T}(\theta) \leq -\frac{d^2(\theta)\nu}{2},
$$

which shows $\dot{d}(\theta) \leq -\frac{d(\theta)\nu}{4}$ and finally

$$
d(\theta) \leq d(0)\, e^{-\frac{\nu}{4}\theta} \leq \|\eta\|\, e^{-\frac{\nu}{4}\theta} \leq \frac{\delta}{2}\, e^{-\frac{\nu}{4}\theta}. \tag{12}
$$

This proves (3) and in particular $d(\theta) \leq d(0) = \|\eta\| \leq \frac{\delta}{2}$ for all $\theta \in [0, T_0]$ and thus that both $T(\theta)$ and $d(\theta)$ are defined for all $\theta \in [0, T_0]$ by a prolongation argument. This concludes the proof of Proposition 2.1. □

In Proposition 2.2 we extend the results of Proposition 2.1 to all points q of a full neighborhood of p.

Proposition 2.2. *Let the assumptions of Theorem 1.2 be satisfied. For $S > 0$ and $T_0 > 0$ there are two positive constants δ^* and ν such that the following holds for all $p \in \overline{G}$, for which $\|S_\theta p\| \leq S$ for all $\theta \in [0, T_0]$:*
For all $q \in \mathbb{R}^n$ with $\|p - q\| \leq \delta^*$ there is a $t_0 = t_0(q)$ with $|t_0| \leq \frac{T_0}{2}$ and a diffeomorphism $\tilde{T}_p^q \colon [t_0, T_0] \longrightarrow \tilde{T}_p^q([t_0, T_0]) \subset \mathbb{R}_0^+$ which satisfies $\tilde{T}_p^q(t_0) = 0$, $\frac{1}{2} \leq \dot{\tilde{T}}_p^q(\theta) \leq \frac{3}{2}$ and

$$
\left(S_{\tilde{T}_p^q(\theta)} q - S_\theta p \right) \perp f(S_\theta p)
$$

for all $\theta \in [t_0, T_0]$. Moreover, $\tilde{T}_p^q(\theta)$ depends continuously on q, and the distance function $\tilde{d}(\theta) := \|S_{\tilde{T}_p^q(\theta)} q - S_\theta p\|$ satisfies

$$
\tilde{d}(\theta) \quad \leq \quad 3\, \|p - q\|\, e^{-\frac{\nu}{4}(\theta - t_0)} \ \textit{for all } \theta \in [t_0, T_0]. \tag{13}
$$

Proof. First, we give the idea of the proof. For a given point q in the neighborhood of p we find a point $S_{t_0} p =: p'$, such that we can write $q = p' + \eta$ with

$\eta \perp f(p')$. Then all statements follow by Proposition 2.1. In the proof we use the notations of Proposition 2.1.

I. Since f is uniformly continuous on the compact set G_S, there is a constant $\delta_2 > 0$ so that for all $\xi \in \mathbb{R}^n$ with $\|\xi\| \leq \delta_2$ and all $p \in G_S$

$$\|f(p) - f(p+\xi)\| \quad \leq \quad \frac{\epsilon_1}{2} \tag{14}$$

holds, where ϵ_1 is the constant of (6). Set $\delta^* := \min\left(\frac{\delta_2}{2}, \frac{\delta}{6}, \frac{\epsilon_1}{8}T_0\right)$, where δ was defined in (8).

Now we fix a point $p \in G_S$ and choose a $0 < \tilde{\delta} \leq \delta^*$. We prove that there are times $-\frac{T_0}{2} \leq t_1 < 0 < t_2 \leq \frac{T_0}{2}$ with $\|p - S_{t_1}p\| = \|p - S_{t_2}p\| = 2\tilde{\delta}$ and $\|p - S_t p\| < 2\tilde{\delta}$ for all $t \in (t_1, t_2)$.

We prove the existence of t_1. Since $\|p - S_t p\|$ is continuous with respect to t, assuming the opposite means that $S_\tau p \in B_{2\tilde{\delta}}(p)$ for all $\tau \in \left[-\frac{T_0}{2}, 0\right]$ and therefore $S_\tau p$ is defined by prolongation for all these τ. This yields

$$\begin{aligned}
\|p - S_t p\| &= \left\|\int_0^t f(S_\tau p)\, d\tau\right\| \\
&= \left\|\int_0^t f(p)\, d\tau + \int_0^t (f(S_\tau p) - f(p))\, d\tau\right\| \\
&\geq |t|\left(\|f(p)\| - \frac{\epsilon_1}{2}\right) \text{ by (14)} \\
&\geq |t|\frac{\epsilon_1}{2} \text{ by (6)}
\end{aligned}$$

for all $t \in \left[-\frac{T_0}{2}, 0\right]$. For $t = -\frac{T_0}{2}$ we conclude $\|p - S_{\frac{T_0}{2}}p\| \geq 2\delta^* \geq 2\tilde{\delta}$, which is a contradiction. This proves the existence of t_1. To show the existence of t_2 we can argue in a similar way.

II. We show that for all points $q \in \overline{B_{\tilde{\delta}}(p)}$ there is a $t_0 \in (t_1, t_2) \subset \left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$ so that $(q - S_{t_0}p) \perp f(S_{t_0}p)$ and $\|q - S_{t_0}p\| \leq 3\tilde{\delta} \leq \frac{\delta}{2}$.

We fix $q \in \overline{B_{\tilde{\delta}}(p)}$ and define the continuous function $a(\tau)$ by $a(\tau) := \|q - S_\tau p\|$. We have $a(0) \leq \tilde{\delta}$ and $a(t_1) \geq \|S_{t_1}p - p\| - \|p - q\| \geq \tilde{\delta}$, $a(t_2) \geq \tilde{\delta}$. The intermediate value theorem yields the existence of $t_1 \leq t_1' < t_2' \leq t_2$ with $a(t_1') = a(t_2')$. Thus $a^2(t_1') = a^2(t_2')$ and there is a $t_0 \in (t_1', t_2')$ with $\frac{d}{d\tau}a^2(t_0) = 0$. This proves $\langle q - S_{t_0}p, f(S_{t_0}p)\rangle = 0$. As $\|q - S_{t_0}p\| \leq \|q - p\| + \|p - S_{t_0}p\| \leq \tilde{\delta} + 2\tilde{\delta} \leq 3\delta^* \leq \frac{\delta}{2}$ the claim is proven.

III. Choose q with $\|p - q\| \leq \delta^*$ and set $\tilde{\delta} := \|p - q\|$. By II. there exists a $t_0(q)$ with $|t_0(q)| \leq \frac{T_0}{2}$ such that $q = S_{t_0}p + \eta$, where $q - S_{t_0}p = \eta \perp f(S_{t_0}p)$ and $\|\eta\| \leq \frac{\delta}{2}$. By Proposition 2.1 and II. we have

$$\begin{aligned}
\left(S_{T^q_{S_{t_0}p}(\theta)}q - S_{t_0+\theta}p\right) &\perp f\left(S_{t_0+\theta}p\right) \\
\text{and } \|S_{T^q_{S_{t_0}p}(\theta)}q - S_{t_0+\theta}p\| &\leq \|q - S_{t_0}p\|\, e^{-\frac{\nu}{4}\theta} \leq 3\,\tilde{\delta}\, e^{-\frac{\nu}{4}\theta}
\end{aligned}$$

for all $\theta \in [0, T_0 - t_0]$. Mind that $\tilde{\delta} = \|p - q\|$. Thus, (13) follows by setting $\tilde{T}_p^q(t_0 + \theta) := T_{S_{t_0}p}^q(\theta)$. □

If the whole positive orbit through p stays in the bounded set $\overline{B_S(0)}$, the statements of Propositions 2.1 and 2.2 hold for all positive times by prolongation. Thus, we get the following results concerning the ω-limit sets of nearby points.

Proposition 2.3. *Let the assumptions of Theorem 1.2 be satisfied. For $S > 0$ there are three positive constants δ, δ^* and ν such that the following holds for all $p \in \overline{G}$, for which $s(p) = \sup_{t \geq 0} \|S_t p\| \leq S$:*
For all $\eta \in \mathbb{R}^n$ with $\eta \perp f(p)$ and $\|\eta\| \leq \frac{\delta}{2}$, there exists a diffeomorphism $T_p^{p+\eta} \colon \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ which satisfies $\frac{1}{2} \leq \dot{T}_p^{p+\eta}(\theta) \leq \frac{3}{2}$ and

$$\left(S_{T_p^{p+\eta}(\theta)}(p + \eta) - S_\theta p\right) \perp f(S_\theta p)$$

for all $\theta \geq 0$. $T_p^{p+\eta}(\theta)$ depends continuously on η, and the distance function $d(\theta) = \|S_{T_p^{p+\eta}(\theta)}(p + \eta) - S_\theta p\|$ satisfies

$$d(\theta) \quad \leq \quad e^{-\frac{\nu}{4}\theta} \|\eta\| \text{ for all } \theta \geq 0. \tag{15}$$

Moreover, for the ω-limit sets we have $\omega(p) = \omega(p + \eta)$.
For all $q \in \mathbb{R}^n$ with $\|p - q\| \leq \delta^*$ we have $\omega(p) = \omega(q)$.
Furthermore, for each $\tau \geq 0$, there is a $\theta \geq 0$ such that (16) holds.

$$\|S_\theta p - S_\tau q\| \quad \leq \quad 3\,\|p - q\|. \tag{16}$$

Also, for each $\theta \geq 0$, there is a $\tau \geq 0$ such that (16) holds.

Proof. We define δ as in (8). Using the notations of Proposition 2.1 we have $S_t p \in G_S$ for all $t \geq 0$. Thus, the proof of Proposition 2.1 shows that we can define $T(\theta)$ and $d(\theta)$ for all $\theta \geq 0$ by a prolongation argument, and also (3) holds for all positive θ, i.e., (15) is proven.

