# COEFFICIENT ESTIMATES IN SUBCLASSES OF THE CARATHÉODORY CLASS RELATED TO CONICAL DOMAINS

S. KANAS

ABSTRACT. We study some properties of subclasses of of the Carathéodory class of functions, related to conic sections, and denoted by $\mathcal{P}(p_k)$. Coefficients bounds, estimates of some functionals are given.

## 1. INTRODUCTION

We denote by $\mathcal{P}$ the class of Carathéodory functions analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$, e.g.

$$\mathcal{P} = \{p : p \text{ analytic in } \mathcal{U},\ p(0) = 1,\ \operatorname{Re} p(z) > 0\}. \tag{1.1}$$

Some special subclasses of $\mathcal{P}$ play an important role in geometric function theory because of their relations with subclasses of univalent functions. Many such classes have been introduced and studied; some became the well-known, for instance, the class of analytic functions $p$ in the unit disk $\mathcal{U}$ such that $p(0) = 1$ and $p \prec (1+Az)/(1+Bz)$, that is the class of functions for which $p(\mathcal{U})$ is a subset of a disk, or a half-plane. The other choice is the class of all $p$ such that $p \prec [(1 + z)/(1 - z)]^\gamma$. In this case $p(\mathcal{U})$ is a subset of a sector, contained in a right half-plane with a vertex at the origin and symmetric about the real axis. Here the symbol "$\prec$" denotes the subordinations (cf. e.g. [7]).

Received March 3, 2003.
2000 *Mathematics Subject Classification.* Primary 34A25, 30C35,33E05, 30C45.
*Key words and phrases.* The Carathéodory class, extremal problems, Fekete-Szegö-Goluzin functional.

Let $k \in [0, \infty)$. For arbitrarily chosen $k$ let $\Omega_k$ denote the following domain

$$\Omega_k = \left\{ u + iv : u^2 > k^2(u-1)^2 + k^2v^2 \right\}. \tag{1.2}$$

Note that $\Omega_k$ is convex and symmetric in the real axis and $1 \in \Omega_k$ for all $k$. $\Omega_0$ is nothing but the right half-plane and when $0 < k < 1$, $\Omega_k$ is an unbounded domain enclosed by the right branch of the hyperbola

$$\left( \frac{(1-k^2)u + k^2}{k} \right)^2 - \left( \frac{(1-k^2)v}{\sqrt{1-k^2}} \right)^2 = 1 \tag{1.3}$$

with foci at 1 and $-(1+k^2)/(1-k^2)$. When $k = 1$, the domain $\Omega_1$ is still unbounded domain enclosed by the parabola

$$2u = v^2 + 1$$

with the focus at 1. When $k > 1$, the domain $\Omega_k$ becomes bounded domain being the interior of the ellipse

$$\left( \frac{(k^2-1)u - k^2}{k} \right)^2 - \left( \frac{(k^2-1)v}{\sqrt{k^2-1}} \right)^2 = 1$$

with foci at 1 and $(k^2+1)/(k^2-1)$. It should be noted that, for no choice of parameter $k$, $\Omega_k$ reduce to a disk. $\{\Omega_k, k \in [0, \infty)\}$ forms the family of domains bounded by conic sections, convergent in the sense of the kernel convergence.

Let $p_k$ denote the conformal mapping of $\mathcal{U}$ onto $\Omega_k$ determined by conditions $p_k(0) = 1$, $p_k'(0) > 0$. The concrete form of $p_k$ was given in [7], [8], [11] and in [5].

**Theorem 1.1.** *Let $k \in [0,\infty)$. The conformal mapping of $\mathcal{U}$ onto $\Omega_k$ is of the form*

$$p_k(z) = \begin{cases} \frac{1+z}{1-z} & \text{for} \quad k = 0, \\ 1 + \frac{2}{1-k^2} \ \sinh^2\left(A(k)\operatorname{arctanh}\sqrt{z}\right) & \text{for} \quad k \in (0,1), \\ 1 + \frac{8}{\pi^2}\left(\operatorname{arctanh}\sqrt{z}\right)^2 & \text{for} \quad k = 1, \\ 1 + \frac{2}{k^2-1}\sin^2\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t}, t)\right) & \text{for} \quad k > 1, \end{cases} \tag{1.4}$$

*where $A(k) = (2/\pi)\arccos k$, $\mathcal{F}(w,t)$ is the Jacobi elliptic integral of the first kind*

$$\mathcal{F}(w,t) = \int_0^w \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}, \tag{1.5}$$

*and $k = \cosh\mu(t) = \cosh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)$, $t \in (0,1)$.*

By $\mathcal{P}(p_k)$ we denote the subclass of the Carathéodory class $\mathcal{P}$, consisting of functions $p$, analytic in $\mathcal{U}$, $p(0) = 1$, $\operatorname{Re} p(z) > 0$ in $\mathcal{U}$, and such that $p \prec p_k$ in $\mathcal{U}$. Observe that when $k$ varies, $\mathcal{P}(p_k)$ generate a number of subclasses of the class $\mathcal{P}$.

