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IRREDUCIBLE IDENTITIES OF n-ALGEBRAS

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Abstract. One can generalize the notion of n-Lie algebra (in the sense of Fillipov) and define "weak n-Lie algebra" to be an anticommutative n-ary algebra $(A, [\cdot, \ldots, \cdot])$ such that the $(n-1)$-ary bracket $[\cdot, \ldots, \cdot]_a = [\cdot, \ldots, \cdot, a]$ is an $(n-1)$-Lie bracket on A for all a in A. It is well known that every n-Lie algebra is weak n-Lie algebra. Under some additional assumptions these notions coincide. We show that it is not the case in general. By means of representation theory of symmetric groups a full description of n-bracket multilinear identities of degree 2 that can be satisfied by an anticommutative n-ary algebra is obtained. This is a solution to the conjectures proposed by M. Bremner. These methods allow us to prove that the dual representation of an n-Lie algebra is in fact a representation in the sense of Kasymov. We also consider the generalizations of n-Lie algebra proposed by A. Vinogradov, M. Vinogradov and Gautheron. Some correlation between these generalizations can be easily seen. We also describe the kernel of the expansion map.

Introduction

Following Fillipov [**F**], an *n-Lie algebra* is a vector space A together with an antisymmetric n-argument operation $[\cdot, \ldots, \cdot] : A \times \ldots \times A \to A$ which satisfies the general Jacobi identity (GJI)

$$[[u_1, u_{2,} \ldots, u_n], u_{n+1}, u_{n+2,} \ldots, u_{2n-1}] =$$

$$= \sum_{i=1}^{n} (-1)^{i+1} [[u_i, u_{n+1}, u_{n+2}, \ldots, u_{2n-1}], u_1, u_{2,} \ldots, \widehat{u_i}, \ldots, u_n]. \tag{GJI}$$

The following are the standard examples of n-Lie algebras.

Example 1. [**F**] Let V be an $(n+1)$-dimensional oriented Euclidian space. Define

$$[v_1, \ldots, v_n] = v_1 \times \ldots \times v_n,$$

where $v_1 \times \ldots \times v_n$ is the vector product of the vectors $v_i \in V$. Then $(V, [\cdot, \ldots, \cdot])$ is an n-Lie algebra.

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Example 2. Let $C^\infty(\mathbb{R}^k)$ be the algebra of C^∞-functions on $\mathbb{R}^k$ and $x_1, \ldots, x_k$ be the coordinates on $\mathbb{R}^k$. Define

$$[f_1, \ldots, f_n] = \det \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j=1,\ldots,n}, \; n \leq k \, . \tag{1}$$

Then $(C^\infty(\mathbb{R}^k), [\cdot, \ldots, \cdot])$ is an n-Lie algebra.

The bracket (1) satisfies also the following Leibniz rule

$$[f \cdot g,\, f_2,\, f_3, \ldots, f_n] = f \cdot [g,\, f_2,\, f_3, \ldots, f_n] + g \cdot [f,\, f_2,\, f_3, \ldots, f_n]. \tag{LR}$$

It gives rise to a definition of Nambu-Poisson manifold to be a pair $(C^\infty(M), [\cdot, \ldots, \cdot])$, where M is a differential manifold and the antisymmetric bracket $[\cdot, \ldots, \cdot]$ satisfies the general Jacobi identity (GJI) and the Leibniz rule (LR). The bracket of the form (1) was considered by Nambu [**N**]. Then Takhtajan [**T**] discovered the identity (GJI) in its physical context in generalized formulations of Hamiltonian Mechanics.

These structures are widely studied. It is easily seen that every Nambu-Poisson structure on M is given by an n-vector field V on M:

$$[f_1, \ldots, f_n]\,(p) = V_p(f_1, \ldots, f_n) \quad \text{for } p \in M.$$

It is known (see [**AG**], [**Ga2**]) that if $V_p \neq 0$ and V defines an n-Nambu-Poisson structure then V, in some neighbourhood of p, can be given in the form

$$V = \partial_{x_1} \wedge \partial_{x_2} \wedge \ldots \wedge \partial_{x_n}$$

for some coordinate functions $x_1, \ldots, x_n$.

Several authors have been investigating the properties of n-Lie algebras. It was already noticed in [**F**] that each n-Lie algebra A gives a family of $(n-1)$-Lie algebras with the operations defined by

$$[x_1, \ldots, x_{n-1}]_a = [x_1, \ldots, x_{n-1}, a], \; a \in A.$$

An anticommutative n-algebra A such that $(A, [\cdot, \ldots, \cdot]_a)$ is an $(n-1)$-Lie algebra for all a in A we call a *weak n-Lie algebra.*

Under some additional assumptions the notions of "weak n-Lie algebra" and "n-Lie algebra" coincide. For example, it is proved in [**GM**] that every weak n-algebra bracket on $C^\infty(M)$ satisfying the Leibniz rule (LR) must automatically satisfy the Jacobi identity (GJI). Thus it gives rise to a Nambu-Poisson manifold.

We will prove that in general a weak n-Lie algebra needs not to be an n-Lie algebra. Our example will use the concept of a *free n-Lie algebra* based on the construction of a free n-algebra given in [**HW**].

By an identity in some n-algebra we mean an n-ary polynomial which is zero when the variables of the polynomial are replaced by elements of the n-algebra. Sometimes, to emphasize a point, we will put "= 0" on the right of the polynomial identity. It is well known that, in case of the characteristic of the ground field equal to zero, every identity is equivalent to the linearizations of its homogenous components. For example, we get the linearization of the identity

$$[[x_1, x_2, x_3], x_1, x_1] + [[x_1, x_2, x_3], x_2, x_3] \tag{2}$$

if we put $x_1 = a_1 + a_2 + a_3$, $x_2 = b_1 + b_2$, $x_3 = c_1 + c_2$ and omit the monomials in which some of the variables a_1, a_2, a_3, b_1, b_2, c_1, c_2 occur at least twice. The obtained identity is equivalent to (2) in case char$A \neq 2, 3$. From now on we assume the ground field to be of characteristic zero.

We will consider multilinear n-ary identities in which each term involves two bracket operations. Our approach is a well-known one, treating the space freely spanned by two-bracket terms as an $\mathbb{S}_d$-module ($d = 2n-1$) acting on the variables of the terms.This gives us a $\binom{2n-1}{n}$-dimensional $\mathbb{S}_{2n-1}$-module, denoted as P_n^2, considered in [**B**]. We will give a list of all simple components of P_n^2 together with a list of the corresponding identities. Note that if A is an anticommutative n-algebra satisfying a multilinear identity I in variables $u_1, \ldots, u_d$ and $\sigma \in \mathbb{S}_d$ then A satisfies also the identity I^σ coming from I by interchanging u_i with $u_{\sigma(i)}$. Thus if we treat I as an element of P_n^2 then A satisfies also every identity which belongs to the $\mathbb{S}_d$ -submodule generated by I. Because, as we will see, the decomposition of P_n^2 into simple submodules is homogeneous we can work out what exactly the submodule generated by I is. The decomposition of P_n^2 is the thesis of conjecture 3 from [**B**] which generalizes the conjecture 1 of [**B**]. This reduces the study of two-bracket identities to a finite set of identities.

Some authors have introduced some generalizations of the n-algebras by imposing on the n-ary skew-symmetric operation ω of an n-algebra (A, ω) some additional conditions. For example Gautheron ([**Ga**]) considered the following identity

$$[\omega_{a_1,\ldots,a_{n-k}}, \omega]^{RN} = 0 \tag{3}$$

where $[\cdot,\cdot]^{RN}$ is the Richardson-Nijenhuis bracket of skew-symmetric maps and $\omega_{a_1,\ldots,a_{n-k}}(x_1, \ldots, x_k) = \omega(x_1, \ldots, x_k, a_1, \ldots, a_{n-k})$. Also in [**V**] the following identity

$$[\omega_{a_1,\ldots,a_k}, \omega_{b_1,\ldots,b_l}]^{RN} = 0$$

is under consideration.

Gautheron has shown that the condition (3) for odd k is equivalent to the identity

$$\sum_\sigma \operatorname{sgn}(\sigma)[a_1, a_2, \ldots, a_{n-k-1}, x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_k}, [x_{\sigma_{k+1}}, \ldots, x_{\sigma_{n+k}}]] = 0$$

for all $a_1, a_2, \ldots, a_{n-k-1}$ in A, where the summation is taken over all permutations of x's. We will achieve description of theses identities in term of submodules of P_n^2. From this one can easily see some correlations between some of these generalizations.

Kasymov in [**K**] considered representations of an n-Lie algebra. Since we have a simple method for proving (or disproving) an identity in some special n-algebras (e.g. n-Lie algebras) we are able to show that the notion of dual representation in the theory of n-Lie algebras makes sense.