Now we show that all points $p + \eta$ with η as above have the same ω-limit set as p itself. Assume $w \in \omega(p)$. Then we have a strictly increasing sequence $\theta_n \to \infty$ satisfying $\|w - S_{\theta_n} p\| \to 0$ as $n \to \infty$. Because of (15) and $\dot{T} := \dot{T}_p^{p+\eta} \geq \frac{1}{2}$ the sequence $T(\theta_n)$ satisfies $T(\theta_n) \to \infty$ and $\|S_{T(\theta_n)}(p + \eta) - S_{\theta_n} p\| = d(\theta_n) \leq$ $\leq \frac{\delta}{2} \exp(-\frac{\nu}{4}\theta_n) \to 0$ as $n \to \infty$. This proves $S_{T(\theta_n)}(p + \eta) \to w$ and $w \in \omega(p + \eta)$. The inclusion $\omega(p + \eta) \subset \omega(p)$ follows similarly.

Now we consider the extension of Proposition 2.2. We set $\delta^* := \min\left(\frac{\delta_2}{2}, \frac{\delta}{6}\right)$. Then by similar arguments as in the proof of Proposition 2.2 there are times $-\frac{4\delta^*}{\epsilon_1} \leq t_1 < 0 < t_2 \leq \frac{4\delta^*}{\epsilon_1}$ such that the statements of I. hold (cf. the proof of Proposition 2.2). II. and III. also hold with $|t_0| \leq \frac{4\delta^*}{\epsilon_1}$. $\tilde{T}(\theta)$ and $\tilde{d}(\theta)$ are defined for all $\theta \geq t_0$ as in Proposition 2.2. Also, (13) holds for all $\theta \geq t_0$ and in particular p and q have the same ω-limit set.

Now we prove (16). For $\tau \geq 0$ we choose $\theta = (\tilde{T}_p^q)^{-1}(\tau)$. If $\theta \geq t_0$, set $\tau = \tilde{T}_p^q(\theta)$. In both cases (16) follows by (13). If $0 \leq \theta < t_0$, then choose $\tau = 0$. We have then

$$\begin{aligned} \|S_\theta p - q\| &\leq \|S_\theta p - p\| + \|p - q\| \\ &\leq 2\tilde{\delta} + \|p - q\| \end{aligned}$$

by I. of Proposition 2.2 since $[0, t_0) \subset (t_1, t_2)$. □

The next proposition is the main step towards unbounded sets G. Recall the definition $s(p) := \sup_{t \geq 0} \|S_t p\|$. We will prove that either $s(p) = \infty$ for all $p \in G$ or $s(p) < \infty$ for all $p \in G$. If $s(p) = \infty$ for all $p \in G$, then the same holds true for all points of the boundary. In the other case, the same is only true if G has a boundary, which is given by the graph of a smooth map. In Section 3.4 we give an example for a dynamical system and a set G which satisfy the assumptions of Theorem 1.2 with $s(p) < \infty$ for all $p \in G$, but there is a $q \in \partial G$ with $s(q) = \infty$.

Proposition 2.4. *Let the assumptions of Theorem 1.2 be satisfied. Then either $s(p) = \infty$ for all $p \in \overline{G}$ or $s(p) < \infty$ for all $p \in G$.*

Proof. Define $G^* := \{p \in G \mid s(p) < \infty\}$ and $G' := \{p \in G \mid s(p) = \infty\}$. Obviously $G = G^* \dot{\cup} G'$. If we can prove that both G^* and G' are open, we have either $G^* = \emptyset$ or $G' = \emptyset$ since G is connected. We will show that G^* is open in the first, and that G' is open in the second step. At the end we will deal with the points of the boundary.

I. In this step we will show: If $q \in \overline{G}$ with $s(q) < \infty$, then for all q' with $\|q - q'\| \leq$ $\leq \delta^*$ where δ^* is chosen as in Proposition 2.3 with $S = s(q)$

$$|s(q) - s(q')| \leq 3\,\|q - q'\| \tag{17}$$

holds. This means that s is a continuous function and that if $s(q) < \infty$ holds for a point $q \in \overline{G}$, than this property holds for all points of a neighborhood of q. Hence, in particular G^* is an open set.

Choose a point q' with $\|q - q'\| \leq \delta^*$. First we show that $s(q') < \infty$. If this was not the case, there would be a $\tau \geq 0$, such that $\|S_\tau q'\| \geq 2\,s(q) + 3\,\|q - q'\|$. But by Proposition 2.3, (16) there is a $\theta \geq 0$ such that $\|S_\theta q - S_\tau q'\| \leq 3\|q - q'\|$ holds. Then

$$\begin{aligned} \|S_\theta q\| &\geq \|S_\tau q'\| - \|S_\theta q - S_\tau q'\| \\ &\geq 2\,s(q), \end{aligned}$$

which is a contradiction to $s(q) = \sup_{\theta \geq 0} \|S_\theta q\|$. Hence, $s(q') < \infty$.

Let $\theta_n \geq 0$ be a sequence of times such that $\Big| s(q) - \|S_{\theta_n} q\| \Big| \leq \frac{1}{n}$. Then by Proposition 2.3 there are times $\tau_n \geq 0$ such that $\|S_{\theta_n} q - S_{\tau_n} q'\| \leq 3\,\|q - q'\|$. Hence,

$$\begin{aligned} s(q') &\geq \|S_{\tau_n} q'\| \\ &\geq s(q) - \Big| s(q) - \|S_{\theta_n} q\| \Big| - \|S_{\theta_n} q - S_{\tau_n} q'\| \\ &\geq s(q) - \frac{1}{n} - 3\,\|q - q'\|. \end{aligned}$$

Hence, $s(q') \geq s(q) - 3\|q - q'\|$. Assume now that $\tau_n \geq 0$ is a sequence such that $\left|s(q') - \|S_{\tau_n} q'\|\right| \leq \frac{1}{n}$. By a similar argument we can show that $s(q') \leq$ $\leq s(q) + 3\|q - q'\|$ and thus $|s(q') - s(q)| \leq 3\|q - q'\|$.

II. We want to show that G' is open. Assuming the opposite there is a $p' \in G'$ such that every neighborhood of p' contains a point of G^*. Since $p' \in G$, which is open, there is a ball $B_\epsilon(p') \subset G$ with $\epsilon > 0$. This is a neighborhood of p' in G and thus it contains a point $q \in G^*$. Consider the line $\tilde{\gamma}(l) = lp' + (1 - l)q$, $l \in [0, 1]$, with $\tilde{\gamma}(0) = q \in G^*$ and $\tilde{\gamma}(1) = p' \notin G^*$. Let l^* be the minimal $0 \leq l \leq 1$ such that $\tilde{\gamma}(l) \notin G^*$. This number exists since G^* is open, and we have $0 < l^* \leq 1$. Denote $p := \tilde{\gamma}(l^*) \in G'$ and $r := \|p - q\| > 0$. Now consider the line $\gamma(\lambda) := \lambda p + (1 - \lambda)q$. We have the following situation: $\gamma(\lambda) \in G^*$ for $\lambda \in [0, 1)$ and $\gamma(1) = p \in G'$.

1. We show the following:

$$s(\gamma(\lambda)) \quad \leq \quad s(q) + 4r =: s^* \text{ for all } \lambda \in [0, 1). \tag{18}$$

Note that the function $h(\lambda) := s(\gamma(\lambda)) - s(q) - 4\|\gamma(\lambda) - q\|$ is continuous for all $\lambda \in [0, 1)$ by I. If the claim was wrong, there would be a $\lambda^* \in [0, 1)$ such that $h(\lambda^*) > 4(r - \|\gamma(\lambda^*) - q\|) \geq 0$. The minimum of $h(\lambda)$ for $\lambda \in [0, \lambda^*]$ is nonpositive since $h(0) = 0$, and thus it is assumed at $\lambda' \neq \lambda^*$. In $\gamma(\lambda') \in G^*$ we can choose a δ^* according to Proposition 2.3, which depends on $S = s(\gamma(\lambda'))$. If $\lambda^* - \lambda' > \alpha > 0$ is chosen so small that $\|\gamma(\lambda') - \gamma(\lambda' + \alpha)\| = r\alpha \leq \delta^*$ then by (17)

$$|s(\gamma(\lambda')) - s(\gamma(\lambda' + \alpha))| \quad \leq \quad 3\|\gamma(\lambda') - \gamma(\lambda' + \alpha)\| = 3\,\alpha\, r\,. \tag{19}$$

Since h assumes its minimum in λ', we have $h(\lambda' + \alpha) \geq h(\lambda')$. Hence,

$$\begin{aligned} s(\gamma(\lambda' + \alpha)) - s(\gamma(\lambda')) &\geq 4\left(\|\gamma(\lambda' + \alpha) - q\| - \|\gamma(\lambda') - q\|\right) \\ &= 4\,\alpha\, r\,. \end{aligned}$$

But this is a contradiction to (19). Thus, we have shown (18).

2. Since $p \in G'$, there is a minimal $T_0 > 0$ such that $\|S_{T_0} p\| = 2s^*$ where s^* was defined in (18). Now choose for this T_0 and $S := 2s^*$ a δ^* with Proposition 2.2. Let $\tilde{\delta} = \min\left(\delta^*, \frac{s^*}{6}, \frac{r}{2}\right)$. Then (13) of Proposition 2.2 yields for $q' := \gamma\left(1 - \frac{\tilde{\delta}}{r}\right)$

$$\begin{aligned} \frac{s^*}{2} &\geq 3\,\tilde{\delta} \\ &= 3\,\|q' - p\| \\ &\geq \|S_{T_0} p - S_{\tilde{T}_p^{q'}(T_0)} q'\| \\ &\geq \|S_{T_0} p\| - s(q') \\ &\geq 2s^* - s^* \end{aligned}$$

by (18), which is a contradiction to $s^* > 0$. Hence, G' is open.

Since G is connected and G^* and G' are open, either $s(p) = \infty$ for all $p \in G$ or $s(p) < \infty$ for all $p \in G$. Now assume that $s(p) = \infty$ for all $p \in G$ and $s(p_0) < \infty$ for a $p_0 \in \partial G$. By I. $s(p) < \infty$ for all $\|p - p_0\| \leq \delta^*$ and thus in particular there is a $p \in G$ with this property, which is a contradiction. This proves the proposition. □

If $s(p) < \infty$ for all $p \in G$ and $p_0 \in \partial G$ with $s(p_0) = \infty$ following the above argumentation we can only show a contradiction, if there is a line $\gamma^*(\lambda) := \lambda p_0 + (1-\lambda)q$ with $\gamma^*([0,1)) \subset G$. In Section 3.4 we give an example where there is no such line. If, however, the boundary of G is the graph of a smooth map, we can always find such a line and thus the points of the boundary behave like the inner points.