The aim of this paper is to present some properties of the class $\mathcal{P}(p_k)$. In Section 2 we prove the continuity of functions "extremal" in $\mathcal{P}(p_k)$ as regards the parameter $k$. Some coefficients problems are treated in Section 3, in particular we obtain the sharp bound on the coefficient functional $|b_2 - \mu b_1^2|$ $(-\infty < \mu < \infty)$.

## 2. General properties of the family $\mathcal{P}(p_k)$

We recall some notation and properties of Jacobi elliptic functions which will be used in next theorems (cf. e.g. [1], [4]).

The *elliptic integral* (or *normal elliptic integral*) *of the first kind* has been defined at (1.5). By $\mathcal{K}(t)$ we denote the *complete elliptic integral of the first kind*

$$\mathcal{K}(t)=\mathcal{F}(1,t),\quad \text{and let}\quad \mathcal{K}'(t)=\mathcal{K}(t'),\quad t'=\sqrt{1-t^2},\quad t\in(0,1).$$

Let $\mathcal{E}(w,t)$ denote the *elliptic integral of the second kind*, e.g.

$$\mathcal{E}(w,t)=\int_0^w\sqrt{\frac{1-t^2x^2}{1-t^2}}dx, \tag{2.1}$$

and let $\mathcal{E}(t)=\mathcal{E}(1,t)$ be the *complete elliptic integral of the second kind*, $t\in(0,1)$. Also, set $\mathcal{E}'(t)=\mathcal{E}(t')$. Changing the variable by $x=\sin\theta$ integrals (1.5) and (2.1) reduce to the Legendre form

$$\mathcal{F}(\varphi,t)=\int_0^\varphi(1-t^2\sin^2\theta)^{-1/2}d\theta,$$

$$\mathcal{E}(\varphi,t)=\int_0^\varphi\sqrt{1-t^2\sin^2\theta}d\theta.$$

The equation $z=\mathcal{F}(\varphi,t)$, where $z$ is assumed to be real, defines $\varphi$ as a function of $z$ which has been called by Jacobi the *amplitude* of $z$ and denoted $\varphi=\mathrm{am}(z,t)$. Further Jacobi introduced sin(amz), and cos(amz) (*sinus* and *cosinus amplitudinus*) that have several applications in geometry and mechanics. Among numerous interesting properties of elliptic functions, the following will be used in the proof:

$$\lim_{t\to0^+}\mathcal{K}(t)=\lim_{t\to0^+}\mathcal{E}(t)=\lim_{t\to1^-}\mathcal{K}'(t)=\frac{\pi}{2}, \tag{2.2}$$

$$\lim_{t\to0^+}\mathcal{K}'(t)=\infty,\quad \lim_{t\to0^+}\mathcal{E}'(t)=1, \tag{2.3}$$

$$\lim_{t\to 1^-} \mathcal{K}(t) = \infty, \qquad \lim_{t\to 1^-} \mathcal{E}(t) = 1,$$
$$\lim_{t\to 1^-} (1-t^2)\mathcal{K}(t) = 0, \quad \lim_{t\to 1^-} \frac{\mathcal{K}'(t)}{\mathcal{K}(t)} = 0, \tag{2.4}$$

$$\lim_{t\to 0^+} \frac{\mathcal{K}'(t)}{\mathcal{K}(t)} = \infty, \quad \text{so that} \quad \lim_{t\to 0^+} \sinh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) = \infty. \tag{2.5}$$

Functions $\mathcal{K}, \mathcal{K}', \mathcal{E}, \mathcal{E}'$ are continuous and differentiable on $(0,1)$, and

$$\frac{d\mathcal{K}(t)}{dt} = \frac{\mathcal{E}(t) - (1-t^2)\mathcal{K}(t)}{t(1-t^2)}, \qquad \frac{d\mathcal{E}(t)}{dt} = \frac{\mathcal{E}(t) - \mathcal{K}(t)}{t}, \qquad \frac{d\mathcal{K}'(t)}{dt} = \frac{t^2\mathcal{K}'(t) - \mathcal{E}(t)}{t(1-t^2)}, \tag{2.6}$$

from which the Legendre identity can be derived (cf. [1, p. 112])

$$\mathcal{E}(t)\mathcal{K}'(t) + \mathcal{E}'(t)\mathcal{K}(t) - \mathcal{K}(t)\mathcal{K}'(t) = \frac{\pi}{2}.$$

Further, (2.6) and the above identity yields the result

$$\begin{aligned} \frac{d[\mathcal{K}'(t)/\mathcal{K}(t)]}{dt} &= \frac{\mathcal{K}(t)\mathcal{K}'(t) - \mathcal{E}(t)\mathcal{K}'(t) - \mathcal{E}'(t)\mathcal{K}(t)}{t(1-t^2)\mathcal{K}^2(t)} \\ &= -\frac{\pi}{2t(1-t^2)\mathcal{K}^2(t)}. \end{aligned} \tag{2.7}$$

Moreover

$$\frac{\pi}{1+\sqrt{1-t^2}} \le \mathcal{K}(t) \le \frac{\pi}{2\sqrt{1-t^2}}, \tag{2.8}$$

(c.f. [2], see also [3]).