At the end we will also give the description of the kernel of the $\mathbb{S}_{2n-1}$-map from P_n^2 to $\mathbb{C}\mathbb{S}_{2n-1}$, called *expansion map* (see [**B**]), moving a basis vector$[[x_1, \ldots, x_n],$ $x_{n+1}, \ldots, x_{2n-1}]$ to

$$\sum_{\substack{\sigma\in\mathbb{S}\{1,2,\ldots,n\}\\ \tau\in\mathbb{S}\{n+1,n+2,\ldots,2n-1\}\\ i=1\ldots n}} A_{\sigma\tau}$$

where

$$A_{\sigma\tau} = \operatorname{sgn}(\sigma\tau)(-1)^{i+1}x_{\tau(n+1)}x_{\tau(n+2)}\ldots x_{\tau(n+i-1)}\cdot x_{\sigma 1}\ldots x_{\sigma n}\cdot x_{\tau(n+i)}\ldots x_{\tau(2n-1)}.$$

Free n-Lie algebra

We fix an integer $n \geq 1$.

Let T be a rooted tree. The set of its nodes splits up into two sets: the set of leaves denoted by $\mathcal{L}(T)$ and the set of interior nodes.

Definition. An *N-labelled tree* is a pair (T,ϕ), where T is a rooted tree in which every interior node has exactly n children and $\phi : \mathcal{L}(T) \to N$ is a function to a set N.

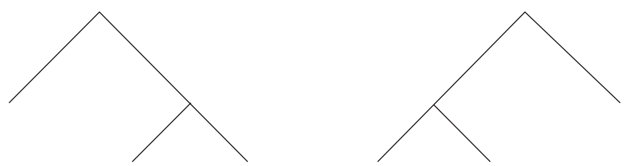

Figure 1

Note that the children of a node in an N-labelled tree are ordered so that the trees in the Figure 1. are different. Every N-labelled tree gives rise to a pure bracket in variables x_i, $i \in N$, e.g. $[[x_1, [x_2, x_1, x_3], x_1], [x_5, x_3, x_2], x_2]$ is represented by the tree in Figure 2.

Consider the space W with the basis consisting of all N-labelled trees, where N is a fixed set. We have an n-ary bracket on W in the obvious way. The space W is graded

$$W = \bigoplus_{k=0}^{\infty} W_k\,,$$

where W_k is a subspace spanned by all N-labelled trees with k interior nodes. Note that

$$[W_{a_1}, W_{a_2}, \ldots, W_{a_n}] \subseteq W_{a_1+\ldots+a_n+1}\,.$$

Moreover, we can decompose $W_2 = W_2^{'} \oplus W_2^{''}$ where $W_2^{'}$ is a subspace of W_2 spanned by all N-labelled trees with 2 interior nodes and ϕ being injective. Here $W_2^{''}$ is spanned by those N-labelled trees with ϕ not being injective.

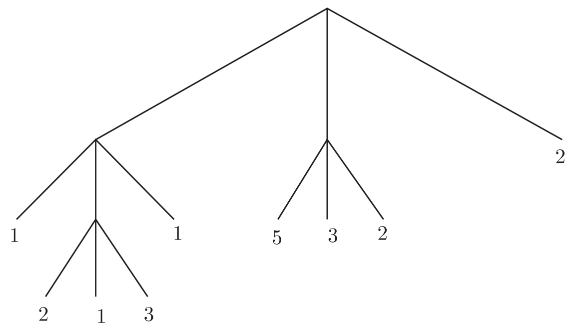

Figure 2

Let $\mathbb{S}_N = \mathbb{S}_{|N|}$ be the full permutation group of the set N. Then $\mathbb{S}_N$ acts on W by changing the labelling of the base vectors. Moreover, the action of $\mathbb{S}_N$ is homogeneous, i.e. $\mathbb{S}_N W_k \subseteq W_k$ and $\mathbb{S}_N W_2' \subseteq W_2'$, $\mathbb{S}_N W_2'' \subseteq W_2''$.

Consider the expressions

$$A(u_1, u_2, \dots, u_n; \sigma) = [u_1, u_2, \dots, u_n] - \operatorname{sgn}(\sigma)\,[u_{\sigma(1)}, u_{\sigma(2)}, \dots, u_{\sigma(n)}],\ \sigma \in \mathbb{S}_N$$

$$\begin{aligned} J(u_1, u_2, \dots, u_{2n-1}) &= [[u_1, u_2, \dots, u_n], u_{n+1}, u_{n+2}, \dots, u_{2n-1}] - \\ &- \sum_{i=1}^{n} (-1)^{i+1} [[u_i, u_{n+1}, u_{n+2}, \dots, u_{2n-1}], u_1, u_2, \dots, \widehat{u_i}, \dots, u_n], \end{aligned} \tag{4}$$

$$\begin{aligned} WJ(u_1, u_2, \dots, u_{2n-1}) &= J(u_1, u_2, \dots, u_{2n-1}) + \\ &+ J(u_1, u_2, \dots, u_{n-1}, u_{2n-1}, u_{n+1}, u_{n+2}, \dots, u_{2n-2}, u_n). \end{aligned} \tag{5}$$

The meaning of u_i's will be explained later.

For the following paragraph take the expression $WJ(x_1, x_2, \dots, x_{2n-1}) = 0$ as an identity in variables $x_1, x_2, \dots, x_{2n-1}$ in an n-algebra. Note that an anticommutative n-algebra A is a weak n-Lie algebra if and only if A satisfies this identity. We will refer to this identity by WJI (weak Jacobi identity).

Let us take an N-labelled tree T and its interior node v and let $u_1, u_2, \dots, u_n$ be the subtrees rooted at the children of v. Let $\sigma \in \mathbb{S}_n$ and T^σ be the tree obtained from T by replacing each u_i with $u_{\sigma(i)}$. Set V_A to be a subspace of W spanned by elements of the form $T - \operatorname{sgn}(\sigma) T^\sigma$. Take again an N-labelled tree T and its interior nodes x and y with y being the left-most child of x. Let u_{n+1}, $u_{n+2}, \dots, u_{2n-1}$ be the trees whose roots are other children of x and u_1, $u_2, \dots, u_n$ be those whose roots are the children of y (See Figure 3). Let $T^{\alpha_1, \dots, \alpha_{2n-1}}$, α_i's in $\{1, 2, \dots, 2n-1\}$, be the tree obtained from T by replacing u_i with u_{α_i}. Consider the subspace V_J spanned by the elements of the form

$$T - \sum_{i=1}^{n} (-1)^{i+1} T^{i, n+1, n+2, \dots, 2n-1, 1, 2, \dots, \hat{\imath}, \dots, n}$$

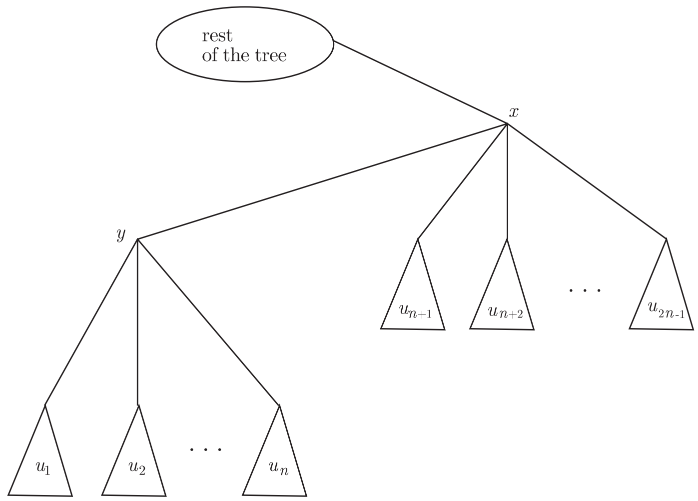

Figure 3

In the same way we can define the subspace V_{WJ} spanned by elements of the form associated with the weak Jacobi identity (WJI).

It is clear from the definitions of V_J and V_{WJ} that $[V_J, W, \ldots, W] \subseteq V_J$ and $[V_{WJ}, W, \ldots, W] \subseteq V_{WJ}$. Hence there is a well defined bracket $[\cdot, \ldots, \cdot]$ on the spaces $A_J = W/V_J$ and $A_{WJ} = W/V_{WJ}$ inherited from W. Thus A_J and A_{WJ} are respectively n-Lie and weak n-Lie algebras. We will call them *free n-Lie* and *free weak n-Lie* algebras with the set of generators equal to N.

Note that the elements which span V_J and V_{WJ} are homogeneous, i.e. belong to a single component W_k. This applies also to the components $W_2^{'}$ and $W_2^{''}$. This implies that the algebras A_J and A_{WJ} are graded algebras with respect to the bracket operation.

The Construction of the example

In this section we construct an example of weak n-Lie algebra which does not satisfies (GJI).

Take $N = \{1, 2, \ldots, 2n-1\}$ and write

$$A_{WJ} = A_0 \oplus A_1 \oplus A_2^{'} \oplus A_2^{''} \oplus \bigoplus_{k=3}^{\infty} A_k,$$

where A_{WJ} is the free weak n-Lie algebra with the set of generators equal to N and $A_i = W_i/W_i \cap V_{WJ}$, $A_2' = W_2'/W_2' \cap V_{WJ}$ and $A_2^{''} = W_2''/W_2'' \cap V_{WJ}$. Note the dimensions:

$$A_0 = \operatorname{span}\{x_1, \ldots, x_{2n-1}\}, \ \dim A_0 = 2n-1,$$

$A_1 = \mathrm{span}\{[x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}] \colon 1 \le \alpha_1 < \ldots < \alpha_n \le 2n{-}1\}$, $\dim A_1 = \binom{2n-1}{n}$.