Using again the fact that G is connected we can now prove Proposition 2.5 showing that all points of G have the same ω-limit set.

Proposition 2.5. *Let the assumptions of Theorem 1.2 be satisfied. Then either $s(p) = \infty$ for all $p \in \overline{G}$. Or $\emptyset \neq \omega(p) = \omega(q) =: \Omega \subset \overline{G}$ for all $p, q \in G$, and Ω is invariant and bounded.*

Proof. Either $s(p) = \infty$ holds for all $p \in \overline{G}$. Or there is a point $p_0 \in G$ such that $s(p_0) =: S < \infty$ by Proposition 2.4. Since for all $\theta \geq 0$ we have $S_\theta p_0 \subset \overline{G} \cap \overline{B_S(0)}$, which is a compact set, $\emptyset \neq \omega(p_0) =: \Omega \subset \overline{G} \cap \overline{B_S(0)}$, and Ω is invariant and bounded.

Now consider an arbitrary point $p \in G$. By Proposition 2.4 $s(p) < \infty$. Thus by Proposition 2.3 with $S = s(p)$ we have $\omega(p) = \omega(q)$ for all q in a neighborhood of p. Hence $V_1 := \{p \in G \mid \omega(p) = \omega(p_0)\}$ and $V_2 := \{p \in G \mid \omega(p) \neq \omega(p_0)\}$ are open sets. Since $G = V_1 \dot\cup V_2$, $p_0 \in V_1$ and G is connected, V_2 must be empty and $V_1 = G$. □

To finally prove Theorem 1.2, it remains to show that Ω is an exponentially asymptotically stable periodic orbit. Proposition B.1 which is stated and proven in the appendix gives a sufficient condition under which a point p belongs to an exponentially asymptotically stable periodic orbit.

Proof of Theorem 1.2. By Proposition 2.5 we either have $s(p) = \infty$ for all $p \in \overline{G}$. Or, by the same proposition, we can choose a point $p_0 \in \Omega$. Since Ω is invariant and bounded, $s(p_0) < \infty$. Also, $\omega(p_0) = \Omega$ by Proposition 2.5 if $p_0 \in G$. If $p_0 \in \partial G$ then by Proposition 2.3 $\omega(p_0) = \omega(q)$ holds for all points q in a neighborhood of p_0 and in particular for a $q \in G$, and then, again by Proposition 2.5, $\omega(q) = \Omega$. Thus we have $p_0 \in \Omega = \omega(p_0)$ in both cases.

By Proposition 2.3 with $S := s(p_0)$ also the other conditions of Proposition B.1 are satisfied with p_0, $g(\theta) := \frac{f(S_\theta p)}{\|f(S_\theta p)\|}$ and $C := 1$. Hence, Ω is an exponentially asymptotically stable periodic orbit and by Proposition 2.5 $\omega(q) = \Omega$ for all $q \in G$. Since Ω is asymptotically stable, $q \in A(\Omega)$ follows for all $q \in G$.

It remains to prove uniqueness. If $\Omega' \in \overline{G}$ is a periodic orbit then for $p' \in \Omega'$ we have $s(p') < \infty$, since Ω' is invariant and bounded. But with the same argumentation as above, $\omega(p') = \omega(q) = \Omega$ for a nearby point $q \in G$ and hence $\Omega' = \Omega$. □

Proof of Corollary 1.4. Note that the connectedness of K does not imply the connectedness of $\mathring{K}$. Hence, we cannot directly apply Theorem 1.2 to $G = \mathring{K}$. Also, we have to show that not only $\mathring{K}$ but the whole set K is a subset of $A(\Omega)$.

Since K is compact and positively invariant, there is a $S \geq 0$ such that $K \subset \overline{B_S(0)}$. Hence, Proposition 2.1 to 2.3 hold for $\overline{G} = K$. We choose a $p_0 \in K$ and then $\emptyset \neq \omega(p_0) =: \Omega \subset K$ since K is compact. As in Proposition 2.5 we can show $\omega(p) = \Omega$ for all $p \in K$. Using Proposition B.1 we show that Ω is an exponentially asymptotically stable periodic orbit (for details cf. [**7**]). □

3. Examples

To apply Theorem 1.2 we first calculate the sign of L in the phase space and then search for a positively invariant set $\overline{G}$ which lies in the part of the phase space where L is negative.

In the first section we will apply Theorem 1.2 to the FitzHugh-Nagumo equation. In the second section we show how to use Theorem 1.2 in order to prove that the whole set $\{x \in \mathbb{R}^2 \mid L(x) \leq 0\}$ belongs to the basin of attraction of a limit cycle. In the third section we consider a three-dimensional system and in the last section we give a two-dimensional example, where G belongs to the basin of attraction of a limit cycle, whereas the positive orbits of some points of the boundary are unbounded.

3.1. FitzHugh-Nagumo equation

The FitzHugh-Nagumo equation was introduced by FitzHugh [**8**] and Nagumo [**14**] as a model for the nerve conduction (cf. (1) and (2) in [**8**]).

$$(20) \qquad \begin{cases} \dot{x} &= c\left(y + x - \dfrac{x^3}{3} + z\right) \\ \dot{y} &= -\dfrac{x - a + by}{c} \end{cases}$$

The existence and uniqueness of limit cycles of (20) have been shown recently in [**12**] for general parameter values using results on the existence and uniqueness of limit cycles of Liénard's equation.

We consider the parameter values $a = 0.7$, $b = 0.8$ and $c = 3$ (cf. [**8**], Figure 5) as model for the break reexcitation in the heart muscle (cf. [**8**], p. 455). We set $z = -0.85$. We use the simple transformation $x \mapsto \kappa x$ with $\kappa = 0.8$ and obtain the equations

$$(21) \qquad \begin{cases} \dot{x} &= c\kappa\left(y + \dfrac{x}{\kappa} - \dfrac{1}{3}\left(\dfrac{x}{\kappa}\right)^3 + z\right) \\ \dot{y} &= -\dfrac{1}{c}\left(\dfrac{x}{\kappa} - a + by\right) \end{cases}$$

There is exactly one (unstable) equilibrium at approximatively $(0.0395, 0.7516)$ which is marked in Figure 2.

Instead of the function L we calculate the function

$$\tilde{L}(x,y) = (f_2(x,y), -f_1(x,y))Df(x,y)\begin{pmatrix} f_2(x,y) \\ -f_1(x,y) \end{pmatrix}$$

which has the same sign as L. Figure 2, left, shows the zero set of $\tilde{L}$ (thick line). Inside $\tilde{L}$ is positive and outside negative. We denote by G the points outside the polygone with edges $(-0.35, 1.6)$, $(0.55, 1.05)$, $(0.55, 0.45)$, $(0.5, 0.23)$, $(0.4, 0.11)$, $(0.21, 0.13)$, $(-0.4, 0.35)$, $(-0.5, 1.42)$, and $(-0.41, 1.58)$. Since G is an open and connected set, $\overline{G}$ is positively invariant and $L(p) < 0$ holds for all points $p \in \overline{G}$, we can apply Theorem 1.2. Since the set $\{(x,y) \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} \leq 2\}$ is positively invariant, there is a bounded positive orbit and thus there is a unique limit cycle in $\overline{G}$ and G belongs to its basin of attraction. The right hand part of Figure 2 shows the approximated periodic orbit and the set G, which belongs to its basin of attraction, as we have proven.

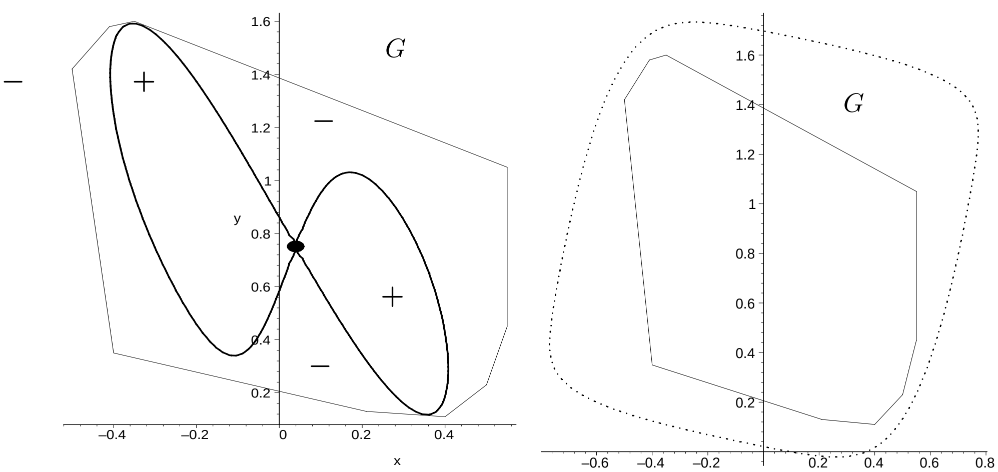

Figure 2. left: the zero set and the sign of L (thick line) and the boundary (thin line) of the set G which belongs to the basin of attraction of a unique limit cycle; right: the approximated limit cycle (dotted line) and G for (21).

3.2. A two-dimensional system with known limit cycle

A problem in applications is to find a positively invariant set G. In this example we use the orbital derivative of the function $\tilde{L}$ to show that sets of the form $\{p \in \mathbb{R}^2 \mid \tilde{L}(p) < -\nu\}$ are positively invariant.

Consider the two-dimensional system

$$\left\{ \begin{array}{rcl} \dot{x} & = & \left(x - \frac{1}{2}\right)(1 - x^2 - y^2) - y \\ \dot{y} & = & y(1 - x^2 - y^2) + x \end{array} \right. \tag{22}$$

There is exactly one equilibrium at approximatively $(-0.2209, 0.2483)$, which is marked in Figure 3. $\Omega = \{(x, y) \mid x^2 + y^2 = 1\}$ is a periodic orbit. In Figure 3 the zero set of $\tilde{L}$ is shown as a thick line. Inside $\tilde{L}$ is positive and outside negative. We claim that $G' := \{p \neq (0,0) \mid \tilde{L}(p) \leq 0\}$ belongs to the basin of attraction of the periodic orbit Ω.