Finally, by a simple computation we arrive at

$$\lim_{t\to 1^-} \mathcal{F}(\sqrt{z/t}, t) = \operatorname{arctanh}\sqrt{z}. \tag{2.9}$$

We now return to functions "extremal" in the class $\mathcal{P}(p_k)$.

**Theorem 2.1.** *Functions $p_k$ is continuous as regards the parameter $k \in [0, \infty)$.*

*Proof.* First we observe that

$$\begin{aligned}\lim_{k\to 0^+} p_k(z) &= \lim_{k\to 0^+} \frac{2}{1-k^2}\sinh^2\left(A(k)\mathrm{arctanh}\sqrt{z}\right)+1\\ &= 2\sinh^2\left(\mathrm{arcsinh}\frac{\sqrt{z}}{\sqrt{1-z}}\right)+1 = p_0(z),\end{aligned}$$

since $\lim_{k\to 0^+} A(k) = \lim_{k\to 0^+}(2/\pi)\arccos k = 1$.

Simultaneously, by (1.4) and setting $t = 2\mathrm{arctanh}\sqrt{z}$ one gets

$$\begin{aligned}\lim_{k\to 1^-} p_k(z) &= \lim_{k\to 1^-}\left(1+\frac{2}{1-k^2}\sinh^2\left(A(k)\mathrm{arctanh}\sqrt{z}\right)\right)\\ &= 1+2\lim_{k\to 1^-}\frac{\sinh^2\frac{A(k)t}{2}}{1-k^2}\\ &= 1+\frac{t^2}{2}\lim_{k\to 1^-}\left(\frac{A(k)}{\sqrt{1-k^2}}\right)^2\left(\frac{\sinh\frac{A(k)t}{2}}{\frac{A(k)t}{2}}\right)^2\\ &= 1+\frac{t^2}{2}\left(\lim_{k\to 1^-}\frac{\frac{-2}{\pi\sqrt{1-k^2}}}{\frac{-k}{\sqrt{1-k^2}}}\right)^2\left(\lim_{k\to 1^-}\frac{\sinh\frac{A(k)t}{2}}{\frac{A(k)t}{2}}\right)^2\\ &= 1+\frac{2}{\pi^2}t^2 = p_1(z).\end{aligned}$$

Here, $A(k) = (2/\pi)\arccos k \to 0^+$ as $k \to 1^-$.

Finally, we will prove the right-hand continuity of $p_k$ at $k = 1$ if we show that

$$\lim_{k\to 1^+} \frac{\sin\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t},t)\right)}{\sqrt{k^2-1}} = \frac{2}{\pi}\operatorname{arctanh}\sqrt{z}. \tag{2.10}$$

Note that $k = \cosh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)$, so that $\sqrt{k^2-1} = \sinh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)$ and if $k \to 1^+$ then $t \to 1^-$, thus (2.10) is equivalent to

$$\lim_{t\to 1^-} \frac{\sin\left(\frac{\pi}{4\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t},t)\right)}{\sinh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)} = \frac{2}{\pi}\operatorname{arctanh}\sqrt{z}. \tag{2.11}$$

Since by (2.4) and (2.9) both, the numerator and the denominator tend to 0 we need to prove, by the l'Hospital rule, that there exists the limit of the quotient of derivatives of (2.11), or equivalently

$$\lim_{t\to 1^-} \left( \cos\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t},t)\right) \cdot \frac{\frac{\pi}{2}\left[\frac{\mathcal{E}(t)-(1-t^2)\mathcal{K}(t)}{t(1-t^2)\mathcal{K}^2(t)}\mathcal{F}(\sqrt{z/t},t) - \frac{1}{\mathcal{K}(t)}\frac{d[\mathcal{F}(\sqrt{z/t},t)]}{dt}\right]}{\cosh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)\frac{\pi^2}{4}\frac{1}{t(1-t^2)\mathcal{K}^2(t)}} \right). \tag{2.12}$$

Set

$$D(z,t) = \left(\mathcal{E}(t) - (1-t^2)\mathcal{K}(t)\right)\mathcal{F}(\sqrt{z/t},t) - t(1-t^2)\mathcal{K}(t)\frac{d[\mathcal{F}(\sqrt{z/t},t)]}{dt}.$$

Then (2.12) reduces to

$$\frac{2}{\pi}\lim_{t\to 1^-} \frac{\cos\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t},t)\right)}{\cosh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)} D(z,t).$$

Differentiating with respect to $t$ we obtain from (1.5)

$$\frac{d[\mathcal{F}(\sqrt{z/t},t)]}{dt} = \int_0^{\sqrt{z/t}} \frac{tx^2}{\sqrt{(1-x^2)(1-t^2x^2)^3}}dx - \frac{\sqrt{z}}{2t\sqrt{t-z}\sqrt{1-tz}}$$

so that

$$\lim_{t\to 1^-} \frac{d[\mathcal{F}(\sqrt{z/t},t)]}{dt} = \frac{1}{4}\log\frac{1-\sqrt{z}}{1+\sqrt{z}} - \frac{1}{2(1-\sqrt{z})}.$$

