ONESIDED INVERSES FOR SEMIGROUPS

MARIO PETRICH

Abstract. In any semigroup S, we say that elements a and b are left inverses of each other if $a = aba$, $b = bab$ and $a \mathcal{L} b$, in which case we write $a \gamma b$. Right inverses are defined dually with the notation δ. Set $\tau = \delta\gamma$. We study the classes of semigroups in which τ has some of the usual properties of a relation. We also consider properties of (maximal) completely simple subsemigroups of S.

In terms of the above concepts, we characterize E–solid, central, (almost) $\mathcal{L}$–unipotent and locally (almost) $\mathcal{L}$–unipotent semigroups in many ways. We define these notions for arbitrary semigroups by extending their definitions from regular semigroups.

1. Introduction and summary

It is the existence of an inverse for each element which distinguishes a group from a monoid and the existence of an identity which distinguishes a monoid from a semigroup. These differences are attenuated by the possibility of defining an inverse of some elements of an arbitrary semigroup S by the following device. Elements a and b are inverses of each other if $a = aba$ and $b = bab$. The standard notation for the set of all inverses of a is $V(a)$. Not all elements of S may have an inverse, but those that do are called regular. Hence a semigroup is called regular if all its elements are regular (this is equivalent to the usual definition). Two other important classes of semigroups are distinguished by the properties of inverses of their elements. First, S is an inverse semigroup if

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every element of S has a unique inverse. Second, an element a of S is completely regular if it has an inverse with which it commutes, and S is completely regular if all its elements are.

Hence some of the most important classes of semigroups can be defined (or characterized) by properties of inverses of elements of their members. The purpose of this paper is to further ramify the above properties of inverse and thereby characterize several of the most important classes of (regular) semigroups. The first task thus consists in introducing some new properties an inverse an element may have while the second amounts to identifying the classes of semigroups in which inverses of their elements have these properties.

The principal novelty here is the following set of concepts. For a semigroup S and $a, b \in S$, we say that a and b are left (respectively, right) inverses of each other if $b \in V(a)$ and $a \mathcal{L} b$ (respectively $a \mathcal{R} b$); if both, we say that they are twosided inverses of each other. It is easy to see that the existence of any kind of these inverses for an element a of S is equivalent to a being a completely regular element, that is contained in a subgroup of S. We are thus led to the notion of the group part of S which we denote by $G(S)$. A further refinement of the concept of an inverse is a bounded (onesided) inverse. We also consider completely simple subsemigroups of an arbitrary semigroup.

We start in Section 2 with a brief list of needed notation and some well-known lemmas. In Section 3 we establish a number of auxiliary results which will be used in the main body of the paper. In the next five sections, we consider E–solid, central, completely regular, almost $\mathcal{L}$–unipotent (every $\mathcal{L}$–class contains at most one idempotent) and locally almost $\mathcal{L}$–unipotent semigroups, respectively.

2. Background

For all notation and terminology not defined in the paper, consult standard texts on semigroups. In addition to the concepts defined in Section 1, we recall briefly the notation that will be used frequently.

Throughout the paper, S *stands for an arbitrary semigroup unless stated otherwise*. For any $A \subseteq S$, we denote by $E(A)$ the set of all idempotents in A. Green's relations are denoted by $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$, and for any

$a \in S, L_a, R_a, H_a$ and D_a are their classes containing the element a. We emphasize that for $a \in S$,

$$V(a) = \{b \in S | a = aba, b = bab\},$$

is the set of all inverses of a. The set $G(S)$ is the union of all subgroups of S; equivalently $G(S) = \bigcup_{e \in E(S)} H_e$, or it is the set of all completely regular elements of S. For $e \in E(S)$ and $a \in H_e$, we denote by a^{-1} the inverse of a in the group H_e and write $a^0 = e$.

For any set X, ε_X denotes the equality relation on X.

The remainder of this section is well known. Since it plays a basic role in our deliberations and for the sake of completeness, we supply (short) proofs.