Since this set G' does not satisfy the condition $L(p) < 0$ for all $p \in G'$ we cannot apply Theorem 1.2 to G'. Instead, we will apply this theorem to sets of the form $G_\nu := \{p \neq (0,0) \mid \tilde{L}(p) < -\nu\}$ with $\nu > 0$. We first calculate the orbital derivative $g(x, y) := \langle \nabla \tilde{L}(x, y), f(x, y)\rangle$ of $\tilde{L}$ (note that $f \in C^2(\mathbb{R}^2, \mathbb{R}^2)$). The zero set of g is plotted in Figure 3 as thin lines. We find that the zero set of $\tilde{L}$ lies in the region, where g is negative.

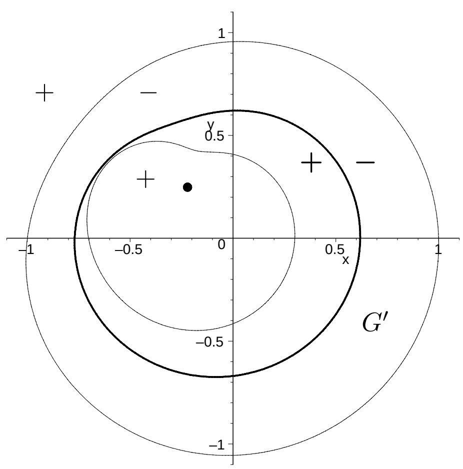

Figure 3. The zero sets and the signs of $\tilde{L}$ (thick line) and g (thin lines) for (22). The set $G' = \{(x, y) \in \mathbb{R}^2 \mid \tilde{L}(x, y) \leq 0\}$ is a subset of the basin of attraction of the limit cycle $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

Choosing $G_\nu := \{p \neq (0,0) \mid \tilde{L}(p) < -\nu\}$ for $\nu > 0$ so small, that the boundary of G_ν lies in the region, where g is negative, we can apply Theorem 1.2 to G_ν. We check that the conditions are fulfilled. G_ν is open and connected. L is strictly negative in $\overline{G_\nu}$, because so is $\tilde{L}$. $\overline{G_\nu}$ does not contain the equilibrium. Since the orbital derivative $g(x, y) := \langle \nabla \tilde{L}(x, y), f(x, y)\rangle$ is strictly negative for $(x, y) \in \partial G_\nu$, $\overline{G_\nu}$ is positively invariant. To show that $G_\nu \subset A(\Omega)$ we have to exclude the first alternative of the theorem. But the points of the periodic orbit which lies in G_ν have bounded positive orbits and thus $G_\nu \subset A(\Omega)$.

For a point $p \in G'$ with $\tilde{L}(p) < 0$, we can find a $\nu > 0$ such that $p \in G_\nu$ and use the above argumentation. If $p \in G'$ with $\tilde{L}(p) = 0$, $g(p) < 0$ guarantees that $\tilde{L}(S_t p) < 0$ for small $t > 0$. We can use the above argumentation for $S_t p$, and hence also $\omega(p) = \omega(S_t p) = \Omega$, i.e. $p \in A(\Omega)$.

3.3. A three-dimensional example

In two-dimensional systems the Poincaré-Bendixson theorem already gives a complete characterization of the possible ω-limit sets. Theorem 1.2, however, is valid in any dimension $n \geq 2$ and thus we give a higher-dimensional example. In three-dimensional systems there is a two-dimensional family of vectors $v \perp f(x)$. It is possible to determine analytically a vector v_0 depending on x with $f(x) \neq 0$ such that $L(x) = \max_{\|v\|=1, v \perp f(x)} L(x,v) = L(x,v_0)$. In the following, however, we will estimate $L(x,v)$ without calculating such a vector v_0 explicitly.

Let us consider the following three-dimensional system

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& (9 - x_1^2 - x_2^2 - x_3^2)\, x_1 - x_2 \\ \dot{x}_2 &=& (9 - x_1^2 - x_2^2 - x_3^2)\, x_2 + x_1 + 0.25 \\ \dot{x}_3 &=& (-x_1^2 - x_2^2 - x_3^2)\, x_3 + 0.25 \end{array} \right. \tag{23}$$

We choose $G = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 > \rho^2, |x_3| < z_0\}$ with $\rho := 2.9$ and $z_0 := 0.1$ and claim that this set is part of the basin of attraction of a unique limit cycle. We check that the assumptions of Theorem 1.2 are satisfied.

We calculate $L(x,v)$ for $\|v\| = 1$ and denote $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

$$L(x,v) \;=\; -\|x\|^2 + 9(v_1^2 + v_2^2) - 2\langle x, v\rangle^2$$

If $\|x\| > 3$, then we have $L(x,v) < 0$ since $v_1^2 + v_2^2 \leq 1$. Now we consider the case $\|x\| \leq 3$. We use $v \perp f(x)$, i.e.

$$0 \;=\; \langle x, v\rangle(9 - \|x\|^2) - 9x_3v_3 + x_1v_2 - x_2v_1 + 0.25v_2 + 0.25v_3. \tag{24}$$

Hence, using $\sqrt{x_1^2 + x_2^2} \leq \|x\| \leq 3$, we get with (24) for any $k_1, k_2 > 0$

$$\begin{aligned} &-\langle x, v\rangle^2[1 + (9 - \|x\|^2)^2] \\ &\quad= -(x_1v_1 + x_2v_2)^2 - 2(x_1v_1 + x_2v_2)x_3v_3 - x_3^2v_3^2 \\ &\qquad - (x_1v_2 - x_2v_1)^2 - 2(x_1v_2 - x_2v_1)[-9x_3v_3 + 0.25v_3 + 0.25v_2] \\ &\qquad - [-9x_3v_3 + 0.25v_3 + 0.25v_2]^2 \\ &\quad\leq -(x_1^2 + x_2^2)(v_1^2 + v_2^2) \\ &\qquad + 2\sqrt{x_1^2 + x_2^2}\sqrt{v_1^2 + v_2^2}\,[10|x_3v_3| + 0.25|v_3| + 0.25|v_2|] \\ &\quad\leq -\rho^2(v_1^2 + v_2^2) + 3\left(k_1(v_1^2 + v_2^2) + \frac{1}{k_1}(10z_0 + 0.25)^2v_3^2\right) \\ &\qquad + 3\left(k_2(v_1^2 + v_2^2) + \frac{1}{k_2}(0.25)^2v_2^2\right). \end{aligned}$$

Setting $k_2 := \frac{1}{4\sqrt{26}}$ and $k_1 := 25k_2$ we get with $v_2^2 + v_3^2 \leq 1$

$$-\langle x, v\rangle^2[1 + (9 - \|x\|^2)^2] \;\leq\; \left(\frac{3}{4}\sqrt{26} - \rho^2\right)(v_1^2 + v_2^2) + \frac{3}{4}\sqrt{26}.$$

We plug this into the formula for $L(x,v)$.

$$\begin{aligned} L(x,v) &\le -\|x\|^2 + 9(v_1^2+v_2^2) + \frac{\frac{3}{2}\sqrt{26}-2\rho^2}{1+(9-\|x\|^2)^2}(v_1^2+v_2^2) \\ &\quad + \frac{\frac{3}{2}\sqrt{26}}{1+(9-\|x\|^2)^2} \\ &= \frac{1}{1+(9-\|x\|^2)^2}\Bigg[-\|x\|^2[1+(9-\|x\|^2)^2] + \frac{3}{2}\sqrt{26} \\ &\quad + (v_1^2+v_2^2)\left(9[1+(9-\|x\|^2)^2] + \frac{3}{2}\sqrt{26} - 2\rho^2\right)\Bigg]. \end{aligned}$$

Depending on the sign of $9[1+(9-\|x\|^2)^2]+\frac{3}{2}\sqrt{26}-2\rho^2$ the maximal value of the right hand side is assumed either for $v_1^2+v_2^2=0$ or for $v_1^2+v_2^2=1$, respectively. In the first case

$$L(x,v) \le \frac{1}{1+(9-\|x\|^2)^2}\left[-\rho^2 + \frac{3}{2}\sqrt{26}\right] < 0$$

since $\rho^2 = 8.41$. In the second case we have with $9 \ge \|x\|^2 \ge x_1^2+x_2^2 \ge \rho^2$

$$\begin{aligned} L(x,v) &\le \frac{1}{1+(9-\|x\|^2)^2}\left[(9-\|x\|^2)[1+(9-\|x\|^2)^2] + 3\sqrt{26} - 2\rho^2\right] \\ &\le \frac{1}{1+(9-\|x\|^2)^2}\left[(9-\rho^2)[1+(9-\rho^2)^2] + 3\sqrt{26} - 16.82\right] < 0. \end{aligned}$$

Thus L is negative in $\overline{G}$.

Moreover we have to prove that $\overline{G}$ is positively invariant. At $x_3 = |z_0|$ we have for $x_1^2+x_2^2 \ge \rho^2$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}x_3^2 &= -z_0^2\|x\|^2 + 0.25x_3 \\ &\le -z_0^2\rho^2 + 0.25z_0 < 0. \end{aligned}$$

We calculate

$$\frac{1}{2}\frac{d}{dt}(x_1^2+x_2^2) = (x_1^2+x_2^2)(9-\|x\|^2) + 0.25x_2.$$

At $x_1^2+x_2^2=\rho^2$ and $|x_3| \le z_0$ we have

$$\frac{1}{2}\frac{d}{dt}(x_1^2+x_2^2) \ge \rho^2(9-\rho^2-z_0^2) - 0.25\rho > 0.$$

To find a bounded positive orbit, we consider the set $\tilde{G} := \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid \rho^2 \le x_1^2+x_2^2 \le R^2, |x_3| \le z_0\}$ with $R = 4$. This bounded set is positively invariant, because at $x_1^2+x_2^2 = R^2$ we have

$$\frac{1}{2}\frac{d}{dt}(x_1^2+x_2^2) \le R^2(9-R^2) + 0.25R < 0.$$

Hence, Theorem 1.2 yields that there is a unique limit cycle in $\overline{G}$ and that G belongs to its basin of attraction.