We can view A_2' as an $\mathbb{S}_{2n-1}$-module isomorphic to P_n^2/M where P_n^2 is a $\binom{2n-1}{n}$ dimensional module spanned by the elements

$$\begin{aligned} [[x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}], x_{\alpha_{n+1}}, x_{\alpha_{n+2}}, \ldots, x_{\alpha_{2n-1}}],\ 1 &\le \alpha_1 < \ldots < \alpha_n \le 2n-1, \\ 1 &\le \alpha_{n+1} < \ldots < \alpha_{2n-1} \le 2n-1 \end{aligned}$$

and M is the $\mathbb{S}_{2n-1}$-module generated by the identity expression $WJ(x_1, \ldots, x_{2n-1})$.

Remark. We also deal with elements of the form $[[x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}], x_{\alpha_{n+1}}, x_{\alpha_{n+2}}, \ldots, x_{\alpha_{2n-1}}]$ as elements of P_n^2, where the condition on α_i's is weakened to $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}\} = \{1, \ldots, 2n-1\}$. Such an element is, of course, equal to one of the base vectors up to a sign. Then any n-bracket multilinear identity in $x_1, \ldots, x_{2n-1}$ in which each term involves two n-bracket operations can be seen as an element of P_n^2, so saying that an identity generates a submodule of P_n^2 makes sense.

The conjecture in the paper [**B**] says that

$$P_n^2 \simeq \bigoplus_{i=1}^{n} [2^{n-i} 1^{2i-1}],$$

where $[2^i 1^j]$ is the module associated with the Young diagram of the partition

$$2i + j = \underbrace{2 + 2 + \ldots + 2}_{i \text{ times}} + \underbrace{1 + \ldots + 1}_{j \text{ times}}.$$

For $n = 3, 4$ the author [**B**] gives a full list of modules and associated identities they are generated by. For example, for $n = 3$ the weak Jacobi identity generates the 4 dimensional module $[2\,1^3]$ while the Jacobi identity – the 5 dimensional module $[21^3] \oplus [1^5]$.

Our example of the weak n-Lie algebra is as follows. Take $B = A_0 \oplus A_1 \oplus A_2'$. This is a quotient algebra of the weak free n-Lie algebra A described above. We will show that it does not satisfy the Jacobi identity. Note that $[[A_1, B, B, \ldots, B], B, B, \ldots, B] = 0$ and $[A_2', B, B, \ldots, B] = 0$. Moreover, $[[x_i, B, B, \ldots, B], x_i, B, B, \ldots, B]] = 0$. Because the Jacobi identity is multilinear, it is enough to verify it on the basis of A_0. Now it is enough to show that the element $J(x_1, \ldots, x_{2n-1})$ does not belong to M, a submodule of P_n^2. This is equivalent to say that the modules generated by the weak Jacobi identity and the Jacobi identity differ. This is the case for $n = 3, 4$, according to the result of [**B**]. The general case is the subject of the next section. Thus B is a weak n-Lie algebra which is not an n-Lie algebra.

The main theorem

In this section we will prove the following theorem:

Theorem 1. *Identifying n-algebra identities involving two brackets with elements of P_n^2 we have the following:*

1. $P_n^2 \simeq \bigoplus_{i=1}^{n} [2^{n-i}1^{2i-1}]$.
2. *The Engel's identity*

$$[[b, a_1, a_2, \ldots, a_{n-1}], a_1, a_2, \ldots, a_{n-1}], \tag{6}$$

(more precise, its linearization) generates the submodule $[2^{n-1}1]$.

3. *The identity for* $[2^{n-i}1^{2i-1}]$ *is*

$$\sum_{\substack{\sigma\in \\ \mathbb{S}(\{1,\ldots,2i-1\})}} \operatorname{sgn}(\sigma)\, [[x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(i)}]_Y, x_{\sigma(i+1)}, \ldots, x_{\sigma(2i-1)}]_Y\ , \tag{7}$$

where $[u_1, u_2, \ldots, u_i]_Y = [u_1, u_2, \ldots, u_i, y_1, y_2, \ldots, y_{n-i}]$,

4. *The Jacobi identity generates the submodule*

$$\bigoplus_{i=2}^{n} [2^{n-i}1^{2i-1}].$$

5. *The weak Jacobi identity generates the submodule*

$$\bigoplus_{i=2}^{n-1} [2^{n-i}1^{2i-1}].$$

We precede the proof with some notation and preliminary results.
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of d, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$, $\lambda_1 + \lambda_2 + \ldots + \lambda_k = d$. To a partition λ we can associate a labelled Young diagram Y_λ. For example, for the partition $\lambda = (5, 3, 3, 1, 1, 1)$ of $d = 14$ we can associate

$$Y_\lambda = \begin{array}{|c|c|c|c|c|}
\hline 1 & 2 & 3 & 4 & 5 \\
\hline 6 & 7 & 8 \\
\cline{1-3} 9 & 10 & 11 \\
\cline{1-3} 12 \\
\cline{1-1} 13 \\
\cline{1-1} 14 \\
\cline{1-1}
\end{array}$$

It has λ_i boxes in the ith row, the rows of boxes lined up to the left. The labelling of boxes is injective.

For a given numbered Young diagram consider the subgroups of $\mathbb{S}_d$:

$$\begin{aligned}
P_\lambda &= \{g \in \mathbb{S}_d : g \text{ preserves rows of } \lambda\}, \\
Q_\lambda &= \{g \in \mathbb{S}_d : g \text{ preserves columns of } \lambda\}
\end{aligned}$$

and the elements

$$\mathbf{a}_\lambda = \sum_{g\in P} g \in \mathbb{C}\mathbb{S}_d,$$
$$\mathbf{b}_\lambda = \sum_{g\in Q} \operatorname{sgn}(g)\, g \in \mathbb{C}\mathbb{S}_d.$$

The *conjugate partition* λ' to the partition λ is defined by interchanging rows and columns in the Young diagram, i.e. reflecting the diagram in the 45° line.

Let us quote some basic results on representations of $\mathbb{S}_d$.

Theorem 2. [**FH**] *The module $V_\lambda = \mathbb{C}\mathbb{S}_d \mathbf{a}_\lambda \mathbf{b}_\lambda$ is irreducible. Every finite dimensional irreducible representation of $\mathbb{S}_d$ can be obtained in this way for a unique partition. Changing the labelling of the Young diagram leads, of course, to equivalent representation.*

Theorem 3. [**FH**] *$V_{\lambda'} \simeq V_\lambda \otimes U$, where U is the one dimensional alternating representation, i.e. $g\cdot u = \operatorname{sgn}(g)\, u$, for $g \in \mathbb{S}_d$, $u\in U$.*

Theorem 4. [**FH**, Exercise 4.48]

$$\mathbb{C}\mathbb{S}_d \mathbf{b}_\lambda \simeq \bigoplus_\tau K_{\tau',\lambda'}\, V_\tau\ ,$$

where $K_{\mu,\tau}$ are so called Kostka's numbers and can be defined combinatorially as the number of ways to fill the boxes of the Young diagram for μ with τ_1 of $1's$, τ_2 of $2's$, up to τ_k of $k's$, in such a way that the entries in each row are non-decreasing, and those in each column are strictly increasing.

Define now a linear function

$$\phi : P_n^2 \to \mathbb{C}\mathbb{S}_{2n-1},$$

$$\phi([[x_{i_1}, x_{i_2}, \ldots, x_{i_n}], x_{j_1},\ldots, x_{j_{n-1}}]) = \sum_{\substack{\sigma\in\mathbb{S}_n\\ \tau\in\mathbb{S}_{n-1}}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\, \mu_{\sigma,\tau}\ , \tag{8}$$

where $[[x_{i_1}, x_{i_2}, \ldots, x_{i_n}], x_{j_1},\ldots, x_{j_{n-1}}]$ is the standard basis vector of P_n^2 and
$\mu_{\sigma,\tau} \in \mathbb{S}_{2n-1}, \qquad \mu_{\sigma,\tau} = \begin{pmatrix} 1 & \ldots & n & n+1 & \ldots & 2n-1 \\ i_{\sigma(1)} & \ldots & i_{\sigma(n)} & j_{\tau(1)} & \ldots & j_{\tau(n-1)} \end{pmatrix}$
$\sigma \in \mathbb{S}_{\{1,\ldots,n\}}, \qquad \tau\in \mathbb{S}_{\{1,\ldots,n-1\}}$.