Since, by (2.4)

$$\lim_{t\to 1^-} t(1-t^2)\mathcal{K}(t) = 0, \quad \text{and} \quad \lim_{t\to 1^-} \mathcal{E}(t) = 1$$

then using (2.9) we obtain

$$\lim_{t\to 1^-} D(z,t) = \operatorname{arctanh}\sqrt{z}.$$

Therefore, the above and first and fourth relation from (2.4) finally yield

$$\frac{2}{\pi}\lim_{t\to 1^-} \frac{\cos\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t},t)\right)}{\cosh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)} D(z,t) = \frac{2}{\pi}\operatorname{arctanh}\sqrt{z},$$

that is equivalent to (2.11). It completes the proof. □

**Theorem 2.2.** *Let $k \in [0,\infty)$ be fixed. The function $p_k(z)$ has the positive Taylor coefficients around the origin.*

The proof of Theorem 2.2 appeared complicated for the case $k \in (1,\infty)$ and has been proved in [11] using the theory of continued fractions. We quote it here for the sake of completeness. Applying Theorem 2.2 estimates of the modulus and the real part of $p \in \mathcal{P}(p_k)$ were derived [11].

## 3. Coefficient bounds

Now, we find some bounds in the family $\mathcal{P}(p_k)$. The first problem, we discus, is the Fekete-Szegö-Goluzin's problem in the class $\mathcal{P}(p_k)$. We begin by proving the theorem that is itself interesting, since it improves the Livingston result in $\mathcal{P}(p_k)$ [12]. For fixed $k$, set

$$p_k(z) = 1 + P_1(k)z + P_2(k)z^2 + \cdots, \quad z \in \mathcal{U}.$$

**Theorem 3.1.** *Let $0 \le k < \infty$ be fixed. Then*

$$|P_1^2(k) - P_2(k)| \le P_1(k). \tag{3.1}$$

*Proof.* The inequality (3.1) is obvious for $k = 0$, by Livingston result [12], therefore we assume $k > 0$. We consider separately cases:

$k \in (0,1)$, $\qquad k = 1$, $\qquad k > 1$.

*Case 1.* By virtue of (1.4) a precise form of coefficients of $p_k$ were derived (cf. [5], [9])

$$P_1(k) = \frac{2A^2(k)}{1-k^2}, \qquad P_2(k) = \frac{2A^2(k)(A^2(k)+2)}{3(1-k^2)} = P_1(k)\frac{A^2(k)+2}{3},$$

$$A(k) = \frac{2}{\pi}\arccos k.$$

Since $P_1(k)$ is positive for $k \in (0,1)$, the inequality (3.1) reduces to proving $|P_1(k) - (A^2(k)+2)/3| \le 1$. Note that

$$P_1(k) - \frac{A^2(k)+2}{3} = \frac{A^2(k)(5+k^2)+1-k^2}{3(1-k^2)} > 0$$

for $k \in (0,1)$, then it suffices to show the inequality $P_1(k) - (A^2(k)+2)/3 \le 1$, or equivalently

$$\frac{5(1-k^2)}{k^2+5} - \frac{4}{\pi^2}\arccos^2 k \ge 0.$$

Set

$$h(k) := \sqrt{\frac{5(1-k^2)}{k^2+5}} - \frac{2}{\pi}\arccos k.$$

Then functions $h$ is well defined on a closed interval $[0,1]$ and

$$h'(k) = \frac{2}{\pi\sqrt{1-k^2}}\left[1 - \frac{3\sqrt{5}\pi k}{(k^2+5)^{3/2}}\right].$$

Note that $h'(0) > 0, h'(1^-) < 0$. Since $g(k) = \frac{3\sqrt{5}\pi k}{(k^2+5)^{3/2}}$ is monotone then there exists the only point $k_0 \in (0,1)$ such that $h'(k_0) = 0$. Then $h'(k) > 0$ in $0 < k < k_0$ and $h'(k) < 0$ in $0 < k_0 < k < 1$. Therefore $h(k) \ge \min\{h(0), h(1)\} = 0$ in $0 \le k < 1$, so that the proof of the case 1. is complete.

*Case 2.* In this instance $P_1(k) = 8/\pi^2$ and $P_2(k) = 16/(3\pi^2) = 2P_1(k)/3$ (cf. [14], [15]). The inequality (3.1) now follows immediately by means of the relation $-1/3 < P_1(k) < 5/3$.

*Case 3.* To this purpose we use the following form of $p_k$ for $k > 1$

$$p_k(z) = \frac{1}{k^2-1}\sin\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(u(z)/\sqrt{t}, t)\right) + \frac{k^2}{k^2-1}, \tag{3.2}$$