3.4. Different behaviour of boundary and inner points

In this section we give an example, where the set G fulfills the conditions of Theorem 1.2 and the second alternative is valid, i.e. all points of G belong to the basin of attraction of a periodic orbit, but at the same time there are points of the boundary, the positive orbits of which tend to infinity.

Let us briefly sketch, how such an example can be constructed. We start with the two-dimensional system, which reads $\dot{r} = r(1-r)$ and $\dot{\phi} = -1$ in polar coordinates. Hence, the circle $r = 1$ is an exponentially asymptotically stable periodic orbit and its basin of attraction is $\mathbb{R}^2 \setminus \{0\}$ (cf. Figure 4, above). We will construct the example by modifying this system. The set G will be the union of tubes G_k and a set G_- which connects the tubes. While all points of G are attracted by the periodic orbit, $\overline{G}$ will include a line, the positive orbits of which are unbounded.

There is a $0 < r_0 < 1$ such that L is strictly negative for all points with $r > r_0$. Choose $-1 < i_0 < -r_0$ and $o_0 < -1$. The image points after time π and 2π, respectively, are called $(o_0', 0) := S_\pi(o_0, 0)$, $(o_0'', 0) := S_{2\pi}(o_0, 0)$, $(i_0', 0) := S_\pi(i_0, 0)$, $(i_0'', 0) := S_{2\pi}(i_0, 0)$. We have $o_0 < o_0''$. Choose a point $(o_\infty'', 0)$ such that $o_0 < < o_\infty'' < o_0''$ and denote its backward image points at times π and 2π, respectively, by $(o_\infty', 0)$ and $(o_\infty, 0)$, such that $S_\pi(o_\infty, 0) = (o_\infty', 0)$ and $S_{2\pi}(o_\infty, 0) = (o_\infty'', 0)$. Denote $G_- := \{q = S_t(x, 0) \mid i_0' < x < o_\infty', t \in (0, \pi]\}$.

All points $(x, 0)$ with $o_\infty < x < 0$ will be mapped to some points $S_\pi(x, 0) = = (x', 0)$ with $0 < x' < o_\infty'$. Choose points i_j, o_j with $j \in \mathbb{N}$ such that $o_0 > i_1 > > o_1 > i_2 > o_2 > \ldots > o_\infty$ and $\lim_{j\to\infty} o_j = o_\infty$ (cf. Figure 4, below). Then for the image points after time π such that $S_\pi(o_j, 0) = (o_j', 0)$ and $S_\pi(i_j, 0) = (i_j', 0)$ for all $j \in \mathbb{N}$ we have $o_0' < i_1' < \ldots < o_\infty'$. We will now smoothly transform the upper halfplane, but still the points $(x, 0)$ will reach their former destination points $(x', 0)$. In particular, $(i_j, 0)$ and $(o_j, 0)$ will eventually reach the points $(i_j', 0)$, $(o_j', 0)$ respectively.

Let the positive orbit of $(o_\infty, 0)$ tend to (o_∞, ∞) parallel to the y-axis (cf. Figure 4, below). Let the points $(x, 0)$ for which $o_\infty < x < i_1$ holds tend to $\left(o_\infty + \frac{|x-o_\infty|}{|i_1-o_\infty|}\frac{|i_1-o_\infty|+|i_1'-o_\infty'|}{2}, \infty\right)$ (dotted line). At the right hand side let the points $(x', 0)$ with $o_\infty' > x' > i_1'$ tend to $\left(o_\infty' - \frac{|x'-o_\infty'|}{|i_1'-o_\infty'|}\frac{|i_1-o_\infty|+|i_1'-o_\infty'|}{2}, \infty\right)$ when $t \to -\infty$. Correcting also the orbits outside these stripes, we can assure that $L(o_\infty, y) < 0$ and $L(o_\infty', y) < 0$ for all $y \geq 0$ and also that L is strictly negative on the two stripes. We can make this change smooth at the points $(x, 0)$ and also assure $L(x, 0) < 0$ for all $o_\infty \leq x \leq i_1$ and $i_1' \leq x \leq o_\infty'$.

Up to now the points of the stripe $(x, 0)$ with $o_\infty \leq x \leq i_1$ tend to infinity. We change the positive orbits so that – as we claimed before – the points $(x, 0)$ with $o_\infty < x < 0$ reach their original destination points $(x', 0)$ with $0 < x' < o_\infty'$. For all points $(x, 0)$ with $o_1 < x < i_1$ we keep the positive orbits up to height $y = 1$ and then lead them to the points $(\tilde{x}, 1)$, which will later reach the points $(x', 0)$. We can do that by further contracting the tube, such that $L < 0$ all the way from $(x, 1)$ to $(\tilde{x}, 1)$. The positive orbits of the points $(x, 0)$ with $i_1 < x < o_0$ are also

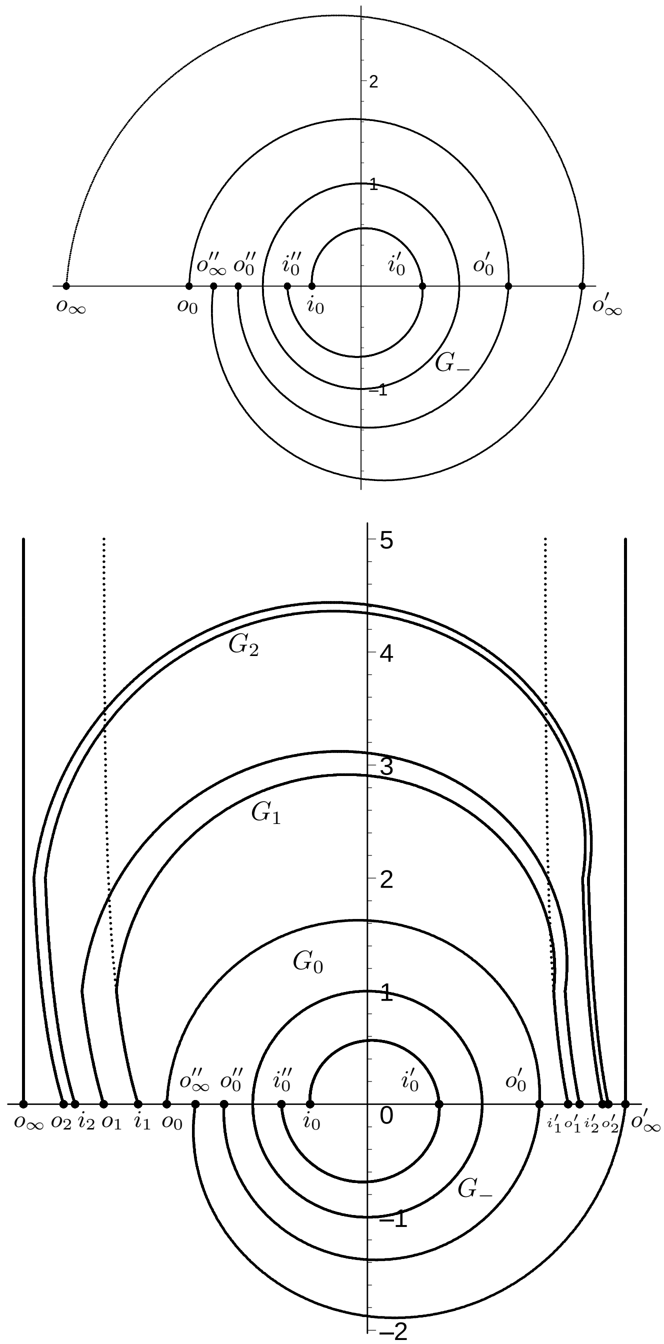

Figure 4. Sketch of the construction of a dynamical system and a set $G = \bigcup_{k=0}^{\infty} G_k \cup G_-$ such that G belongs to the basin of attraction of the unit circle which is a limit cycle and there are points of ∂G the positive orbit of which tend to infinity.

changed so that they will reach the points $(x', 0)$ as before. We can make this change smooth, which has not been done in the sketch in Figure 4, below.

Now proceed for $k = 2, 3, \ldots$ by changing the positive orbits for all points $(x, 0)$ with $o_k < x < i_k$. We keep the positive orbits up to height $y = k$ and then lead them to the points $(\tilde{x}, k)$, which will later reach the points $(x', 0)$. We can do that by further contracting the tube, such that $L < 0$ all the way from (x, k) to $(\tilde{x}, k)$. The positive orbits of the points $(x, 0)$ with $i_k < x < o_{k-1}$ are also changed so that they will eventually reach the points $(x', 0)$ as before. We can make this change smooth. Hence, all points $(x, 0)$ with $o_\infty < x \leq o_0$ reach points $(x', 0)$ with $o'_0 \leq x' < o'_\infty$.

We define $G_k := \left\{S_t(x, 0) \mid \frac{3}{4}o_k + \frac{1}{4}i_k < x < \frac{1}{4}o_k + \frac{3}{4}i_k, t \in (0, T_k(x)]\right\}$, where $T_k(x)$ is the time, when the positive orbit reaches the point $S_{T_k(x)}(x, 0) = (x', 0)$, and $k \in \mathbb{N}$. Moreover let us define the set

$$G_0 := \left\{S_t(x, 0) \mid \frac{1}{2}o_0 + \frac{1}{2}o''_\infty < x < i_0, t \in (0, T_0(x)]\right\}$$

where $T_0(x)$ is the time, when the positive orbit reaches the point $S_{T_0(x)}(x, 0) = = (x', 0)$.

Now set $G := \bigcup_{k=0}^\infty G_k \cup G_-$. We claim that G satisfies the conditions of Theorem 1.2. G is open. It is arcwise connected, because two points of G_k and G_l can be connected by a path leading to G_- and back. $\overline{G}$ contains the half-axes $\{(o_\infty, y) \mid y \geq 0\}$ and $\{(o'_\infty, y) \mid y \geq 0\}$. As we showed above, L is strictly negative on $\overline{G}$, this set is positively invariant by construction and does not contain the only equilibrium, which is situated in the origin. Since the periodic orbit $r = 1$ belongs to G there is a bounded positive orbit, and, by Theorem 1.2, G belongs to the basin of attraction of this periodic orbit. However, the points (o_∞, y) with $y \geq 0$ have unbounded positive orbits and hence $\overline{G}$ does not belong to the basin of attraction of the periodic orbit. Note that for a point $p_0 = (o_\infty, y) \in \partial G$ there is no line $\gamma^*(\lambda) := \lambda p_0 + (1 - \lambda)q$ such that $\gamma^*([0, 1)) \subset G$. This corresponds to the remark after Proposition 2.4.