Lemma 1. *The following holds:*

(i) *ϕ is injective,*

(ii) *ϕ is a homomorphism of $\mathbb{S}_{2n-1}$-modules, where $\mathbb{S}_{2n-1}$ acts by re-numbering, i.e. for $g\in\mathbb{S}_{2n-1}$*

$$g\cdot[[x_{i_1}, x_{i_2}, \ldots, x_{i_n}], x_{j_1},\ldots, x_{j_{n-1}}] = [[x_{g(i_1)}, x_{g(i_2)}, \ldots, x_{g(i_n)}], x_{g(j_1)}, \ldots, x_{g(j_{n-1})}].$$

Proof. Note that the definition of ϕ in (8) is still correct if we drop the assumption that i_k's and j_k's are in the increasing order.

(i) Follows from the fact that for different basis vectors of P_n^2, x and x', we have $\operatorname{supp}(\phi(x)) \cap \operatorname{supp}(\phi(x')) = \emptyset$.
(ii) This is obvious. One may identify $\mu \in \mathbb{S}_{2n-1}$with the element $x_{\mu 1}x_{\mu 2}\cdots x_{\mu(2n-1)}$ in the free associative algebra. Then $\mathbb{S}_{2n-1}$ acts by permuting the generators $x_1, \ldots, x_{2n-1}$.

□

We will write the same symbol for a partition and for the unique (up to isomorphism) irreducible $\mathbb{S}_{2n-1}$-module associated with this partition (e.g. $[2^{n-1}1]$). We hope it will not lead to misunderstanding.

Next step is to see that $\phi(P_n^2) = \mathbb{C}\mathbb{S}_{2n-1}\mathbf{b}_\lambda$, where $\lambda = (\underbrace{2, 2, \ldots, 2}_{n-1}, 1) = [2^{n-1}1]$,

and

$$Y_\lambda = \begin{array}{|c|c|} \hline 1 & n+1 \\ \hline 2 & n+2 \\ \hline \vdots & \vdots \\ \hline n-1 & 2n-1 \\ \hline n & \\ \cline{1-1} \end{array}$$

It is true, since P_n^2 as an $\mathbb{S}_{2n-1}$-module is generated by $t = [[x_1, x_2, \ldots, x_n], x_{n+1,\ldots}, x_{2n-1}]$ and we have $\phi(t) = b_\lambda$.

Now we are ready to prove the main theorem.

Proof of Theorem 1. 1. We need to prove that $\mathbb{C}\mathbb{S}_{2n-1}\mathbf{b}_\lambda \simeq \bigoplus_{i=1}^n [2^{n-i}1^{2i-1}]$, where $\lambda = [2^{n-1}1]$.
Introduce ([**FH**]) the standard ordering on the set of partitions of a given number d :

$$\tau > \mu \text{ iff the first non vanishing } \tau_i - \mu_i \text{ is greater than } 0\ .$$

It is easy to see that $K_{\mu\tau} = 0$ for $\mu < \tau$ and

$$K_{\tau',(n,n-1)} \neq 0 \text{ iff } \tau = (2^{n-i}1^{2i-1}) \text{ for some } i = 1, 2, \ldots, n.$$

For such a τ we have $K_{\tau',(n,n-1)} = 1$. This, in view of Theorem 4, justifies the decomposition $\mathbb{C}\mathbb{S}_{2n-1}\mathbf{b}_\lambda \simeq \bigoplus_{i=1}^n [2^{n-i}1^{2i-1}]$.
2. The Engel's identity (6) has the following multilinear form

$$E \stackrel{def}{=} \sum_{\substack{\varepsilon:\{1,\ldots,n-1\} \\ \to\{0,1\}}} [[x_0, x_{i_1}, x_{i_2}, \ldots, x_{i_{n-1}}], x_{j_1,\ldots}, x_{j_{n-1}}]\ ,$$

where $i_k = k + \varepsilon(k)\,(n-1)$, $j_k = k + (1-\varepsilon(k))\,(n-1)$, $k = 1, 2, \ldots, n-1$, so $i_k \equiv j_k \,(\operatorname{mod} n-1)$, $i_k \neq j_k$.
Note that

$$\phi(E) = \mathbf{a}_\lambda \mathbf{b}_\lambda\ ,$$

where

$$Y_\lambda = \begin{array}{|c|c|} \hline 1 & n \\ \hline 2 & n+1 \\ \hline \vdots & \vdots \\ \hline n-1 & 2n-2 \\ \hline 0 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Indeed, since

$$\phi([[x_0, x_{i_1}, x_{i_2}, \ldots, x_{i_{n-1}}], x_{j_1,\ldots}, x_{j_{n-1}}]) = \begin{pmatrix} 0 & 1 & \ldots & n & \ldots & 2n-2 \\ 0 & i_1 & \ldots & j_1 & \ldots & j_{n-1} \end{pmatrix} \cdot b_\lambda$$

and $\begin{pmatrix} 0 & 1 & \ldots & n & \ldots & 2n-2 \\ 0 & i_1 & \ldots & j_1 & \ldots & j_{n-1} \end{pmatrix}$ is a generic element of $\operatorname{supp} \mathbf{a}_\lambda = P_\lambda$, we get (2).

3. For $i = 1$ this reduces to the statement (2).

Let I be an ordered set of symbols

$$I = \{s_1, s_2, \ldots, s_{2i-1}, t_1, t_2, \ldots, t_{n-i}, t'_1, t'_2, \ldots, t'_{n-i}\}.$$

The linearization of the identity (7) has the following form

$$T' = \sum_{\substack{\sigma \in \mathbb{S}(\{s_1, s_2, \ldots, s_{2i-1}\}), \\ \varepsilon : \{1, \ldots, n-i\} \to \{0,1\}}} B_{\sigma,\varepsilon} \tag{9}$$

where

$$B_{\sigma,\varepsilon} = \operatorname{sgn}(\sigma) \left[[x_{\sigma(s_1)}, \ldots, x_{\sigma(s_i)}, y_{1,\ldots}, y_{n-i}], x_{\sigma(s_{i+1})}, \ldots, x_{\sigma(s_{2i-1})}, y'_{1,\ldots}, y'_{n-i} \right],$$

and

$$y_k = \begin{cases} x_{t_k} & \text{if } \varepsilon(k) = 0 \\ x_{t'_k} & \text{if } \varepsilon(k) = 1 \end{cases} \text{ and } y'_k = \begin{cases} x_{t_k} & \text{if } \varepsilon(k) = 1 \\ x_{t'_k} & \text{if } \varepsilon(k) = 0 \end{cases}.$$

Consider the group $\mathbb{S}(I) = \mathbb{S}_{2n-1} \simeq \mathbb{S}(\{1, \ldots, 2n-1\})$, where the pairing is as follows

$$\begin{aligned} I &\leftrightarrow \{1, \ldots, 2n-1\}, \\ s_k &\leftrightarrow k, \text{ for } k = 1, \ldots, i \\ t_k &\leftrightarrow i + k, \text{ for } k = 1, \ldots, n-i \\ s_{i+k} &\leftrightarrow n + k, \text{ for } k = 1, \ldots, i-1 \\ t'_k &\leftrightarrow n + k + i - 1, \text{ for } k = 1, \ldots, n-i\ . \end{aligned}$$

Let

$$\lambda = [2^{n-i}1^{2i-1}], \quad Y_\lambda = \begin{array}{|c|c|} \hline t_1 & t'_1 \\ \hline t_2 & t'_2 \\ \hline \vdots & \vdots \\ \hline t_{n-i} & t'_{n-i} \\ \hline s_1 \\ \cline{1-1} \vdots \\ \cline{1-1} s_{2i-1} \\ \cline{1-1} \end{array}$$

with the above identifications. Consider the map defined by:

$$\psi : \mathbb{CS}_{2n-1} \to P_n^2,$$

$$\psi(x_{i_1} x_{i_2} \dots x_{i_{2n-1}}) = [[x_{i_1}, x_{i_2}, \dots, x_{i_n}], x_{i_{n+1}}, \dots, x_{i_{2n-1}}]. \tag{10}$$

This is obviously homomorphism of left $\mathbb{CS}_{2n-1}$-modules. One can verify that $\psi(\mathbf{a}_\lambda \mathbf{b}_\lambda)$ is a non-zero scalar multiple of T'. (Notice that some terms in $\psi(\mathbf{a}_\lambda \mathbf{b}_\lambda)$ reduces, since $[[\cdots], \cdots]$ is anticommutative on the last $n-1$ positions.) Now $\psi([2^{n-i}1^{2i-1}])$ must be an irreducible module, i.e. the module generated by T'.

4. We will prove this in two steps.

(a) For each $i = 2, \dots, n$, $[2^{n-i}1^{2i-1}] \subseteq \mathbb{CS}_{2n-1} J$, where $J = J(x_1, x_2, \dots, x_{2n-1})$ as in (4).
By (3) it remains to prove that (GJI) implies the identity (7).

Note that it is enough to show this for $i = n$, as if $(A, [\cdot, \dots, \cdot])$ satisfies (GJI), then $(A, [\cdot, \dots, \cdot]_y)$ so. Let $H = \mathbb{S}_n \times \mathbb{S}_{n-1} \subset \mathbb{S}_{2n-1}$. For $h \in H$ we have

$$h \cdot J = \operatorname{sgn}(h)\, J\ .$$

Hence,

$$\sum_{g \in \mathbb{S}_{2n-1}} \operatorname{sgn}(g)\, g \cdot J = |H| \sum_{\substack{\sigma_1 < \dots < \sigma_n \\ \sigma_{n+1} < \dots < \sigma_{2n-1}}} \operatorname{sgn}(\sigma)\, [[x_{\sigma_1}, \dots, x_{\sigma_n}], x_{\sigma_{n+1}}, \dots, x_{\sigma_{2n-1}}]\ ,$$

what ends the proof of (a).

