Acta Math. Univ. Comenianae
Vol. LXXII, 2(2003), pp. 191–196

ON THE RANGE AND THE KERNEL OF THE ELEMENTARY OPERATORS $\sum_{i=1}^{n} A_i X B_i - X$

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Abstract. Let $B(H)$ denote the algebra of all bounded linear operators on a separable infinite dimensional complex Hilbert space H into itself. For $A = (A_1, A_2...A_n)$ and $B = (B_1, B_2...B_n)$ n-tuples in $B(H)$, we define the elementary operator $\Delta_{A,B}X : B(H) \mapsto B(H)$ by $\Delta_{A,B} = \sum A_i X B_i - X$. In this paper we show that if $\Delta_{A,B} = 0 = \Delta^*_{A,B}$, then

$$\left\|T + \Delta_{A,B}(X)\right\|_{\mathcal{I}} \geq \|T\|_{\mathcal{I}}$$

for all $X \in \mathcal{I}$ (proper bilateral ideal) and for all $T \in \ker(\Delta_{A,B} \mid \mathcal{I})$.

1. Introduction

Let H be a separable infinite dimensional complex Hilbert space, and let $B(H)$ denote the algebra of operators on H into itself. Given $A, B \in B(H)$, we define the generalized derivation $\delta_{A,B} : B(H) \mapsto B(H)$ by $\delta_{AB}(X) = AX - XB$, and the elementary operator $\Delta_{AB} : B(H) \mapsto B(H)$ by $\Delta_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X$, where $A = (A_1, A_2...A_n)$ and $B = (B_1, B_2...B_n)$ are n-tuples in $B(H)$. Note $\delta_{A,A} = \delta_A, \Delta_{A,A} = \Delta_A$. Let

$$B(H) \supset K(H) \supset C_p \supset F(H) (0 < p < \infty)$$

denote, respectively, the class of all bounded linear operators, the class of compact operators, the Schatten $p-$ class, and the class of finite rank operators on H. All operators herein are assumed to be linear and bounded. Let $\|.\|_p$, $\|.\|_\infty$ denote, respectively, the C_p-norm and the $K(H)$-norm. Let $\mathcal{I}$ be a proper bilateral ideal of $B(H)$. It is well known that if $\mathcal{I} \neq \{0\}$, then $K(H) \supset \mathcal{I} \supset F(H)$.

In [**1**, Theorem 1.7], J. Anderson shows that if A is normal and commutes with T, then for all $X \in B(H)$,

$$\|T + \delta_A(X)\| \geq \|T\| . \tag{1.1}$$

Received January 3, 2003.
2000 *Mathematics Subject Classification.* Primary 47B47, 47A30, 47B20; Secondary 47B10.
Key words and phrases. elementary operators, Schatten p-classes, unitairy invariant norm, orthogonality.
This Work was Supported by the Research Center Project No. Math/1422/10.

Over the years, Anderson's result has been generalized in various ways. Some results concern elementary operators on $B(H)$ such as $X \to AXB - X$ or $\delta_{A,B}(X) = AX - XB$; since these are not normal derivations, some extra condition is needed in each case to obtain the orthogonality result. In [**4**], P. B. Duggal established the orthogonality result for Δ_{AB} under the hypothesis that (A, B) satisfies a generalized Putnam-Fuglede property (which is one way to generalize normality).

Another way to generalize Anderson's result is to consider the restriction of an elementary operator (e.g., $X \to AXB - X$, $\delta_{A,B}(X) = AX - XB$ or Δ_{AB}) to a norm ideal $(\mathcal{I}, \|\|_{\mathcal{I}})$ of $B(H)$. Among the results in this direction, Duggal [**6**] has obtained the orthogonality result for $\Delta_{AB} \mid C_p$ (the restriction to the Schatten p-class C_p) under the Putnam-Fuglede hypothesis on (A, B), and F. Kittaneh [**8**], [**9**] proved the orthogonality result for restricted generalized derivations $\delta_{A,B} \mid \mathcal{I}$ (with the Putnam-Fuglede condition for (A, B)).

In [**16**], A. Turnsek initiated a different approach to generalize Anderson's theorem, one which does not rely on the normality via the Putnam-Fuglede condition.