Lemma 2.1. *Let $a, b \in S$. Then H_b contains an inverse of a if and only if there exist $e, f \in E(S)$ such that*

$$a \; \mathcal{R} \; e \; \mathcal{L} \; b \; \mathcal{R} \; f \; \mathcal{L} \; a.$$

If this is the case, then the inverse of a in H_b is unique.

Proof. Necessity. If $x \in V(a) \cap H_b$, then $e = ax$ and $f = xa$ satisfy the requisite conditions.

Sufficiency. By hypothesis, we have

$$a = ea = af, \;\; b = be = fb, \;\; e = au, \;\; f = va$$

for some $e, f \in E(S)$ and $u, v \in S^1$. For $x = fuave$, we obtain

$$axa = afuavea = auava = eaf = a,$$

$$xax = fuaveafuave = fua(va)uave = fu(au)ave = fuave = x,$$

$$e = au = afuau = afuafu = afuavau = afuave \in Sx,$$

and similarly $f \in xS$ which together with $x \in fS \cap Se$ implies that

$$x \in V(a) \cap L_e \cap R_f = V(a) \cap H_b.$$

For the final assertion, let $x, y \in V(a)$ and $x \; \mathcal{H} \; y$. Then $x = yu = vy$ for some $u, v \in S^1$ which implies that

$$x = yu = yayu = yax$$

and similarly $x = xay$ which yields

$$x = xax = (yax)a(xay) = yay = y,$$

as asserted. $\square$

Lemma 2.2. *Let $a, b \in S$. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ is a group.*

Proof. Necessity. By hypothesis, $a = abx$ and $b = yab$ for some $x, y \in S^1$. Then letting $e = bx$, we get $e = ya$ and thus $e \in L_a \cap R_b \cap E(S)$, as required.

Sufficiency. Let $a \,\mathcal{L}\, e \,\mathcal{R}\, b$ where $e \in E(S)$. Then

$$a = ae, \; e = xa, \; b = eb, \; e = by$$

for some $x, y \in S^1$. Hence

$$a = ae = aby, \; b = eb = xab$$

so that $a \,\mathcal{R}\, ab \,\mathcal{L}\, b$. $\square$

Corollary 2.3. *Let $a, b, c \in G(S)$ be such that $a \,\mathcal{R}\, b \,\mathcal{L}\, c$. Then the following conditions are equivalent*

(i) $ac \,\mathcal{H}\, b$.

(ii) $a \,\mathcal{L}\, h \,\mathcal{R}\, c$ *for some* $h \in E(S)$.

(iii) $ca \in G(S)$.

Proof. The equivalence of parts (i) and (ii) follows directly from Lemma 2.2. The same reference also implies that $ca \in L_a \cap R_c$. This implies the equivalence of parts (ii) and (iii). $\square$

Lemma 2.4. *Let $e, f \in E(S)$, $x \in V(ef)$, $g = efxe$ and $h = fxef$. Then $g, h \in E(S)$ and*

$$g \,\mathcal{R}\, ef = gh \,\mathcal{L}\, h.$$

Proof. Indeed,

$$g^2 = (efxe)(efxe) = ef(xefx)e = efxe = g,$$

and similarly $h^2 = h$,

$$g \in efS,\ ef \in gS \cap Sh,\ h \in Sef,$$
$$ef = ef(xefx)ef = (efxe)(fxef) = gh,$$

as required. □

3. Preliminaries

The following are our basic concepts and notation. For $a, b \in S$,

$$\begin{aligned}
&b \text{ is a } \textit{left inverse} \text{ of } a \text{ if } b \in V_l(a) = V(a) \cap L_a,\\
&b \text{ is a } \textit{right inverse} \text{ of } a \text{ if } b \in V_r(a) = V(a) \cap R_a,\\
&b \text{ is a } \textit{twosided inverse} \text{ of } a \text{ if } b \in V_t(a) = V(a) \cap H_a.
\end{aligned}$$

We start by ellucidating the nature of these concepts. Hall [1, Theorem 2], proved that a regular semigroup S is orthodox if and only if it has the property

$$V(a) \cap V(b) \neq \phi \Rightarrow V(a) = V(b).$$

The next proposition shows that the sets $V_l(a)$ have an analogous property.