(b) $J \in [2^{n-1}1]^\perp = \bigoplus_{i=2}^n [2^{n-i}1^{2i-1}]$.
If it isn't true, we get $[2^{n-1}1] \subseteq \mathbb{CS}_{2n-1} J$ by the simplicity of $[2^{n-1}1]$ and part 1. By (a) we get $\mathbb{CS}_{2n-1} J = P_n^2$. This implies that every n-Lie algebra satisfies

$$[[u_1, u_2, \dots, u_n], u_{n+1}, u_{n+2}, \dots, u_{2n-1}] = 0,$$

thus a contradiction.

5. Part 4. shows that the identity (GJI) is equivalent to the system of $n-1$ identities (7). Hence the weak Jacobi identity, as being the general Jacobi identity for the bracket $[\cdot,\dots,\cdot]_a$, is equivalent to the system of $n-2$ identities (7) for the bracket $[\cdot,\dots,\cdot]_a$, so to the identities (7) for the bracket $[\cdot,\dots,\cdot]$ and $i=2,3,\dots,n-1$. This ends the proof of the theorem. $\square$

Dual Representation

Following Kasymov [**K**], a representation of an n-Lie algebra is a linear map

$$\rho : \bigwedge^{n-1} A \longrightarrow \mathrm{End}(V),$$

where V is a vector space and ρ satisfies the following two conditions

$$[\rho_{(a)}, \rho_{(b)}] = \sum_{i=1}^{n-1} \rho(a_1,\dots,[a_i,b_1,\dots,b_{n-1}],\dots,a_{n-1}) \tag{11}$$

$$\rho([a_1,\dots,a_n],b_2,\dots,b_{n-1}) = \sum_{i=1}^{n}(-1)^{i+1}\rho(a_i,b_2,\dots,b_{n-1})\rho(a_1,\dots,\widehat{a_i},\dots,a_n) \tag{12}$$

where we used the notation $(x)=(x_1,\dots,x_{n-1})$.
One can check that the adjoint map $\rho = ad$, $ad(a_1,\dots,a_{n-1})(a_0) = [a_0,a_1,a_2,\dots,a_{n-1}]$ is a representation of an n-Lie algebra.

There is a Loday algebra structure on $\bigwedge^{n-1} A$ (see [**Ga2**]) given by

$$\{(a),(b)\} = \sum_{i=1}^{n-1} a_1\wedge\ldots\wedge[a_i,b_1,\dots,b_{n-1}]\wedge\ldots\wedge a_{n-1}\ .$$

This means $\{\cdot,\cdot\}$ is not necessary skew-symmetric but satisfies $\{Z,\{X,Y\}\} = \{\{Z,X\},Y\} - \{\{Z,Y\},X\}$.
The condition (11) says that ρ is a representation of Loday algebra $(\bigwedge^{n-1} A, \{\cdot,\cdot\})$ or equivalently a representation of Lie algebra $\frac{(A,\{\cdot,\cdot\})}{\langle\{x,y\}+\{y,x\}\rangle}$, where $\langle\{x,y\}+\{y,x\}\rangle$ stands for the two-sided ideal in Loday algebra generated by elements of the form $\{x,y\}+\{y,x\}$.

Let $\rho : \bigwedge^{n-1} A \longrightarrow \mathrm{End}(V)$ be a representation of an n-Lie algebra A. Then ρ induces a map on dual space V^* given by $-\rho^T$, where T stands for the transposition map, which certainly satisfies the condition (11). We will show that the condition (12) is also satisfied.

Theorem 5. *If ρ is a representation of an n-Lie algebra then the same is true for the dual $-\rho^T$.*

Proof. We need to show that

$$-\rho^T([a_1,\dots,a_n],b_2,\dots,b_{n-1}]) =$$

$$= \sum_{i=1}^{n} (-1)^{i+1} \rho^T(a_i, b_2, \ldots, b_{n-1}) \rho^T(a_1, \ldots, \widehat{a_i}, \ldots, a_n) \; .$$

Let L and R be, respectively, the left and the right side of the above equation. We have

$$\begin{aligned} R^T &= \sum_{i=1}^{n} (-1)^{i+1} \rho(a_1, \ldots, \widehat{a_i}, \ldots, a_n) \rho(a_i, b_2, \ldots, b_{n-1}) = \\ &= \sum_{i=1}^{n} (-1)^{i+1} \rho(a_i, b_2, \ldots, b_{n-1}) \rho(a_1, \ldots, \widehat{a_i}, \ldots, a_n) + \\ &\quad + \sum_{i=1}^{n} (-1)^{i+1} [\rho(a_1, \ldots, \widehat{a_i}, \ldots, a_n), \rho(a_i, b_2, \ldots, b_{n-1})] = \\ &= \rho([a_1, \ldots, a_n], b_2, \ldots, b_{n-1}) + \\ &\quad + 2 \sum_{s<t} (-1)^{s+t} \rho([a_s, a_t, b_2, \ldots, b_{n-1}], a_1, \ldots, \widehat{a_s}, \ldots, \widehat{a_t}, \ldots, a_n) \; . \end{aligned}$$

We are left to show that

$$\rho([a_1, \ldots, a_n], b_2, \ldots, b_{n-1}]) +$$

$$+ \sum_{s<t} (-1)^{s+t} \rho([a_s, a_t, b_2, \ldots, b_{n-1}], a_1, \ldots, \widehat{a_s}, \ldots, \widehat{a_t}, \ldots, a_n) = 0 \; . \qquad (\mathrm{T}_0)$$

First we will show (T_0) for the adjoint representation i.e. we will prove that the following identity holds in any n-Lie algebra:

$$[c, [a_1, \ldots, a_n], b_2, \ldots, b_{n-1}] +$$

$$+ \sum_{s<t} (-1)^{s+t} [c, [a_s, a_t, b_2, \ldots, b_{n-1}], a_1, \ldots, \widehat{a_s}, \ldots, \widehat{a_t}, \ldots, a_n] = 0. \qquad (\mathrm{T})$$

Let U be an n-ary identity of degree 2. If necessary, we can linearize U and treat U as an element in P_n^2. Denote M_U for the $\mathbb{S}_{2n-1}$-submodule of P_n^2 generated by the identity U. Then M_U is a direct sum of its simple components. Thanks to homogenity of P_n^2 we know exactly what identities generate these simple components. Thus U is equivalent to the system of some identities of the form specified in Theorem 1.

As we know $P_2^n \cong M_E \oplus M_J$, where $M_E \cong [2^{n-1}1]$ is associated with the Engel's identity

$$[[b, a_1, \ldots, a_{n-1}], a_1, \ldots, a_{n-1}]] = 0 \qquad (\mathrm{E})$$

and $M_J \cong \bigoplus_{i=2}^{n} [2^{n-i} 1^{2i-1}]$ is associated with (GJI).

We want to show that (GJI) implies (T) or equivalently that $M_T \subseteq M_J$. It is enough to show that $M_E \not\subseteq M_T$ or equivalently that (T) does not implies the Engel's identity. We will be done if we find an n-algebra which satisfies (T) but not (E).

We take the n-algebra V_n from Example 1 with the bracket being the vector product. Let $\{e_1,\dots,e_{n+1}\}$ be the basis of V_n. Taking $b=e_n$ and $a_i=e_i$ for $i=1,2,\dots,n-1$ in (E) we see that V_n does not satisfies the Engel's identity.

Next we will show that V_n satisfies (T). It is easy to see that T is antisymmetric with respect to a_i's and also to b_i's. Thus it is enough to check (T) in case a_i's being different basis vectors and the same for b_i's. Take $b_i=e_i$ for $i=2,\dots,n-1$. Then $[c,\,[a_s,a_t,b_2,\dots,b_{n-1}],\,a_1,\dots,\widehat{a_s},\dots,\widehat{a_t},\dots,a_n]\neq 0$ only if $[a_1,\dots,a_n]=\pm e_1$ or $\pm e_n$ or $\pm e_{n+1}$. From symmetry it is enough to verify only one of these cases. Let for example $a_i=e_i$ for $i=1,2,\dots,n$. Then the only non zero summand in the sum of (T) occurs when $s=1$ and $t=n$. But

$$[c,[e_1,e_n,e_2,\dots,e_{n-1}],e_2,\dots,e_{n-1}]=-[c,e_{n+1},e_2,\dots,e_{n-1}].$$

We continue the proof of (T_0) for any ρ.