Turnsek [**16**, Theorem 1.1] proved that if ϕ is a contractive map on a (fairly general normed algebra $\mathcal{A}$, then $\phi(s) = s$ implies $\|\phi(x) - x + s\| \geq \|s\|$ for every x in A. Let $\phi(X) = \sum_{i=1}^{n} A_i X B_i$; thus, if $\|\phi\| \leq 1$, then $\Delta_{AB}(S) = 0$ implies $\|\Delta_{AB}(X) - S\| \geq \|S\|$ for every operator X in $B(H)$, i.e., the range and the kernel of Δ_{AB} are orthogonal [**16**, Proposition 1.2]. Turnsek also obtained an analogue of the orthogonality result for $\Delta_{AB} \mid C_p$. Let $\Delta_{AB}^*(X) = \sum_{i=1}^{n} A_i^* X B_i^* - X$. Turnsek's result [16,Theorem 2.4] is that if $\sum_{i=1}^{n} A_i^* A_i \leq 1$, $\sum_{i=1}^{n} A_i A_i^* \leq 1$, $\sum_{i=1}^{n} B_i^* B_i \leq 1$, $\sum_{i=1}^{n} B_i B_i^* \leq 1$, then for $S \in C_p$, $\Delta_{AB}^*(S) = \Delta_{AB}(S) = 0$ implies that $\|\Delta_{AB}(X) - S\|_p \geq \|S\|_p$. The main result of this note is a direct extension of Turnesek's theorem from C_p to a general norm ideal $\mathcal{I}$. Other related results are also given.

2. Prelimenaries

Let $T \in B(H)$ be compact, and let $s_1(X) \geq s_2(X) \geq \ldots \geq 0$ denote the singular values of T, i.e., the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator T is said to be belong to the Schatten p-class C_p if

$$\|T\|_p = \left[\sum_{i=1}^{\infty} s_j(T)^p\right]^{\frac{1}{p}} = [\mathrm{tr}(T)^p]^{\frac{1}{p}}, \qquad 1 \leq p < \infty,$$

where tr denotes the trace functional. Hence C_1 is the trace class, C_2 is the Hilbert-Schmidt class, and C_∞ is the class of compact operators with

$$\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|$$

denoting the usual operator norm. For the general theory of the Schatten p-classes the reader is referred to [**7**], [**14**], [**15**].

Note that the Ky Fan norm $\|T\|_n$ is defined by

$$\|T\|_n = \sum_{j=1}^{n} s_j(T)$$

for $n \geq 1$.

Each unitarily invariant norm $\|.\|_{\mathcal{I}}$ satisfies

$$\|UA\|_{\mathcal{I}} = \|AV\|_{\mathcal{I}}$$

for all unitaries U and V (provided that $\|A\|_{\mathcal{I}} < \infty$), and is defined on a natural subclass $\mathcal{I}_{\|.\|_{\mathcal{I}}}$ of $B(H)$, called the norm ideal associated with $\|.\|_{\mathcal{I}}$. Whereas the (unitarily invariant) usual norm $\|.\|$ is defined on all of $B(H)$, other invariant norms are defined on norm ideals contained in the ideal $K(H)$ of compact operators in $B(H)$, see [**7**].

Definition 2.1. let C be complex numbers and let E be a normed linear space. Let $x, y \in E$, if $\|x - \lambda y\| \geq \|\lambda y\|$ for all $\lambda \in C$, then x is said to be orthogonal to y. Let F and G be two subspaces in E. If $\|x + y\| \geq \|y\|$, for all $x \in F$ and for all $y \in G$, then F is said to be orthogonal to G.

3. Main results

Our main results are the following

Theorem 3.1. *Let $\mathcal{I}$ be a bilateral ideal of $B(H)$ and $C = (C_1, C_2, \dots, C_n)$ n-tuple of operators in $B(H)$. If $\sum_{i=1}^{n} C_i C_i^* \leq 1$, $\sum_{i=1}^{n} C_i^* C_i \leq 1$ and $\Delta_c(T) = 0 = \Delta_c^*(T)$, then*

$$\|T + \Delta_c(X)\|_{\mathcal{I}} \geq \|T\|_{\mathcal{I}} \tag{3.1}$$

for all $X \in \mathcal{I}$ and for all $T \in \ker \Delta_c \cap \mathcal{I}$.