Proposition 3.1. *The following implication holds in S:*

$$V_l(a) \cap V_l(b) \neq \phi \Rightarrow V_l(a) = V_l(b).$$

Proof. Let $a, b, c, d \in S$ be such that $c \in V_l(a) \cap V_l(b)$, $d \in V_l(a)$. Then

$$a \mathcal{L} b \mathcal{L} c \mathcal{L} d,$$

$$c = cac, \quad a = aca, \quad c = cbc, \quad b = bcb, \quad d = dad, \quad a = ada,$$

so that $a = sb$, $d = tc$, $c = ud$, $b = va$ for some $s, t, u, v \in S^1$. Hence

$$d = dad = dsbd = dsbcbd = dacbd = tcacbd = tcbd = dbd,$$

$$b = bcb = budb = budadb = bcadb = vacadb = vadb = bdb$$

and thus $d \in V_l(b)$. Therefore $V_l(a) \subseteq V_l(b)$ and by symmetry, the equality prevails. □

An element of S may have an inverse without having a left or a right inverse. The next lemma specifies exactly which elements of S have onesided inverses. We shall generally use this lemma without express reference.

Lemma 3.2. *The following conditions on an element a of S are equivalent.*

(i) $a \in G(S)$.
(ii) *a has a twosided inverse.*
(iii) *a has a left inverse.*
(iv) *a has a right inverse.*

Proof. That (i) implies (ii) implies (iii) is trivial.
(iii) *implies* (i): Let $x \in V_l(a)$. Then

$$a = axa, \; x = xax, \; a = ux, \; x = va$$

for some $u, v \in S^1$. We always have that $ax \in E(S) \cap R_a$. In this case also

$$a = ux = uxax \in Sax, \quad ax = ava \in Sa$$

so that $a \mathcal{L} ax$. Therefore $ax \in E(S) \cap H_a$ which implies that H_a is a group so $a \in G(S)$.
The equivalence of parts (i) and (iv) now follows by left–right duality. □

As a consequence we have that an element a of S has an (onesided) inverse if and only if a is (completely) regular. For an element of $G(S)$, the next lemma provides the sets of all such inverses.

Lemma 3.3. *Let $a \in G(S)$.*

(i) $V_l(a) = \{fa^{-1} \mid f \in E(L_a)\}$
$\quad = \{x \in S \mid a = axa = a^2x,\ x = xax = x^2a\}$.

(ii) $V_r(a) = \{a^{-1}f \mid f \in E(R_a)\}$
$\quad = \{x \in S \mid a = axa = xa^2,\ x = xax = ax^2\}$.

(iii) $V_t(a) = \{a^{-1}\}$
$\quad = \{x \in S \mid a = axa,\ x = xax,\ ax = xa\}$.

Proof. We only prove part (i); part (ii) is its dual and part (iii) requires a straightforward argument. Let $e = a^0$ and denote the three sets by $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively.

Let $x \in \mathcal{A}$. By Lemma 3.2, we have that $x \in G(S)$, so let $f = x^0$. By Lemma 2.1, $fa^{-1} \in R_f \cap L_{a^{-1}}$ if and only if $L_f \cap R_{a^{-1}}$ is a group. But $L_f \cap R_{a^{-1}} = H_a$ is a group and thus $fa^{-1} \in H_x$. Further, since $e \mathrel{\mathcal{L}} f$, we get

$$(fa^{-1})a(fa^{-1}) = (fe)fa^{-1} = fa^{-1},$$
$$a(fa^{-1})a = (af)e = ae = a$$

and thus $fa^{-1} \in V(a)$. Hence $x, fa^{-1} \in V(a) \cap H_x$ which by Lemma 2.1 yields that $x = fa^{-1}$. Therefore $x \in \mathcal{B}$ which proves that $\mathcal{A} \subseteq \mathcal{B}$.