Consider the vector space Q with basis given by

$$\begin{aligned}&\rho([x_{i_1},\dots,x_{i_n}],x_{i_{n+1}},\dots,x_{i_{2n-2}}),\qquad i_1<\ldots<i_n,\quad i_{n+1}<\ldots<i_{2n-2},\\ &\{i_1,\dots,i_{2n-2}\}=\{1,\dots,2n-2\},\\ &\rho(x_{j_1},\dots,x_{j_{n-1}})\rho(x_{j_n},\dots,x_{j_{2n-2}}),\qquad j_1<\ldots<j_{n-1},\ j_n<\ldots<j_{2n-2},\\ &\{j_1,\dots,j_{2n-2}\}=\{1,\dots,2n-2\}\ .\end{aligned}$$

Notice that $\dim Q=\binom{2n-2}{n}+\binom{2n-2}{n-1}=\binom{2n-1}{n}=\dim P_2^n$. Moreover, there is an isomorphism $\Phi_{x_0}:Q\longrightarrow P_2^n$ defined on basis by

$$\rho([x_{i_1},\dots,x_{i_n}],x_{i_{n+1}},\dots,x_{i_{2n-2}})\xmapsto{\Phi_{x_0}}[x_0,[x_{i_1},\dots,x_{i_n}],,x_{i_{n+1}},\dots,x_{i_{2n-2}}],$$

$$\rho(x_{j_1},\dots,x_{j_{n-1}})\rho(x_{j_n},\dots,x_{j_{2n-2}})\longmapsto[[x_0,x_{j_1},\dots,x_{j_{n-1}}],x_{j_n},\dots,x_{j_{2n-2}}]\ .$$

We have already shown that the Jacobi identity implies the identity $\Phi_{x_0}(T_0)=T_{x_0}$. Hence

$$T_{x_0}=\sum_{\sigma\in\mathbb{S}_{2n-1}}u_\sigma J^\sigma,$$

for some scalars u_σ, where J^σ is the Jacobi identity with variables permuted accordingly to σ. It is easy to see that $J^\sigma=\Phi_{x_0}(J_1^\tau)$ or $\Phi_{x_0}(J_2^\tau)$ for some $\tau\in\mathbb{S}_{2n-2}$, where J_1 and J_2 are the identities (11) i (12) with possibly permuted variables. Hence

$$T_0=\sum_\tau a_\tau\,J_1^\tau\ +\sum_\tau b_\tau\,J_2^\tau\ ,$$

with scalars a_τ,b_τ. This completes our proof. □

Other Identities

Let us recall the definition of the Richardson-Nijenhuis bracket. Let A be a vector space and let Alt^nA be the set of all skew-symmetric n-linear maps from A to A.

Definition. We fix a map $L \in Alt^l A$ and define

$$L[\cdot] : Alt^n A \to Alt^{n+l-1} A,$$

$$L[N](a_1, a_2, \ldots, a_{l+n-1}) = \sum_{\substack{|I| = n, |J| = l-1, \\ I \cup J = \{1, \ldots, l+n-1\}}} (-1)^{(I,J)} L(N(a_I), a_J),$$

where $a_I = (a_{i_1}, \ldots, a_{i_k})$ for $I = \{i_1 < i_2 < \ldots < i_k\}$. The summation is over all ordered subsets $I \subset \{1, \ldots, n+l-1\}$ and $(-1)^{(I,J)}$ stands for the sign of the permutation $(1, \ldots, n+l-1) \mapsto (i_1, \ldots, i_n, j_1, \ldots, j_{l-1})$.

Definition. Let L (respectively N) be a skew-symmetric l-linear (n-linear) map from A to A. The Richardson-Nijenhuis bracket $[[L, N]]^{RN} \in Alt^{l+n-1}$ is defined by

$$[[L, N]]^{RN} = (-1)^{(l-1)(n-1)} L[N] - N[L].$$

Definition. If $t \in \mathbb{C}\mathbb{S}_{2n-1}$, we will sometimes write $\overline{t} = \psi(t)$ for the element in P_n^2 represented by t. For an element $\overline{t} \in P_n^2$ and $\sigma \in \mathbb{S}_{2n-1}$ one can define $\overline{t}^{\sigma} \in P_n^2$ by

$$\overline{t}^{\sigma} = \psi(tbc\sigma),$$

where

$$b = \sum_{g \in \mathbb{S}\{1, \ldots, n\}} \operatorname{sgn}(g) g, \quad c = \sum_{g \in \mathbb{S}\{n+1, \ldots, 2n-1\}} \operatorname{sgn}(g) g. \tag{13}$$

We extend the definition of $\overline{t}^{\sigma}$ by linearity for all $\sigma \in \mathbb{C}\mathbb{S}_{2n-1}$. For example

$$\begin{aligned} [[x, y], z]^{1+(23)} &= [[x, y], z] - [[y, x], z] + [[x, z], y] - [[y, z], x] = \\ &= 2[[x, y], z] + [[x, z], y] - [[y, z], x]. \end{aligned}$$

Notice that this definition is correct, i.e. $\overline{t}^{\sigma}$ does not depend on the choice of the representant $t \in \mathbb{C}\mathbb{S}_{2n-1}$ but on $\bar{t}$.

The following lemma will be useful in determining the simple modules generated by an identity.

Lemma 2. *Let*

$$d_u = \left(\sum_{g \in \mathbb{S}\{1, \ldots, u, n+1, \ldots, n+u-1\}} \operatorname{sgn}(g) g \right) \cdot \prod_{i=u+1}^{n} (1 + (i, n+i-1)),$$

$d_u \in \mathbb{C}\mathbb{S}_{2n-1}$, $u \in \{1, \ldots, n\}$, b, c *as in (13). Then*

1. *The element* $\phi(\overline{d_u}) = d_u bc$ *generates the module isomorphic to* $[2^{n-u} 1^{2u-1}]$.
2. *The module generated by an identity* $T \in P_n^2$ *contains the module* $[2^{n-u} 1^{2u-1}]$ *if and only if* $T^{d_u} \neq 0$.

Proof.

1. In fact $\overline{d_u} = T$ and $\phi(T) = d_u bc$, where T is the linearization of the identity (7). Since ϕ is a monomorphism, the thesis follows from Theorem 1 (3).

2. We already know that

$$\mathbb{C}\mathbb{S}_{2n-1}\phi(P_n^2) = \bigoplus_{u=1}^{n} \mathbb{C}\mathbb{S}_{2n-1} d_u bc\,.$$

For non-isomorphic simple modules $M, N \subset \mathbb{C}\mathbb{S}_{2n-1}$ we have $M{\cdot}N = \{mn | m \in M, n \in N\} = 0$. Write

$$\phi(T) = \sum_{u=1}^{n} t_u d_u bc$$

for some $t_u \in \mathbb{C}\mathbb{S}_{2n-1}$. Notice $d_u \cdot d_u$ is a scalar multiple of d_u, so

$$d_i bc d_j bc = 0$$

for $i \neq j$ and is nonzero for $i = j$ (apply trace map). Hence

$$[2^{n-u}1^{2u-1}] \subset M_T \iff t_u d_u bc \neq 0 \iff t_u d_u bc d_u bc \neq 0,$$

because $\mathbb{C}\mathbb{S}_{2n-1} d_u bc = \mathbb{C}\mathbb{S}_{2n-1} t_u d_u bc$, so

$$\mathbb{C}\mathbb{S}_{2n-1} t_u d_u bc d_u bc = \mathbb{C}\mathbb{S}_{2n-1} d_u bc d_u bc = \mathbb{C}\mathbb{S}_{2n-1} d_u bc.$$

The last condition is equivalent to

$$\phi(T) d_u bc \neq 0 \iff tbc d_u bc \neq 0,$$

where $\overline{t} = T \iff \psi(tbcd_u) \neq 0$ (since ψ is iso on the image of ϕ) $\iff T^{d_u} \neq 0$.
$\square$

Let (A, ω) be an n-algebra with ω being skew-symmetric. Now we are going to deal with n-algebras (see [**Ga**]) satisfying identity

$$[\omega_{a_1,\ldots,a_{n-k}}, \omega]^{RN} = 0. \tag{14}$$

Let us denote $T = \omega_{a_1,\ldots,a_{n-k}}[\omega]$ and $U = \omega[\omega_{a_1,\ldots,a_{n-k}}]$, $T, U \in P_n^2$.

Theorem 6.

1. *The $\mathbb{S}_{2n-1}$-module generated by T is equal to one generated by U. They are isomorphic to $\oplus_{i=k}^{n} [2^{n-i}1^{2i-1}]$.*
2. *The $\mathbb{S}_{2n-1}$-module generated by $[\omega_{a_1,\ldots,a_{n-k}}, \omega]$, i.e. by $(-1)^{(k-1)(n-1)}T - U$, is isomorphic to $\oplus_{i=k}^{n} [2^{n-i}1^{2i-1}]$ if k is even and to $\oplus_{i=k+1}^{n} [2^{n-i}1^{2i-1}]$ if k is odd.*

Proof.