Appendix A. Continuity of L

In Proposition A.2 we will show that the function $L(p)$ depends continuously on p. In order to do so we prove in Lemma A.1 the existence of a linear map N_ξ, which maps the hyperplane $f(p)^\perp$ to $f(p + \xi)^\perp$.

Lemma A.1. *Assume $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Let $D := \{x \in \mathbb{R}^n \mid f(x) \neq 0\}$ and fix $p \in D$. There is a $\delta > 0$ such that $B_\delta(p) := \{q \in \mathbb{R}^n \mid \|p - q\| < \delta\} \subset D$. For each $\xi \in B_\delta(0)$ there is a linear diffeomorphism $N_\xi \colon \mathbb{R}^n \to \mathbb{R}^n$ which depends continuously on ξ and satisfies $N_0 = id$, $\|N_\xi v\| = \|v\|$ and $N_\xi v \perp f(p + \xi) \Leftrightarrow v \perp f(p)$.*

Proof. For $\xi \in B_\delta(0)$ set

$$\begin{aligned} f &:= \frac{f(p)}{\|f(p)\|} \\ f_1(\xi) &:= \frac{f(p+\xi)}{\|f(p+\xi)\|} \\ s(\xi) &:= \langle f_1(\xi), f\rangle \in [-1,1] \\ f_2(\xi) &:= f_1(\xi) - s(\xi) f \end{aligned}$$

$s(\xi)$ denotes the cosine of the angle β between f and $f_1(\xi)$. If $s(\xi) \neq \pm 1$, $\{f, f_2(\xi)\}$ is an orthogonal basis in the $f, f_1(\xi)$-plane. Now we define $N_\xi v :=$ $:= v + [(s(\xi)-1)\langle f, v\rangle - \langle f_2(\xi), v\rangle] f + \left[-\frac{\langle f_2(\xi), v\rangle}{1+s(\xi)} + \langle f, v\rangle\right] f_2(\xi)$, if $s(\xi) \neq -1$, and $N_\xi v := v - 2\langle f, v\rangle f$, if $s(\xi) = -1$. N_ξ rotates in the $f, f_1(\xi)$-plane by β and keeps the other components. N_ξ^{-1} rotates in the same plane by $-\beta$. The statements of the lemma follow by simple calculations. □

Proposition A.2. *Assume $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Let $D := \{x \in \mathbb{R}^n \mid f(x) \neq 0\}$. Then* $L\colon \begin{cases} D \longrightarrow \mathbb{R} \\ p \longmapsto \max_{\|v\|=1, v\perp f(p)} \langle Df(p)v, v\rangle \end{cases}$ *is a continuous function.*

Proof. We assume in contradiction that there is a $p \in D$, an $\epsilon > 0$ and a sequence ξ_n with $p + \xi_n \in D$ and $\xi_n \to 0$, so that $|L(p+\xi_n) - L(p)| \geq \epsilon$ for all $n \in \mathbb{N}$. So there is either a subsequence satisfying $L(p+\xi_n) \geq L(p) + \epsilon$ or a subsequence satisfying $L(p+\xi_n) \leq L(p) - \epsilon$. These cases will be dealt with separately. There is a $\delta > 0$ such that $B_\delta(p) \subset D$ and we can assume that $\xi_n \in B_\delta(0)$ for all $n \in \mathbb{N}$. We consider the linear mapping $M_\xi := N_\xi^T Df(p+\xi) N_\xi$ with $\xi \in B_\delta(0)$ (cf. Lemma A.1). Since f is C^1 and because of Lemma A.1 M_ξ depends continuously on ξ. Therefore $M_{\xi_n} \stackrel{n\to\infty}{\longrightarrow} Df(p)$. Hence, there is a $N_0 \in \mathbb{N}$, so that

$$|v^T M_{\xi_n} v - v^T Df(p) v| \leq \frac{\epsilon}{2} \text{ for all } n \geq N_0 \text{ and } \|v\| = 1 \tag{25}$$

1. Case Let ξ_n be a subsequence with $L(p+\xi_n) \geq L(p) + \epsilon$. Let $N \geq N_0$. For $L(p+\xi_N)$ there exists by definition a w with $w \perp f(p+\xi_N)$, $\|w\| = 1$ and $L(p+\xi_N, w) = L(p+\xi_N) \geq L(p) + \epsilon$. Now we set $v := N_{\xi_N}^{-1} w$. By Lemma A.1 we have $v \perp f(p)$ and $\|v\| = 1$. Moreover,

$$\begin{aligned} L(p,v) &= v^T Df(p) v \\ &\geq v^T N_{\xi_N}^T Df(p+\xi_N) N_{\xi_N} v - \frac{\epsilon}{2} \text{ by (25)} \\ &= L(p+\xi_N, w) - \frac{\epsilon}{2} \\ &\geq L(p) + \frac{\epsilon}{2} \end{aligned}$$

in contradiction to $L(p) = \max_{\|v\|=1, v\perp f(p)} L(p,v)$.

2. Case Now let ξ_n be a subsequence satisfying $L(p+\xi_n) \leq L(p) - \epsilon$ and v be a vector with $\|v\| = 1$ and $v \perp f(p)$ such that $L(p,v) = L(p)$. Let $N \geq N_0$. Setting $w := N_{\xi_N} v$ we have $w \perp f(p+\xi_N)$ and $\|w\| = 1$ by Lemma A.1. Moreover,

$$\begin{aligned} L(p) - \epsilon &\geq L(p+\xi_N) \\ &\geq L(p+\xi_N, w) \\ &= (N_{\xi_N} v)^T Df(p+\xi_N) N_{\xi_N} v \\ &= v^T M_{\xi_N} v \\ &\geq L(p,v) - \frac{\epsilon}{2} \text{ by (25)} \\ &= L(p) - \frac{\epsilon}{2} \end{aligned}$$

That is a contradiction and thus we proved the continuity of $L(p)$. □

Appendix B. A sufficient condition for the existence of a limit cycle

In this appendix we prove Proposition B.1, which gives a more general condition for a point to belong to an exponentially asymptotically stable periodic orbit than needed in this paper. This proposition can also be used to prove analogous results in the time-periodic case, which will be published elsewhere. $g(\theta)$ defines a direction which is not perpendicular to $f(S_\theta p)$ and changes smoothly. In the proof of Theorem 1.2 we choose $g(\theta) = \frac{f(S_\theta p)}{\|f(S_\theta p)\|}$.

Proposition B.1. *Let $p \in \omega(p)$ where ω denotes the ω-limit set, and let p be no equilibrium point. Assume there is a continuous map $g\colon \mathbb{R}_0^+ \to \mathbb{R}^n$ with $\|g(\theta)\| = 1$ and $\langle g(\theta), f(S_\theta p)\rangle > 0$ for all $\theta \geq 0$. Moreover assume that there are constants $\delta, \nu > 0$ and $C \geq 1$ such that for all $\eta \in \mathbb{R}^n$ with $\eta \perp g(0)$ and $\|\eta\| \leq \delta$ there is a diffeomorphism $T_p^{p+\eta}\colon \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$, such that $T_p^{p+\eta}(\theta)$ depends continuously on η and satisfies $\frac{1}{2} \leq \dot{T}_p^{p+\eta}(\theta) \leq \frac{3}{2}$,*

$$\langle S_{T_p^{p+\eta}(\theta)}(p+\eta) - S_\theta p, g(\theta)\rangle = 0 \tag{26}$$

$$\text{and} \quad \|S_{T_p^{p+\eta}(\theta)}(p+\eta) - S_\theta p\| \leq Ce^{-\nu\theta}\|\eta\| \tag{27}$$

for all $\theta \geq 0$.

Then p is a point of an exponentially asymptotically stable periodic orbit.

Proof. The proof consists of four parts. First we choose new coordinates in order to characterize the behavior of solutions near p, which will be determined by $f(p)$. In the second step we define the hyperplane $H = p + g(0)^\perp$ and the operator π which maps nearby points to H along orbits. In the third step we define a Poincaré-like map P from the compact set U_0 into itself, where U_0 is a subset of the hyperplane H. We prove that the map is contracting and thus we can show the existence of a periodic orbit in the fourth step.

I. New coordinates

We will define a new coordinate system centered in p. We define by y the amount in $g(0)$-direction and by x the vectorial amount perpendicular to $g(0)$. The hyperplane $H := p + g(0)^\perp$ consists of precisely the points q, where $y(q) = 0$. We define for arbitrary $q \in \mathbb{R}^n$

$$y(q) := \langle q - p, g(0)\rangle \in \mathbb{R} \tag{28}$$

$$x(q) := q - p - y(q)g(0) \in g(0)^\perp. \tag{29}$$

Then we can express $q = p + y(q)g(0) + x(q)$ and $\|q - p\|^2 = |y(q)|^2 + \|x(q)\|^2$. We write the vectors $f(q)$ also in the new coordinates. We set

$$\lambda(q) := \langle f(q) - f(p), g(0)\rangle \in \mathbb{R} \tag{30}$$

$$u(q) := f(q) - f(p) - \lambda(q)g(0) \in g(0)^\perp. \tag{31}$$

Thus we can write

$$f(q) = f(p) + \lambda(q)\, g(0) + u(q). \tag{32}$$

Set $\alpha_0 := \langle g(0), f(p)\rangle > 0$. As f is continuous in p, there is a $0 < \delta_1 \le \delta$ such that for all $q \in B_{\delta_1}(p)$ the following inequalities hold

$$|\lambda(q)| \le \frac{1}{2}\alpha_0 \tag{33}$$

$$\|u(q)\| \le \|f(p)\|. \tag{34}$$

Inside the ball $B_{\delta_1}(p)$ orbits only can move within a cone. This is shown in the next lemma.