Proof. By virtue of [**7**, p. 82], it suffices to show that for all $n \geq 1$,

$$\|(\Delta_c(X)) + T\|_n = \sum_{j=1}^{n} s_j(\Delta_c(X)) + T) \geq \sum_{j=1}^{n} s_j(T) = \|T\|_n \,.$$

Let $T = U\,|T|$ be the polar decomposition of T where U is a partial isometry and $\ker U = \ker |T|$. Then for all $j \geq 1$ the result of Gohberg and Krein [**7**, p. 27] guarentees that

$$s_j(\Delta_c(X) + T) \geq s_j(U^*[\Delta_c(X) + T]) = s_j(U^*(\Delta_c(X)) + |T|).$$

Recall that if $\{g_n\}_{n\geq 1}$is an orthonormal basis of H, then it results from [**7**, p. 47] that for all $n \geq 1$,

$$\sum_{j=1}^{n} s_j(U^*(\Delta_c(X)) + |T|) \geq \sum_{j=1}^{n} |\langle [U^*(\Delta_c(X)) + |T|] g_j, g_j\rangle| \,.$$

Consequently we get,

$$\sum_{j=1}^{n} s_j(\Delta_c(X)+T) \geq \sum_{j=1}^{n} |\langle [U^*(\Delta_c(X))+|T|]g_j, g_j\rangle| = \sum . \tag{3.2}$$

It is known that if $\sum_{i=1}^{n} C_i C_i^* \leq 1$, $\sum_{i=1}^{n} C_i^* C_i \leq 1$ and $\Delta_c(T) = 0 = \Delta_c^*(T)$ then the eigenspaces corresponding to distinct non-zero eigenvalues of the compact positive operator $|T|^2$ reduce each C_i, see ([**3**, Theorem 8], [**16**, Lemma 2.3]). In particular $|T|$ commutes with C_i for all $1 \leq i \leq n$. Hence

$$C_i\, |T| = |T|\, C_i.$$

This shows the existence of an orthonormal basis $\{e_{k_i}\} \cup \{f_m\}$ of H such that $\{f_m\}$ is an orthonormal basis of $\ker |T|$ and $\{e_{k_i}\}$ consists of common eigenvectors of C_i and $|T|$. If

$$\{g_n\} = \{e_{k_i}\} \cup \{f_m\},$$

since for all $m \geq 1$,

$$U f_m = |T|\, f_m = 0,$$

(3.2) becomes

$$\sum = \sum_{j=1}^{n} \left\langle |T|\, e_{k_j}, e_{k_j} \right\rangle + \sum_{j=1}^{n} \left| \left\langle [U^*(\Delta_C(X))] e_{k_j}, e_{k_j} \right\rangle \right| .$$

Therefore for all $n \geq 1$,

$$\begin{aligned} \sum_{j=1}^{n} s_j((\Delta_C(X)+T)) &\geq \sum_{j=1}^{\inf(n,\ \mathrm{card}(e_{k_i}))} \langle |T|\, e_j, e_j \rangle \\ &= \sum_{j=1}^{\inf(n,\ \mathrm{card}(e_{k_i}))} s_j(T) \geq \sum_{j=1}^{n} s_j(T) = \|T\|_n . \end{aligned}$$

□

Theorem 3.2. *Let $\mathcal{I}$ be a bilateral ideal of $B(H)$ and $A = (A_1, A_2, \ldots, A_n)$, $B = (A_1, A_2, \ldots, A_n)$ n-tuples of operators in $B(H)$. If $\sum_{i=1}^{n} A_i A_i^* \leq 1$, $\sum_{i=1}^{n} A_i^* A_i \leq 1$, $\sum_{i=1}^{n} B_i B_i^* \leq 1$, $\sum_{i=1}^{n} B_i^* B_i \leq 1$ and $\ker \Delta_{A,B} \subseteq \ker \Delta_{A,B}^*$, then*

$$\|S + \Delta_{A,B}(X)\|_{\mathcal{I}} \geq \|S\|_{\mathcal{I}}, \tag{3.3}$$

for all $X \in \mathcal{I}$ and for all $S \in \ker(\Delta_c \mid \mathcal{I})$.