(cf. [5, p. 20]), where $u(z) = (z-\sqrt{t})/(1-\sqrt{t}z)$ and $k = \cosh(\mu(t)/2)$ for $t \in (0,1)$. In view of [5]

$$P_1(k) = \frac{\pi^2}{4(k^2-1)\mathcal{K}^2(t)\sqrt{t}(1+t)},$$

$$P_2(k) = P_1(k)\frac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathcal{K}^2(t)\sqrt{t}(1+t)} =: P_1(k)D(k),$$

see also [9]. Since $P_1(k)$ is positive for all $t \in (0,1)$, the inequality (3.1) will hold if $|P_1(k)-D(k)| \le 1$, equivalently $P_1(k) \le D(k)+1$ and $D(k) \le P_1(k)+1$. Now, we will show that the inequality $D(k) \le P_1(k)+1$ holds. We rewrite the inequality $D(k) \le P_1(k)+1$ into the form

$$\frac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{4\mathcal{K}^2(t)} \le \frac{3\pi^2}{2(k^2-1)\mathcal{K}^2(t)} + 6\sqrt{t}(1+t). \tag{3.3}$$

Observing that $k^2-1 = \sinh^2(\pi\mathcal{K}'(t)/(4\mathcal{K}(t)))$, the relation (3.3) becomes

$$t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - \frac{\pi^2}{4\mathcal{K}^2(t)} \le \frac{3\pi^2}{2\mathcal{K}^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)}. \tag{3.4}$$

Observe next, that if $k \to 1^+$ then $t \to 1^-$ and the case $k \to \infty$ corresponds to the case $t \to 0^+$. Thus, we may study the inequality (3.4) as regards $t \in (0,1)$. The left-hand side function satisfies

$$t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - \frac{\pi^2}{4\mathcal{K}^2(t)} \le t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - (1-t^2). \tag{3.5}$$

by means of the right-hand estimation in (2.8). Set

$$w(t) := t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - (1-t^2) = 2t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t}.$$

The function $w(t)$ is defined on the closed interval $[0,1]$ and it decreases continuously from $w(0)=0$ to $w(1)=-4$. Indeed $w'(t) = 4t - 9\sqrt{t} + 6 - 3/\sqrt{t} = (t-3) + 3(\sqrt{t}-1)^3/\sqrt{t} < 0$ on $(0,1)$. Therefore $w(t) < 0$ on $(0,1)$.

Now we show that the right-hand function of (3.4) is positive in $(0,1)$ and increases from 0 to $96/\pi^2$. Let

$$W(t) := \frac{3\pi^2}{2\mathcal{K}^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)}.$$

Making use the first relation in (2.2) and the last relation in (2.5) we find that

$$\lim_{t\to 0^+} W(t) = 0.$$

Also, after necessary transformations, we obtain

$$\lim_{t\to 1^-} W(t) = \lim_{t\to 1^-} \frac{24\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)^2}{(\mathcal{K}')^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)} = \frac{96}{\pi^2},$$

because of the last formula in (2.4) and the fact that $\lim\limits_{x\to 0} \sinh x/x = 1$. Now, we will show that $W(t)$ is increasing. Differentiating $W(t)$ and using (2.6) and (2.7) one gets

$$W'(t) = -\frac{3\pi^2\left[(\mathcal{E}(t)-(1-t^2)\mathcal{K}(t))\sinh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - \frac{\pi^2}{8\mathcal{K}(t)}\cosh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)\right]}{t(1-t^2)\mathcal{K}^3(t)\sinh^3\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)}.$$

In order to show that $W'(t) > 0$ it suffices to prove that the expression in the square brackets of $W'(t)$ is negative for $t \in (0,1)$. Such relation may be rewritten in the form

$$(\mathcal{E}(t)-(1-t^2)\mathcal{K}(t))\sinh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) < \frac{\pi^2}{8\mathcal{K}(t)}\cosh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)$$

or

$$\frac{8}{\pi^2}\left[\frac{\mathcal{E}(t)-(1-t^2)\mathcal{K}(t)}{\mathcal{K}(t)}\right] < \coth\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right). \tag{3.6}$$

Set

$$\phi(t) = \mathcal{E}(t) - (1-t^2)\mathcal{K}(t).$$

Then, in view of (2.6) we have

$$\frac{d\phi(t)}{dt} = \frac{\mathcal{E}(t) - \mathcal{K}(t)}{t} - \frac{-2t^2\mathcal{K}(t) + \mathcal{E}(t) - \mathcal{K}(t) + t^2\mathcal{K}(t)}{t} = t\mathcal{K}(t) > 0$$

in $(0, 1)$. Moreover $\phi(0^+) = 0$ and $\phi(1^-) = 1$ by the second and third relation in (2.4). Thus $0 < \phi(t) < 1$ in $(0, 1)$. Note also that $\mathcal{K}(t)$ is increasing from $\pi/2$ to $\infty$, when $t \in (0, 1)$. Therefore

$$\frac{8}{\pi^2}\left[\frac{\mathcal{E}(t) - (1 - t^2)\mathcal{K}(t)}{\mathcal{K}(t)}\right] < \frac{8}{\pi^2}\frac{1}{\pi/2} = \frac{16}{\pi^3} < 1,$$

whereas the right-hand side of (3.6) is greater than 1 since $\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) > 0$. Then the inequality (3.6) holds, equivalently $W'(t) > 0$ on $(0, 1)$ so that $W$ increases on $(0, 1)$. Thus, having in view properties of $w(t)$ and $W(t)$ we conclude that $w(t) \le W(t)$ for all $t \in (0, 1)$, so that (3.4) is satisfied.