Lemma B.2. *Let $S_t q \in B_{\delta_1}(p)$ hold for all $t \in [0, \tilde\tau]$ with a constant $\tilde\tau > 0$. Then for all $t \in [0, \tilde\tau]$ and all τ_1, τ_2 with $0 \le \tau_1 \le \tau_2 \le \tilde\tau$ the following inequalities hold:*

$$\frac{1}{2}\alpha_0 \le \frac{d}{dt} y(S_t q) \le \frac{3}{2}\alpha_0 \tag{35}$$

$$\frac{1}{2}\alpha_0(\tau_2 - \tau_1) \le y(S_{\tau_2} q) - y(S_{\tau_1} q) \le \frac{3}{2}\alpha_0(\tau_2 - \tau_1) \tag{36}$$

$$\textit{and}\quad \|x(S_{\tau_2} q) - x(S_{\tau_1} q)\| \le k_0\big(y(S_{\tau_2} q) - y(S_{\tau_1} q)\big), \tag{37}$$

where $k_0 := 4\frac{\|f(p)\|}{\alpha_0}$.

Proof. Writing $S_t q = p + y(S_t q)\ g(0) + x(S_t q)$ we conclude

$$\begin{aligned} f(S_t q) &= \frac{d}{dt} S_t q \\ &= \frac{d}{dt} y(S_t q) \cdot g(0) + \frac{d}{dt} x(S_t q)\ . \end{aligned} \tag{38}$$

As $x(S_t q) \perp g(0)$ for all $t \in [0, \tilde\tau]$, we have $\frac{d}{dt} x(S_t q) \perp g(0)$ for all $t \in [0, \tilde\tau]$, too. Multiplying (38) with $g(0)$ yields because of (32)

$$\alpha_0 + \lambda(S_t q) = \frac{d}{dt} y(S_t q)\ .$$

Now we can conclude (35) by (33). Since $\int_{\tau_1}^{\tau_2} \frac{d}{dt} y\,(S_t q)\; dt = y\,(S_{\tau_2} q) - y\,(S_{\tau_1} q)$, (36) is shown.

Multiplying (38) by $\frac{d}{dt} x\,(S_t q)$ we get with $\frac{d}{dt} x\,(S_t q) \perp g(0)$

$$\begin{aligned} \left\| \frac{d}{dt} x\,(S_t q) \right\|^2 &= \left\langle f(S_t q), \frac{d}{dt} x\,(S_t q) \right\rangle \\ &= \left\langle f(p) + u(S_t q), \frac{d}{dt} x\,(S_t q) \right\rangle \text{ by (32), i.e.,} \end{aligned}$$

$$\left\| \frac{d}{dt} x(S_t q) \right\| \leq \|f(p)\| + \|u(S_t q)\|. \tag{39}$$

Hence,

$$\begin{aligned} \|x(S_{\tau_2} q) - x(S_{\tau_1} q)\| &= \left\| \int_{\tau_1}^{\tau_2} \frac{d}{dt} x(S_t q)\, dt \right\| \\ &\leq \int_{\tau_1}^{\tau_2} \left\| \frac{d}{dt} x(S_t q) \right\| dt \\ &\leq \int_{\tau_1}^{\tau_2} (\|f(p)\| + \|u(S_t q)\|)\; dt \text{ by (39)} \\ &\leq 2(\tau_2 - \tau_1)\|f(p)\| \text{ by (34)} \\ &\leq k_0 \big(y(S_{\tau_2} q) - y(S_{\tau_1} q) \big) \text{ by (36) .} \end{aligned}$$

This proves (37) and the lemma. □

II. Correction to the hyperplane H

Orbits through points in a small ball around p intersect the hyperplane $H := := p + g(0)^\perp$ once in a short time interval. This enables us to define the projection π on H in the following lemma.

Lemma B.3. *Set* $\epsilon_0 := 2\|f(p)\| + \frac{1}{2}\alpha_0$ *and* $\delta_2 := \frac{1}{2} \frac{\delta_1}{\frac{2\epsilon_0}{\alpha_0} + 1}$.

We define the continuous map

$$\pi \colon \begin{cases} B_{\delta_2}(p) & \longrightarrow \quad H := p + g(0)^\perp \\ \quad q & \longmapsto \quad \pi(q) \end{cases} \tag{40}$$

by $\pi(q) = S_{t^*(q)} q$ *with a continuous function* $t^*(q)$ *satisfying* $|t^*(q)| \leq \frac{2\delta_2}{\alpha_0} =: t_0$ *for all* $q \in B_{\delta_2}(p)$.
For $p^* \in H \cap B_{\delta_1}(p)$ *we have*

$$\|\pi(q) - p^*\| \quad \leq \quad (k_0 + 1)\|q - p^*\| \,. \tag{41}$$

Proof. We show that for $q \in B_{\delta_2}(p)$ the continuous function $y(S_\tau q)$ vanishes for some $\tau = t^*$. We prove this using (36) and show afterwards that t^* is close enough to zero so that $S_\tau q$ remains in $B_{\delta_1}(p)$ for all τ between 0 and t^*.

We consider only the case $y(q) \leq 0$. As long as $S_\tau q \in B_{\delta_1}(p)$ with $\tau \geq 0$, (36) implies $y(S_\tau q) \geq y(q) + \frac{\tau}{2}\alpha_0$. For $\tilde{\tau} = -\frac{2}{\alpha_0}y(q) \geq 0$ we have $y(S_{\tilde{\tau}} q) \geq 0$. Note that $|\tilde{\tau}| \leq \frac{2}{\alpha_0}\delta_2 = t_0$. The intermediate value theorem implies the existence of a $t^* \in [0, \tilde{\tau}]$ satisfying $y(S_{t^*}q) = 0$. As $y(S_\tau q)$ is monotonously increasing in τ by Lemma B.2, t^* is unique. By the implicit function theorem we can define $t^*(q)$ by $y(S_{t^*(q)}q) = 0$. As y and S_t are continuous functions, so is t^*, which also proves the continuity of π.

Now we check that $S_\tau q$ remains in $B_{\delta_1}(p)$ for all $\tau \in [0, \tilde{\tau}]$. Assuming the opposite, there is a $\tau_0 \in [0, \tilde{\tau}]$ with $\|S_{\tau_0}q - p\| = \delta_1$ and $\|S_\tau q - p\| < \delta_1$ for all $\tau \in [0, \tau_0)$. By (32), (33) and (34) we have $\|f(q)\| \leq 2\|f(p)\| + \frac{1}{2}\alpha_0 = \epsilon_0$ for all $q \in B_{\delta_1}(p)$. This yields

$$\begin{aligned}
\delta_1 &= \|S_{\tau_0}q - p\| \\
&\leq \left\| \int_0^{\tau_0} f(S_\tau q)\, d\tau \right\| + \|q - p\| \\
&\leq |\tilde{\tau}|\epsilon_0 + \delta_2 \\
&\leq \delta_2 \left(\frac{2\epsilon_0}{\alpha_0} + 1 \right) = \frac{\delta_1}{2},
\end{aligned}$$

which is a contradiction.

To finally prove (41) we have by (28)

$$\begin{aligned}
y(q) &= \langle q - p, g(0) \rangle \\
&= \langle q - p^*, g(0) \rangle + \underbrace{\langle p^* - p, g(0) \rangle}_{=0}.
\end{aligned}$$

Hence, $|y(q)| \leq \|q - p^*\|$. (37) implies

$$\|x(\pi(q)) - x(q)\| \leq k_0\, |y(q)| \leq k_0 \|q - p^*\|.$$

We conclude

$$\begin{aligned}
\|\pi(q) - p^*\| &= \|x(\pi(q)) - x(p^*)\| \\
&\leq \|x(\pi(q)) - x(q)\| + \|x(q) - x(p^*)\| \\
&\leq (k_0 + 1)\|q - p^*\|.
\end{aligned}$$

This completes the proof of Lemma B.3. □

III. A Poincaré-like map

We have $p \in \omega(p)$. Thus, there is a $T^* \geq 3t_0 + 2\frac{\ln[2C(k_0+1)]}{\nu}$, so that $S_{T^*}p \in$ $\in B_{\delta^*}(p)$, where $\delta^* := \frac{\delta_2}{2(k_0+1)}$. Lemma B.3 shows that there is a $T_1 \in [T^* - t_0,$ $T^* + t_0]$ so that $p_1 := S_{T_1}p = \pi(S_{T^*}p)$ satisfies $p_1 \in H$ and $\|p_1 - p\| \leq \frac{\delta_2}{2}$ by (41). Note that $T_1 \geq 2\left(t_0 + \frac{\ln[2C(k_0+1)]}{\nu}\right) \geq \frac{\ln[2C(k_0+1)]}{\nu}$. Set $\delta_3 := 2\|p_1 - p\| \leq \delta_2 \leq$ $\leq \delta_1 \leq \delta$. By assumption (27) a point q of the compact set $U_0 := H \cap \overline{B_{\delta_3}(p)}$ will reach a point $q_1 := S_{T_p^q(T_1)}q$ which satisfies $\|q_1 - p_1\| \leq Ce^{-\nu T_1}\|q - p\| \leq$

$\leq \frac{\delta_3}{2(k_0+1)} < \delta_2$ (cf. the definition of T_1). By Lemma B.3 the point $\pi(q_1)$ fulfills $\pi(q_1) \in H$ and by (41) $\|\pi(q_1) - p_1\| \leq (k_0+1)\|q_1 - p_1\| \leq \frac{\delta_3}{2}$.

Hence, we can define a continuous Poincaré-like map

$$P\colon \begin{cases} U_0 & \longrightarrow & U_0 \\ q & \longmapsto & \pi\left(S_{T_p^q(T_1)} q\right) \end{cases} \tag{42}$$

It is a return map, but not necessarily the *first* return to the hyperplane H. To prove $P(U_0) \subset U_0$ we calculate $\|P(q)-p\| \leq \|P(q)-p_1\| + \|p_1 - p\| \leq \frac{\delta_3}{2} + \frac{\delta_3}{2}$. P is continuous, because so are π, S_t and T_p^q. By definition of π we have $P(q) = S_{\tau(q)}q$ for a continuous map τ with $\tau(q) \geq \frac{T_1}{2} - t_0 \geq \frac{\ln[2C(k_0+1)]}{\nu} > 0$ for all $q \in U_0$. In Lemma B.4 we will show that the diameter of $P^k(U_0) =: U_k$ decreases.