1. Expanding (14) we have

$$(-1)^{(k-1)(n-1)}T - U =$$
$$= (-1)^{(k-1)(n-1)} \sum \operatorname{sgn}(x\ldots)\, \omega(\omega(x_1,\ldots,x_n), x_{n+1},\ldots,x_{n+k-1}, a_1, a_2, \ldots, a_{n-k})$$
$$- \sum \operatorname{sgn}(x\ldots)\, \omega(\omega(x_1,\ldots,x_k, a_1,\ldots,a_{n-k}), x_{k+1},\ldots,x_{n+k-1}),$$

where the above sums range over $\binom{n+k-1}{n}$ and $\binom{n+k-1}{k}$ permutations in x's. Consider the Young diagram for $\lambda_u = [2^{n-u}1^{2u-1}]$:

$$\begin{array}{|c|c|}
\hline
1 & n+u \\
\hline
2 & n+u+1 \\
\hline
\vdots & \vdots \\
\hline
n-u & 2n-1 \\
\hline
n-u+1 \\
\cline{1-1}
\vdots \\
\cline{1-1}
n+u-1 \\
\cline{1-1}
\end{array}$$

Let us think of x's as of the numbers $1, 2, \dots, n+u-1$ and a's as of the numbers $n+u, \dots, 2n-1$.

As we have already seen (see proof of Theorem 1 part 3.), $\psi(\mathbf{a}_{\lambda_u}\mathbf{b}_{\lambda_u}) \in P_n^2$ is non-zero, hence it must generate the module $[2^{n-u}1^{2u-1}]$. Moreover $\psi(\mathbf{b}_{\lambda_k})$ is T up to a nonzero scalar. By Theorem 4 we have:

$$\mathbb{C}\mathbb{S}_{2n-1}\mathbf{b}_{\lambda_k} = \bigoplus_{u=k}^{n} \mathbb{C}\mathbb{S}_{2n-1}\mathbf{a}_{\lambda_u}\mathbf{b}_{\lambda_u}.$$

By applying ψ we get the thesis for the identity (T).

The proof for U is the same. All we only change is considering x's as the numbers $1, 2, \dots, k, n+1, n+2, \dots$, i.e. the positions which x's occupy. Once again by considering the Young diagram with the numbers $1, 2, \dots, k, n+1, n+2, \dots$ in the first column we will achieve $\psi(\mathbf{b}_\lambda)$ is U up to a sign and the same decomposition to the simple modules is valid.

2. Consider the permutation changing a's with some of x's, moving the element $\omega(\omega(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+k-1}, a_1, a_2, \dots, a_{n-k})$ to the element $\omega(\omega(x_1, \dots, x_k, a_1, \dots, a_{n-k}), x_{n+1}, \dots, x_{n+k-1}, x_{k+1}, \dots, x_{n+k-1})$. The sign of this permutation in x's is $(-1)^{(n-k)(k-1)}$, which is equal to $(-1)^{(n-1)(k-1)}$ if and only if k is odd. One can check that $T^{d_k} = (-1)^{(n-k)(k-1)}U^{d_k} \neq 0$, so $((-1)^{(k-1)(n-1)}T - U)^{d_k} = 0 \iff k$ is odd.

For $u > k$, it is easy to see that T^{d_u} is not proportional to U^{d_u}, since after symmetrization by the element $\prod_{i=u+1}^{n}(1 + (i, n+i-1))$ one gets different number of a's in the interior bracket when applying to T and U. Hence $((-1)^{(k-1)(n-1)}T - U)^{d_u} \neq 0$. The thesis follows from the pervious lemma.

$\square$

We are going to deal with the generalization of n-algebras proposed in [**V**].

Theorem 7.

1. *The $\mathbb{S}_{2n-1}$-module generated in P_n^2 by the element*

$$\omega_{a_1,\dots,a_r}[\omega_{a_{r+1},\dots,a_k}] \tag{15}$$

($k \leq n-1$) is isomorphic to $\oplus_{i=n-k}^{n}[2^{n-i}1^{2i-1}]$, i.e. is isomorphic to one generated by $\omega_{a_1,\ldots,a_k}[\omega]$.

2. *The $\mathbb{S}_{2n-1}$-module generated in P_n^2 by the element*

$$[\omega_{a_1,\ldots,a_r}, \omega_{b_1,\ldots,b_s}]^{RN} \tag{16}$$

($r+s \leq n-1$) is isomorphic to one generated by

$$[\omega_{a_1,\ldots,a_r,b_1,\ldots,b_s}, \omega]^{RN}. \tag{17}$$

Corollary 1. *The classes of skew-symmetric n-algebras satisfying (16) and (17) are identical.*

Proof.

1. Expanding the expression $\omega_{b_1,\ldots,b_s}[\omega_{a_1,\ldots,a_r}]$ we get

$$Z = \sum \operatorname{sgn}(x\ldots)\omega(\omega(x,\ldots,a_1,\ldots,a_r),x,\ldots,b_1,\ldots,b_s).$$

Let us consider the partition $\lambda = [(s+2)^1 2^{r-1} 1^{2n-1-2r-s}]$ and the Young diagram

$$Y_\lambda = \begin{array}{|c|c|c|c|c|c|}
\hline
x_1 & a_1 & b_1 & b_2 & \ldots & b_s \\
\hline
x_2 & a_2 \\
\cline{1-2}
\vdots & \vdots \\
\cline{1-2}
\vdots & a_r \\
\cline{1-2}
\vdots \\
\cline{1-1}
x_{2n-1-r-s} \\
\cline{1-1}
\end{array}$$

and the elements $\mathbf{a}_\lambda, \mathbf{b}_\lambda \in \mathbb{C}\mathbb{S}_{2n-1}$. We will always write some symbols instead of the places these symbols occupy. For example, in the above diagram $x_1,\ldots,x_{n-r},x_{n-r+1},\ldots,x_n$ correspond to the numbers $1,\ldots,n-r,n+1,$ $n+2,\ldots$. It is easily seen that $\psi(\mathbf{b}_\lambda) = m \cdot Z$ for some $0 \neq m \in \mathbb{R}$, where ψ is the map defined in (10). By Theorem 4

$$\mathbb{C}\mathbb{S}_{2n-1}\mathbf{b}_\lambda \simeq \bigoplus_\tau K_{\tau',\lambda'}V_\tau \simeq \bigoplus_{i=n-r-s}^{n} m_i[2^{n-i}1^{2i-1}] \oplus \bigoplus_\mu V_\mu$$

where all m_i's are greater or equal 0 and none of V_μ is isomorphic to $[2^{n-i}1^{2i-1}]$ for some i. Therefore $\psi(\mathbb{C}\mathbb{S}_{2n-1}\mathbf{b}_\lambda) \subseteq \bigoplus_{i=n-r-s}^{n}[2^{n-i}1^{2i-1}] \subseteq P_2^n$. We want to establish the equality. Let us consider

$$\tilde{Z}_{i,j} = \sum \operatorname{sgn}(x\ldots)\operatorname{sgn}(a\ldots)\omega(\omega(x,\ldots,\underbrace{a,\ldots,a}_{i}),x,\ldots,\underbrace{a,\ldots,a}_{j}).$$

where the number of a's is i and j respectively, in the interior and exterior bracket. The summation is taken over all permutation of x's and a's. It is obvious that if $i \leq r$ and $j \leq s$ and an n-algebra satisfies $Z = 0$ then also $\tilde{Z}_{i,j} = 0$, so the inclusion $M_{\tilde{Z}_{i,j}} \subseteq M_Z$ holds between the modules generated by these identities. The module $M_{\tilde{Z}_{i,j}}$ corresponds to the Young diagram

$$Y_\mu = \begin{array}{|c|c|} \hline x_1 & a_1 \\ \hline x_2 & a_2 \\ \hline \vdots & \vdots \\ \hline \vdots & a_{i+j} \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} x_{2n-1-i-j} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

It is not difficult to see that $\psi(\mathbf{a}_\mu \mathbf{b}_\mu) \neq 0$ in P_2^n, so $\psi(\mathbf{a}_\mu \mathbf{b}_\mu)$ generates the module $[2^{n-u}1^{2u-1}]$, where $i+j = n-u$. Hence $[2^{n-u}1^{2u-1}] \subseteq M_{\tilde{Z}_{i,j}} \subseteq M_Z$ for $n-u = i+j \leq r+s$ and we are done.

2. Write

$$z = (-1)^{rs} \sum \operatorname{sgn}(x\ldots)\ \underbrace{x\ldots}_{n-r-s}\ a_1 \ldots a_r \underbrace{x\ldots}_{s}\ \underbrace{x\ldots}_{n-1-r-s}\ \underbrace{x\ldots}_{r} b_1 \ldots b_s\ ,$$

$$\tilde{z} = (-1)^{rs} \sum \operatorname{sgn}(x\ldots)\ \underbrace{x\ldots}_{n-r-s}\ \underbrace{x\ldots}_{r} b_1 \ldots b_s\ \underbrace{x\ldots}_{n-1-r-s}\ a_1 \ldots a_r \underbrace{x\ldots}_{s}\ ,$$

$z, \tilde{z} \in \mathbb{C}\mathbb{S}_{2n-1}$ and the summations are taken over all permutations in x's. We easily see that $\overline{z} = m\,Z$ and $m\,\tilde{Z} = \overline{\tilde{z}} = m\,\omega_{a_1,\ldots,a_r}[\omega_{b_1,\ldots,b_s}]$, $0 \neq m \in \mathbb{R}$. Note also that $\tilde{z} = \epsilon z \pi$, where $\pi = \prod_{i=n-r-s+1}^{n} (i, n+i-1)$ and $\epsilon = (-1)^{r(n-1-r-s)+s(n-1-s)}$.