Proof. It suffices to take the Hilbert space $H \oplus H$, and operators

$$C_i = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$$

and apply Theorem 3.1 and use the fact that the norm of a matrix is greater than or equal to the norm of an entry along the main diagonal of the matrix [**7**]. □

Corollary 3.1. *Let $\mathcal{I}$ be a bilateral ideal of $B(H)$ and $A = (A_1, A_2, \dots, A_n)$, $B = (A_1, A_2, \dots, A_n)$ n-tuples of operators in $B(H)$. If $\sum_{i=1}^{n} A_i A_i^* \leq 1$, $\sum_{i=1}^{n} A_i^* A_i \leq 1$, $\sum_{i=1}^{n} B_i B_i^* \leq 1$, $\sum_{i=1}^{n} B_i^* B_i \leq 1$ and $\ker \Delta_{A,B} \subseteq \ker \Delta_{A,B}^*(S)$, then $\ker(\Delta_{A,B}^n \mid \mathcal{I}) = \ker(\Delta_{A,B} \mid \mathcal{I})$.*

Proof. The result of S. Bouali and S. Cherki [**2**] guarentees that

$$R(A^n) = \ker(A)$$

if, and only if,

$$R(A) \cap \ker A = \{0\},$$

where $A \in B(E)$ and E is a complex vector space. In particular

$$R(\Delta_{A,B}^n \mid \mathcal{I}) = \ker(\Delta_{A,B} \mid \mathcal{I})$$

if and only if,

$$R(\Delta_{A,B} \mid \mathcal{I}) \cap \ker(\Delta_{A,B} \mid \mathcal{I}) = \{0\}$$

which holds from the above theorem. □

4. A Comment and some open questions

(1) It is well known that the Hilbert-Schmidt class C_2 is a Hilbert space under the inner product

$$\langle Y, Z \rangle = \mathrm{tr} Z^* Y.$$

We remark here that for the Hilbert Schmidt norm $\|.\|_2$, the orthogonality results in Theorem 3.2 is to be understood in the usual Hilbert space sense. Note in the case where $\mathcal{I} = C_2$, then

$$\|T + \Delta_{A,B}(X)\|_2^2 = \|\Delta_{A,B}(X)\|_2^2 + \|T\|_2^2,$$

if and only if $\ker \Delta_{A,B} \subseteq \ker \Delta_{A,B}^*$, for all $X \in C_2$ and for all $T \in \ker \Delta_{A,B} \cap \mathcal{I}$. This can be seen as an immediate consequence of the fact that

$$R(\Delta_{A,B} \mid C_2)^{\perp} = \ker(\Delta_{A,B} \mid C_2)^* = \ker(\Delta_{A,B}^* \mid C_2),$$

(2) If the assumptions of Theorem 3.2 holds, then $\overline{\mathrm{ran}} \Delta_{A,B} \cap \ker \Delta_{AB} = \{0\}$, where the closure can be taken in the most weak (uniform) norm. Hence $\Delta_{A,B}(\Delta_{A,B}(X)) = 0$ implies $\Delta_{A,B}(X) = 0$.
Indeed if $Z \in \overline{\mathrm{ran}} \Delta_{A,B} \cap \ker \Delta_{A,B}$, then $Z = \lim_{n \longrightarrow \infty} \Delta_{A,B}(X_n)$ and $\Delta_{A,B}(Z) = 0$. By applying Theorem 3.2 we get

$$\|\Delta_{A,B}(X_n) - Z\|_{\mathcal{I}} \geq \|Z\|_{\mathcal{I}}$$

so,

$$\|Z - Z\|_{\mathcal{I}} \geq \|Z\|_{\mathcal{I}}.$$

Then $Z = 0$. We deduce that $\overline{R(\Delta_{A,B})} \oplus \ker \Delta_{AB} = \mathcal{I}$.
(3) I s the sufficient condition in Thorem 3.2 necessary?

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