Lemma B.4. *We define the compact sets* $U_k \subset H$ *for all* $k \in \mathbb{N}$ *by* $U_k := P^k(U_0)$ *and the points* $p_k := P^k(p) \in U_k$.

Then the following statements hold for all $k \in \mathbb{N}$.

$$U_k \subset U_{k-1} \tag{43}$$

$$\operatorname{diam} U_k \leq \frac{\delta_3}{2^{k-1}}. \tag{44}$$

Proof. (43) follows easily from $U_1 = P(U_0) \subset U_0$. Indeed, for $k \geq 2$ we have $P^k(U_0) = P^{k-1}P(U_0) \subset P^{k-1}(U_0)$. The sets U_k are compact by induction, because they are images of the compact set U_{k-1} under the continuous map P.

In order to prove (44) we give a new characterization of U_k. We show that we reach the same points no matter whether we apply π after each return or only once at the end. We make this precise.

The points p_k belong to the positive orbit through p. So we define T_k satisfying $p_k = P(p_{k-1}) = S_{T_k} p_{k-1}$ for all $k \in \mathbb{N}$, denoting $p_0 := p$. From above we know $T_k \geq \frac{\ln[2C(k_0+1)]}{\nu}$. We define

$$V_k := \{S_{T_p^q(\sum_{i=1}^k T_i)} q \mid q \in U_0\}.$$

We claim $P^k(q) = \pi(q_k)$ for all $q \in U_0$ and all $k \in \mathbb{N}$, where $q_k := S_{T_p^q(\sum_{i=1}^k T_i)} q$. In particular we have $U_k = \pi(V_k)$.

Fix a $k \in \mathbb{N}$. We already know $\pi(p_k) = p_k = P^k(p)$ and $p_k \in U_k \cap \pi(V_k)$ which is the claim for $q = p$. Moreover, we have $U_k, \pi(V_k) \subset H$, and all points of both U_k and $\pi(V_k)$ can be written as $S_{\tau_i(q)} q$ with $q \in U_0$ and a continuous function τ_1, τ_2 respectively. We will use this to prove the above claim. For $q \in U_0$ we consider

$$Q(\tau, q) = \langle S_\tau q - p, g(0)\rangle.$$

We have $Q(\sum_{i=1}^k T_i, p) = \langle p_k - p, g(0)\rangle = 0$ and $\partial_\tau Q(\tau, q) = \langle f(S_\tau q), g(0)\rangle \geq \geq \frac{\alpha_0}{2} > 0$ for all $S_\tau q \in B_{\delta_1}(p)$ by (32) and (33). In particular $\partial_\tau Q(\sum_{i=1}^k T_i, p) \neq 0$, so the implicit function theorem yields a *unique* continuous function $\tau(q)$ near p such that $Q(\tau(q), q) = 0$, which is equivalent to $S_{\tau(q)} q \in H$. As τ_1 and τ_2 are such functions, they have to coincide near p. By prolongation we get $\tau_1 = \tau_2$ on $U_0 \subset B_{\delta_1}(p)$. Thus for $q_k = S_{T_p^q(\sum_{i=1}^k T_i)} q$ we have $P^k(q) = \pi(q_k)$.

In order to prove (44) we consider a point $q \in U_0$. Setting $q_k := S_{T_p^q(\sum_{i=1}^k T_i)} q$ we have $P^k(q) = \pi(q_k)$, as we have shown above. (27) yields $\|q_k - p_k\| \leq$ $\leq Ce^{-\nu \sum_{i=1}^k T_i} \|q - p\| \leq \frac{C\delta_3}{[2C(k_0+1)]^k}$. Note that $C, (k_0 + 1) \geq 1$. This yields $\|P^k(q) - p_k\| = \|\pi(q_k) - p_k\| \leq (k_0 + 1) \frac{\delta_3}{2^k(k_0+1)}$ for all $k \in \mathbb{N}$ by (41). As

$$\begin{aligned} \operatorname{diam} U_k &= \max_{q', q'' \in U_0} \|P^k(q') - P^k(q'')\| \\ &\leq \max_{q' \in U_0} \|P^k(q') - p_k\| + \max_{q'' \in U_0} \|p_k - P^k(q'')\| \\ &\leq 2 \frac{\delta_3}{2^k}, \end{aligned}$$

(44) is proven. □

IV. The periodic orbit

In Lemma B.4 we have constructed a sequence of compact sets U_k with decreasing diameter, so we know that there is one and only one point $\tilde{p}$ which lies in all U_k, $k \in \mathbb{N}_0$. Since $P(\tilde{p})$ lies in all U_k as well, $\tilde{p}$ is a fixed point of P. Hence, there is a time $T > 0$ with $S_T \tilde{p} = \tilde{p}$, and thus $\tilde{p}$ is a point of a periodic orbit Ω. Let T be the minimal period. Since $\tilde{p} = p + \eta$ with $\eta \perp g(0)$ and $\|\eta\| \leq \delta$, by assumption (27) the ω-limit sets of p and $\tilde{p}$ are equal. This is shown as in the proof of Proposition 2.3. Thus $p \in \omega(p) = \omega(\tilde{p}) = \Omega$ and p is a point of the periodic orbit Ω.

Finally we show that the periodic orbit $\Omega := \{S_\theta p \mid \theta \in [0, T]\}$ is exponentially asymptotically stable. Define the set $H_0 := H \cap \overline{B_\delta(p)}$, $P_\theta H_0 := \{S_{T_p^q(\theta)} q \mid q \in H_0\}$ and $E := \bigcup_{\theta \in [0,T]} P_\theta H_0$. Obviously the points of E are attracted by Ω exponentially fast (cf. (27)). We show now that the trajectory of each point of a neighborhood of Ω meets a point of E in a finite time.

Lemma B.5. *Define* $\delta_m := \min_{q \in H, \|q\| = \delta, \theta \in [0,T]} \left\| S_{T_p^q(\theta)} q - S_\theta p \right\| > 0$. *There are* $\delta_2', t_0' > 0$ *such that for each* $q \in \mathbb{R}^n$ *with* $\operatorname{dist}(q, \Omega) \leq \delta_2'$ *there is a* t *with* $|t| \leq t_0'$ *such that* $S_t q = S_\theta p + \eta$ *with* $\theta \in [0, T]$, $\|\eta\| \leq \delta_m$ *and* $\langle \eta, g(\theta) \rangle = 0$.

Proof. We define new coordinates like in I., but this time for all points $S_\theta p$, $\theta \in [0, T]$. We call these coordinates $y_\theta(q)$, $x_\theta(q)$, $\lambda_\theta(q)$ and $u_\theta(q)$. Since $[0, T]$ is a compact set, $f_M := \max_{\theta \in [0,T]} \|f(S_\theta p)\|$, $\alpha_m := \min_{\theta \in [0,T]} \langle g(\theta), f(S_\theta p) \rangle > 0$ and $\alpha_M := \max_{\theta \in [0,T]} \langle g(\theta), f(S_\theta p) \rangle > 0$ exist. Choose $\delta_1' > 0$ such that we have $|\lambda_\theta(q)| \leq \frac{1}{2} \alpha_m$ and $\|u_\theta(q)\| \leq f_M$ for all q such that there is a $\theta \in [0, T]$ with $\|q - S_\theta p\| \leq \delta_1'$. Set $k_0' := \frac{4}{\alpha_m} f_M$, $\epsilon_0' := 2 f_M + \frac{1}{2} \alpha_m$, $\delta_2' := \min \left(\frac{1}{2} \frac{\delta_1'}{\frac{2\epsilon_0'}{\alpha_m} + 1}, \frac{\delta_m}{k_0' + 1} \right)$ and $t_0' := \frac{2\delta_2'}{\alpha_m}$.

Given a point q with $\operatorname{dist}(q, \Omega) \leq \delta_2'$, choose a time $\theta \in [0, T]$ such that $q \in \overline{B_{\delta_2'}(S_\theta p)}$. Lemma B.2 holds in the following form (cf. the proof of Lemma B.2):

Lemma B.6. *Fix* $\theta \in [0, T]$. *Let* $S_t q \in B_{\delta_1'}(S_\theta p)$ *hold for all* $t \in [0, \tilde{\tau}]$ *with a constant* $\tilde{\tau} > 0$.

Then for all $t \in [0, \tilde{\tau}]$ and all τ_1, τ_2 with $0 \leq \tau_1 \leq \tau_2 \leq \tilde{\tau}$ the following inequalities hold:

$$\begin{aligned} \frac{1}{2}\alpha_m \leq \quad & \tfrac{d}{dt} y_\theta(S_t q) & \leq \frac{1}{2}\alpha_m + \alpha_M \\ \frac{1}{2}\alpha_m(\tau_2 - \tau_1) \leq \quad & y_\theta(S_{\tau_2} q) - y_\theta(S_{\tau_1} q) & \leq \left(\frac{1}{2}\alpha_m + \alpha_M\right)(\tau_2 - \tau_1) \end{aligned}$$

$$\text{and} \quad \|x_\theta(S_{\tau_2} q) - x_\theta(S_{\tau_1} q)\| \quad \leq \quad k_0'\big(y_\theta(S_{\tau_2} q) - y_\theta(S_{\tau_1} q)\big).$$

Lemma B.3 also holds in a modified form, defining $\pi'_\theta \colon B_{\delta'_2}(S_\theta p) \to S_\theta p + g(\theta)^\perp$. Thus we can write $\pi'_\theta(q) = S_{t'(q)} q = S_\theta p + \eta$ with $\eta \perp g(\theta)$ and $|t'(q)| \leq t'_0$. By the equivalent of (41) we have $\|\eta\| \leq (k'_0 + 1)\delta'_2 \leq \delta_m$. This proves Lemma B.5. □

We have $P_\theta H_0 \supset \{S_\theta p + \eta \mid \|\eta\| \leq \delta_m, \eta \perp g(\theta)\}$ for all $\theta \in [0, T]$. Hence, Lemma B.5 shows that for all points q of the neighborhood $\Omega_{\delta'_2}$ we have $S_t q \in \bigcup_{\theta \in [0,T]} P_\theta H_0 \subset E$ for a $|t| \leq t'_0$. This shows that the periodic orbit is exponentially asymptotically stable and concludes the proof of Proposition B.1. □

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