We claim that $Z^{d_{n-r-s}} = \epsilon\, \tilde{Z}^{d_{n-r-s}}$.

Let $H = \{g \in \mathbb{S}_{2n-1} : g \text{ has } x\text{'s on positions } i \text{ and } n+i-1 \text{ for some } i \in \{n-r-s+1, \ldots, n\}\}$.

Let $w = zbc$, b, c as in (13), and v be the image of w under the canonical restriction linear map $\mathbb{C}\mathbb{S}_{2n-1} \to \mathbb{C}H$. We will show that $\psi(v d_{n-r-s}) = 0$. Let $\emptyset \neq I \subset \{n-r-s+1, \ldots, n\}$

$$\begin{aligned} H_I = \{g \in \mathbb{S}_{2n-1} : \{&i : n-r-s+1 \leq i \leq n,\ g \text{ has } x\text{'s on positions } i \\ &\text{and } n+i-1\} = I\}, \end{aligned}$$

and w_I be the image of w (or v) under the canonical linear map $\mathbb{C}\mathbb{S}_{2n-1} \to \mathbb{C}H_I$. Note that $w_I\,(i, n+i-1) = -w_I$ for any $i \in I$, since the transposition $(i, n+i-1)$ only changes the positions of x's and w_I is antisymmetric in x's. Moreover, for any $i \in \{n-r-s+1, \ldots, n\}$, $(i, n+i-1)\, d_{n-r-s} = d_{n-r-s}$. Hence $\psi(w_I d_{n-r-s}) = \psi(w_I\,(i, n+i-1)\, d_{n-r-s}) = -\psi(w_I d_{n-r-s})$, so $\psi(w_I d_{n-r-s}) = 0$ and $\psi(v d_{n-r-s}) = 0$ since v is a linear combination of w_I's, in fact $v = \sum_I w_I$. Hence $m\,Z^{d_n-r-s} = \psi((w-v) d_{n-r-s})$. The support of $w-v$ has to have x's on the positions $1, 2, \ldots, n-r-s, n+1, n+2, \ldots, 2n-1-r-s$ and the positions i and $i+n-1$ are occupied by exactly one of x's for any $i \in \{n-r-s+1, \ldots, n\}$.

The last sentence remains true if we repeat above reasoning with the element $\epsilon\,\tilde{z}$ instead of z.

Now, the element $\sum_{g\in\mathbb{S}\{1,\ldots,u,n+1,\ldots,n+u-1\}}\operatorname{sgn}(g)g$, $u=n-r-s$, changes only positions of x's. The element $e=\prod_{i=u+1}^{n}(1+(i,n+i-1))$ has the same effect on $w-v$ as on the analog element constructed starting from $\epsilon\,\tilde{z}$, because $\pi\,e=e$. Hence $\psi(zbcd_{n-r-s})=\psi(\epsilon\,\tilde{z}bcd_{n-r-s})$, so $Z^{d_{n-r-s}}=\epsilon\,\tilde{Z}^{d_{n-r-s}}$ as we have claimed.

Comparing ϵ with $(-1)^{(n-1-r)(n-1-s)}$ occurring in the bracket $[\omega_{a\ldots},\omega_{b\ldots}]^{RN}$ one find that they agree if and only if $n-r-s$ is odd. Hence $[2^{n-u}1^{2u-1}]\subset M_{[\omega_{a\ldots},\omega_{b\ldots}]^{RN}}\iff n-r-s$ is even, where $n-u=r+s$.

For $u<r+s$ one can verify that $\tilde{Z}^{d_{n-u}}$ and $Z^{d_{n-u}}$ have different support and thus are not proportional, so $[2^{n-u}1^{2u-1}]\subset M_{[\omega_{a\ldots},\omega_{b\ldots}]^{RN}}$. Thus, from part (1), $M_{[\omega_{a\ldots},\omega_{b\ldots}]^{RN}}=N\oplus\bigoplus_{i=n-r-s+1}^{n}[2^{n-i}1^{2i-1}]$ where $N=[2^{r+s}1^{2n-2r-2s+1}]$ if $n-r-s$ is even and zero otherwise. This coincides with the thesis of part 2. and Theorem6. $\square$

Expansion map

We are going to finish the paper with the proof of the conjecture 2 of [**B**].

Let

$$\varepsilon: P_n^2\to Ass(x_1,\ldots,x_{2n-1}),$$

$$\begin{aligned}\varepsilon([[x_{i_1},x_{i_2},\ldots,x_{i_n}],x_{j_1},\ldots,x_{j_{n-1}}]) &= \varepsilon([x_{j_0},x_{j_1},\ldots,x_{j_{n-1}}])=\\ &=\sum_{\sigma\in\mathbb{S}(\{0,\ldots,n-1\})}\operatorname{sgn}(\sigma)\;x_{j_{\sigma(0)}}x_{j_{\sigma(1)}}\cdots x_{j_{\sigma(n-1)}}\,,\end{aligned}$$

where

$$x_{j_0}=\sum_{\sigma\in\mathbb{S}(\{1,\ldots,n\})}\operatorname{sgn}(\sigma)\;x_{i_{\sigma(1)}}x_{i_{\sigma(2)}}\cdots x_{i_{\sigma(n)}}$$

and $Ass(x_1,\ldots,x_{2n-1})$ is the free associative algebra generated by $x_1,\ldots,x_{2n-1}$. The map ε is called *expansion map.*

Theorem 8. $\ker\varepsilon$ *is an* $\mathbb{S}_{2n-1}$*-submodule of* P_n^2 *and*

$$\ker\varepsilon=\begin{cases}0 & \text{if } n \text{ is odd}\\ [1^{2n-1}] & \text{if } n \text{ is even}\end{cases}$$

Proof. We will examine the coefficient at $x_{s_1}\cdots x_{s_i}x_{t_1}\cdots x_{t_{n-i}}x_{s_{i+1}}\cdots\cdots x_{s_{2i-1}}x_{t'_1}\cdots x_{t'_{n-i}}$ in the expression $\varepsilon(T)$, where T is T' in (9). We changed the set of indices from $\{1,\ldots,2n-1\}$ to $\{s_1,\ s_2,\ldots,s_{2i-1},\ t_1,t_2,\ldots,t_{n-i},\ t'_1,\ t'_2,\ldots,t'_{n-i}\}$.

The non-vanishing coefficient at $x_1\cdots x_{2n-1}$ appears only in the expressions of the form $\tilde{\varepsilon}([x_1,\ldots,x_n])x_{n+1}\cdots x_{2n-1}$, $x_1\tilde{\varepsilon}([x_2,\ldots,x_{n+1}])x_{n+2}\cdots x_{2n-1}$, $\ldots$, $\ldots,x_1\cdots x_{n-1}\,\tilde{\varepsilon}([x_n,\ldots,x_{2n-1}])$, where

$$\tilde{\varepsilon}([x_{i_1},x_{i_2},\ldots,x_{i_n}])=\sum_{\sigma\in\mathbb{S}(\{1,\ldots,n\})}\operatorname{sgn}(\sigma)\;x_{i_{\sigma(1)}}x_{i_{\sigma(2)}}\cdots x_{i_{\sigma(n)}}\,.$$

In (9) the interior bracket has always i variables x_s's. In the case $i < n$, only in one of the following sequences

$$x_1, \ldots, x_n;\ x_2, \ldots, x_{n+1};\ \ldots\ ; x_n, \ldots, x_{2n-1}$$

there are exactly i variables of x_s's, namely in the first of them. So we are only concerned at the items in (9) for which $[x_{\sigma(s_1)}, \ldots, x_{\sigma(s_i)}, y_{1,\ldots}, y_{n-i}] = [x_1, \ldots, x_n]$, so $\{x_{\sigma(s_1)}, \ldots, x_{\sigma(s_i)}\} = \{1, 2, \ldots, i\}$. It happens $i!(i-1)!$ times with the constant contribution $(= 1)$ to the coefficient at $x_1 \cdots x_{2n-1}$ when evaluated on ε. It implies that

$$\varepsilon([2^{n-i}1^{2i-1}]) \neq 0 \text{ for } i < n.$$

Let now $i = n$. Now the needed coefficient is a sum of n numbers a, where a is the contribution of

$$\sum \operatorname{sgn}(\sigma)[[x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma(n)}], x_{\sigma(n+1)}, \ldots, x_{\sigma(2n-1)}]$$

with the summation taken over all $\sigma \in \mathbb{S}_{2n-1}$ such that $\sigma(\{1, 2, \ldots, n\}) = \{k+1, k+2, \ldots, k+n\}$. Then the contribution is constant and equals to $a = (-1)^k n!(n-1)!$. So the total contribution is zero if and only if $2|n$. Hence $\varepsilon([1^{2n-1}]) \neq 0$ only for even n. □

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