Next, we will show that $P_1(k) \le D(k) + 1$ holds for $t \in (0, 1)$, or equivalently

$$\frac{3\pi^2}{2\mathcal{K}^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)} \le t^2 + 6t\sqrt{t} + 6t + 6\sqrt{t} + 1 - \frac{\pi^2}{2\mathcal{K}^2(t)},$$

by reversing the inequality in (3.4). Since $\mathcal{K}(t) > \frac{\pi}{2}$ we have $-\frac{\pi^2}{4\mathcal{K}^2(t)} > -1$ so that it suffices to show that

$$\frac{3\pi^2}{2\mathcal{K}^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)} \le t^2 + 6t\sqrt{t} + 6t + 6\sqrt{t}. \tag{3.7}$$

Set

$$r(t) := t^2 + 6t\sqrt{t} + 6t + 6\sqrt{t}.$$

Then $r'(t) = 2t + 9\sqrt{t} + 6 + 3/\sqrt{t} > 0$ on $(0, 1)$ and $r(0) = 0$, $r(1) = 19$. Moreover $r(1/\sqrt{2}) \approx 14.158379 > \frac{96}{\pi^2}$ whereas the value $\frac{96}{\pi^2}$ is the supremum of the left hand side of (3.7) as was shown in the first part of the proof of that case. Thus, it suffices to show (3.7) for $t \in (0, 1/\sqrt{2})$. Since $\mathcal{K}(t) > \frac{\pi}{2}$ then

$$\frac{3\pi^2}{2\mathcal{K}^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)} < \frac{6}{\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)}.$$

Now, we will show that

$$\frac{6}{\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)} \leq 6(t+\sqrt{t}) \leq r(t)$$

for $t \in (0, 1/\sqrt{2}]$, which concludes the desired result. The last inequality is obvious therefore it suffices to show that

$$(t+\sqrt{t})\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - 1 \geq 0. \tag{3.8}$$

Let

$$s(t) := (t+\sqrt{t})\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - 1.$$

Now we prove that $s(t)$ decreases in $(0, 1/\sqrt{2})$ to $s(1/\sqrt{2}) > 0$. Differentiating, we obtain

$$s'(t) = \frac{1}{2}\sinh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)\left[\left(1+\frac{1}{2\sqrt{t}}\right)\tanh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - \frac{\pi^2(\sqrt{t}+1)}{4\sqrt{t}(1-t^2)\mathcal{K}^2(t)}\right].$$

Since $\sinh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)$ is positive for $t \in (0, 1/\sqrt{2}]$ then $s'(t) < 0$ if and only if the expression in square brackets of $s'(t)$ is negative, or equivalently

$$\left(1+\frac{1}{2\sqrt{t}}\right)\tanh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - \frac{\pi^2(\sqrt{t}+1)}{4\sqrt{t}(1-t^2)\mathcal{K}^2(t)} < 0.$$

The above will be fulfilled if

$$\frac{2\sqrt{t}+1}{2(\sqrt{t}+1)}\tanh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)-\frac{\pi^2}{4(1-t^2)\mathcal{K}^2(t)}<0,$$

or, by means of the relation $\tanh x<1$, when the inequality

$$\frac{2\sqrt{t}+1}{2(\sqrt{t}+1)}-\frac{\pi^2}{4(1-t^2)\mathcal{K}^2(t)}<0$$

holds. It is easy to see that $b(t)=\frac{2\sqrt{t}+1}{2(1+\sqrt{t})}$ is increasing on $(0,1/\sqrt{2})$ with the maximal value $b(1/\sqrt{2})\approx 0.73$. Let

$$c(t):=\frac{\pi^2}{4(1-t^2)\mathcal{K}^2(t)}.$$

Since, by (2.6),

$$c'(t)=\frac{\pi^2}{2}\frac{\mathcal{K}(t)-\mathcal{E}(t)}{t(1-t^2)^2\mathcal{K}^3(t)}$$

and $\mathcal{K}(t)>\mathcal{E}(t)$ on $(0,1)$, then $c'(t)>0$ on $(0,1)$ so does on $(0,1/\sqrt{2})$ and therefore $c(t)>c(0^+)=1$ for all $t\in(0,1/\sqrt{2})$. Thus $b(t)-c(t)<0$ for all $t\in(0,1/\sqrt{2})$ and hence $s(t)$ decreases on $(0,1/\sqrt{2})$.

Next, we show that $s(0^+)=\infty$. Note that

$$\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)=\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)^2\left[1+\frac{1}{3!}\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)^2+\cdots\right]^2.$$

Then

$$\lim_{t\to0^+}(t+\sqrt{t})\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)=\lim_{t\to0^+}\frac{\pi^2(1+\sqrt{t})}{4\mathcal{K}^2(t)}\sqrt{t}\,(\mathcal{K}'(t))^2\left[1+\frac{1}{3!}\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)^2+\cdots\right]^2.$$

By properties of $\mathcal{K}(t)$ at $0^+$ (the relation (2.2)) we have

$$\lim_{t\to 0^+} \frac{\pi^2(1+\sqrt{t})}{4\mathcal{K}^2(t)} = 1,$$

so that we need to calculate the limit

$$\lim_{t\to 0^+} \sqrt{t}\,(\mathcal{K}'(t))^2 .$$

Applying (2.8) to the value $\sqrt{1-t^2}$ we obtain

$$\frac{\pi}{1+t} \le \mathcal{K}(\sqrt{1-t^2}) = \mathcal{K}'(t) \le \frac{\pi}{2t}, \tag{3.9}$$

and since $\sqrt{t}/(1+t)^2$ and $\sqrt{t}/t^2$ tend to $\infty$, as $t \to 0^+$, we conclude that

$$\lim_{t\to 0^+} \sqrt{t}\,(\mathcal{K}'(t))^2 = \infty. \tag{3.10}$$

Thus also $\lim_{t\to 0^+} s(t) = \infty$. Moreover $s(1/\sqrt{2}) \approx 0.9614 > 0$ so that we obtain the desired result. Hence the proof of the case 3. is complete. □

**Theorem 3.2.** *Let* $0 \le k < \infty$ *be fixed, and let a function* $p \in \mathcal{P}(p_k)$ *be such that* $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$. *Then*

$$|b_1^2 - b_2| \le P_1(k). \tag{3.11}$$

*The equality holds if* $p(z)$ *is* $p_k(z^2)$ *or one of its rotation.*

*Proof.* Since $p \prec p_k$ then, in view of a definition of the subordination, there exists a function $\omega(z) = \alpha_1 z + \alpha_2 z^2 + \cdots$, $|\omega(z)| < 1$ such that $p(z) = p_k(\omega(z))$, therefore

$$1 + b_1 z + b_2 z^2 + \cdots = 1 + P_1(k)\alpha_1 z + z^2(P_1(k)\alpha_2 + P_2(k)\alpha_1^2) + \cdots .$$

Comparing the coefficients of $z$ and $z^2$ we have $b_1 = P_1(k)\alpha_1$ and $b_2 = P_1(k)\alpha_2 + P_2(k)\alpha_1^2$, thus

$$\begin{aligned} |b_1^2 - b_2| &= |P_1^2(k)\alpha_1^2 - P_1(k)\alpha_2 - P_2(k)\alpha_1^2| \\ &= |\alpha_1^2(P_1^2(k) - P_2(k)) - P_1(k)\alpha_2| \\ &\le |\alpha_1|^2|P_1^2(k) - P_2(k)| + |P_1(k)||\alpha_2|. \end{aligned}$$

For the Schwarz' function $\omega$ the classical inequality $|\alpha_2| \le 1 - |\alpha_1|^2$ holds then, on account (3.1), we conclude

$$\begin{aligned} |b_1^2 - b_2| &\le |\alpha_1|^2|P_1^2(k) - P_2(k)| + |P_1(k)|(1 - |\alpha_1|^2) \\ &= |\alpha_1|^2 \left[|P_1^2(k) - P_2(k)| - P_1(k)\right] + P_1(k) \\ &\le P_1(k), \end{aligned}$$

and the proof of the inequality of (3.11) is complete.

The equality in (3.11) holds if $|b_1| = 0$ and $|b_2| = P_1(k)$, or equivalently, $p(z)$ is $p_k(z^2)$ or one of its rotations. □

**Remark.** Observe that the bound like (3.11) can be used in those subclasses of Carathéodory class for which the inequality similar to (3.1) holds. Let

(3.12) $$q(z) = 1 + c_1 z + c_2 z^2 + \cdots,$$

be such that

(3.13) $$|c_1^2 - c_2| \le c_1, \quad \text{with} \quad c_1 \ge 0,$$

it means $q$ satisfies the bounds similar to (3.1). Then, reasoning along the same line as in Theorem 3.2 we may prove that for $p \in \mathcal{P}(q)$, $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ the inequality

(3.14) $$|b_1^2 - b_2| \le c_1,$$

is satisfied.

For instance, if $0 < \gamma \leq 1$ then the function $\varphi(z) = [(1+z)/(1-z)]^\gamma$ maps the unit disk onto an angle, symmetric with respect to real axis, of width $\gamma\pi$ and contained in the right half-plane. Moreover, $\varphi(z) = 1+2\gamma z+2\gamma^2 z^2+\cdots$. Then, $|c_1^2 - c_2| = 2\gamma^2 \leq 2\gamma = c_1$ therefore (3.13) is satisfied, so that (3.14) applies. Concluding, if $p \in \mathcal{P}(\varphi) = \{q : q \prec \varphi\}$ and $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$, then $|b_1^2 - b_2| \leq 2\gamma$. This inequality remarkable improves the result $|b_1^2 - b_2| \leq 2$, due to Ma and Minda [13]. Similarly, the family $\mathcal{P}((1 + (1 - 2\beta)z)/(1 - z))$ can be treated. In this instance we obtain the inequality $|b_1^2 - b_2| \leq 2(1 - \beta)$ for $p \in \mathcal{P}((1 + (1 - 2\beta)z)/(1 - z))$.

**Theorem 3.3.** *Let $0 \leq k < \infty$ be fixed, and let a function $p \in \mathcal{P}(p_k)$ be of the form $p(z) = 1+b_1 z+b_2 z^2+\cdots$. Then*

$$|b_2 - \mu b_1^2| \leq \begin{cases} P_1(k) - \mu P_1^2(k) & \mu \leq 0, \\ P_1(k) & \mu \in (0,1], \\ P_1(k) + (\mu - 1)P_1^2(k) & \mu \geq 1. \end{cases} \tag{3.15}$$

*When $\mu < 0$ or $\mu > 1$, the equality holds if $p(z)$ is $p_k(z)$ or one of its rotations. If $0 < \mu < 1$ then the equality holds if $p(z) = p_k(z^2)$ or one of its rotation.*

*Proof.* Since $p \prec p_k$ then by Rogosinski Subordination Theorem we have $|b_n| \leq P_1(k)$ for $n \geq 1$ and each fixed $k \in [0, \infty)$ First assume that $\mu \geq 1$. In view of Theorem 3.2, we have $|b_2 - b_1^2| \leq P_1(k)$ therefore we obtain

$$|b_2 - \mu b_1^2| \leq |b_1^2 - b_2| + (\mu - 1)|b_1|^2 \leq P_1(k) + (\mu - 1)P_1^2(k).$$

Next, suppose that $\mu \leq 0$. Then

$$|b_2 - \mu b_1^2| \leq |b_2| + (-\mu)|b_1|^2 \leq P_1(k) - \mu P_1^2(k).$$

Finally, if $0 < \mu \le 1$ then $\mu = 1/t$ with $t \ge 1$. Hence one gets

$$\begin{aligned} |b_2 - \mu b_1^2| &= |b_2 - \frac{1}{t} b_1^2| = \frac{1}{t}|t b_2 - b_1^2| = \frac{1}{t}|(t-1)b_2 + b_2 - b_1^2| \\ &\le \frac{1}{t}[(t-1)|b_2| + |b_2 - b_1^2|] \\ &\le \frac{1}{t}[(t-1)P_1(k) + P_1(k)] = P_1(k), \end{aligned}$$

and the proof of all cases of (3.15) is complete.

When $\mu < 0$ or $\mu > 1$, the equality holds if and only if $|b_1| = P_1(k)$, that is, $p(z) = p_k(z)$ or one of its rotation. If $0 < \mu < 1$ then the equality holds if $|b_1| = 0$ and $|b_2| = P_1(k)$, or equivalently, $p(z)$ is $p_k(z^2)$ or one of its rotations. □

**Remark.** In the paper [13] Ma and Minda proved similar bounds in the class $\mathcal{P}$. For instance, when $\mu \le 0$ authors obtained the estimate $|b_2 - \mu b_1^2| \le 2 - 4\mu$. Observe that in the case of $\mathcal{P}(p_k)$ the result is far better; the same it holds in the remaining range of the constant $\mu$.


1. Ahiezer N. I., *Elements of Theory of Elliptic Functions* (Russian), Moscow 1970.
2. Anderson G. D., Vamanamurthy M. K. and Vuorinen M., *Topics in special functions*, Rep. Univ. Jyväskylä **83** (2001), 5–26.
3. Anderson G. D., Vamanamurthy M. K. and Vuorinen M. K, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, Wiley-Interscience, 1997.
4. Gerretsen J. and Sansone G., *Lectures on the Theory of Functions of a Complex Variable*, Wolters-Noordhooff Publishing Groningen, The Netherlands 1969.
5. Kanas S., *Generalized Notion of Starlikeness and Convexity* (dissertation), Rzeszów University of Technology, 2003.
6. ______, *Differential subordination related to conic sections*, to appear in JMAA (2005).
7. ______, *Techniques of the differential subordination for domains bounded by conic sections*, IJMMS **38** (2003), 2389–2400.

**8.** Kanas S. and Wiśniowska A., *Conic regions and k-uniform convexity,* Journal of Computational and Applied Mathematics **105** (1999), 327—336.
**9.** ______, *Conic regions and k-uniform convexity, II,* Folia. Sci. Tech. Resov. **170** (1998), 65–78.
**10.** ______, *Conic domains and starlike functions,* Revue Roum. Math. Pures Appl. **45**(4) (2000), 647–657.
**11.** Kanas S. and Sugawa T., *On conformal representations of the interior of an ellipse,* to appear in AASF Math., 2005.
**12.** Livingston A. E., *The coefficients of multivalent close-to-convex functions,* Proc. Amer. Math. Soc. **21** (1969), 545–552.
**13.** Ma W. and Minda D., *A unified treatment of some special classes of univalent functions,* Proc. Inter. Conf. on Complex Anal. of the Nankai Inst. of Math. (1992), 157–169.
**14.** ______, *Uniformly convex functions, II,* Ann. Polon. Math. **58**(3) (1993), 275–285.
**15.** Rønning F., *Uniformly convex functions and a corresponding class of starlike functions,* Proc. Amer. Math. Soc. **118**(1) (1993), 189–196.

S. Kanas, Department of Mathematics Rzeszów University of Technology, W. Pola 2, PL-35-959 Rzeszów Poland,
*e-mail*: `skanas@prz.rzeszow.pl`