Next let $x \in \mathcal{B}$. Then $x = fa^{-1}$ for some $f \in E(L_a)$. We have seen above that then $fa^{-1} \in V(a)$. Also

$$a^2x = a^2(fa^{-1}) = (a^2f)a^{-1} = a^2a^{-1} = a,$$
$$x^2a = (fa^{-1})(fa^{-1})a = f(a^{-1}f)e = fa^{-1}e = fa^{-1} = x.$$

Hence $x \in \mathcal{C}$ and thus $\mathcal{B} \subseteq \mathcal{C}$.

Finally, if $x \in \mathcal{C}$, then $x \in V(a)$ and $x \mathrel{\mathcal{L}} a$ so that $x \in \mathcal{A}$. Consequently $\mathcal{C} \subseteq \mathcal{A}$. □

We deduce the following basic consequence of Lemma 3.3 which we shall use repeatedly.

Corollary 3.4. *Let $e, f \in E(S)$ and $a \in H_e$. If $e \; \mathcal{L} \; f$ (respectively $e \; \mathcal{R} \; f$), then fa^{-1} (respectively $a^{-1}f$) is the unique left (respectively right) inverse of a in H_f. This exhausts all left (respectively right) inverses. In particular, $V_l(e) = E(L_e)$ and $V_r(e) = E(R_e)$.*

Proof. The first two assertions follow from Lemma 3.3, the third one follows easily from the first. □

We introduce the following relations on $G(S)$:

$$a \; \gamma \; b \;\text{ if }\; b \in V_l(a), \quad a \; \delta \; b \;\text{ if }\; b \in V_r(a), \quad \tau = \delta\gamma.$$

By Lemma 3.3, for $e, f \in E(S)$ and $a \in H_e$, $b \in H_f$, we have

$$a \; \gamma \; b \Leftrightarrow e \; \mathcal{L} \; f, \; b = fa^{-1} \Leftrightarrow e \; \mathcal{L} \; f, \; a = eb^{-1}$$

and similarly for δ. For any relation θ on S, we write $\operatorname{tr} \theta = \theta|_{E(S)}$ and call it the *trace* of θ.

Clearly both γ and δ are symmetric, τ is reflexive (on $G(S)$), $\gamma \subseteq \mathcal{L}$, $\operatorname{tr} \gamma = \operatorname{tr} \mathcal{L}$, $\delta \subseteq \mathcal{R}$, $\operatorname{tr} \delta = \operatorname{tr} \mathcal{R}$, and $\tau \subseteq \mathcal{D}$. We shall consider the question: for which classes of semigroups do γ, δ or τ or their traces have some of the usual properties of a relation? The following concept will prove useful.

If $e, f, g, h \in E(S)$ are such that $e \; \mathcal{R} \; f \; \mathcal{L} \; g \; \mathcal{R} \; h \; \mathcal{L} \; e$, we say that they form an *idempotent square in S* and write $[e, f, g, h]$ *in S*.

4. E-solid semigroups

We now extend a well-known concept from regular to arbitrary semigroups. A semigroup S is E-*solid* if for any $e, f, g \in E(S)$ such that $f \; \mathcal{R} \; e \; \mathcal{L} \; g$, there exists $h \in E(S)$ satisfying $f \; \mathcal{L} \; h \; \mathcal{R} \; g$. We may "localize" this concept by introducing the following notion. If $e \in E(S)$, then S is e-*solid* if for any $f, g \in E(S)$ such that $f \; \mathcal{R} \; e \; \mathcal{L} \; g$, there exists $h \in E(S)$ such that $f \; \mathcal{L} \; h \; \mathcal{R} \; g$. Note that in both cases, the conclusion "there exists $h \in E(S)$ such that $f \; \mathcal{L} \; h \; \mathcal{R} \; g$" is equivalent to $gf \in H_e$, in view of Lemma 2.2.

The following lemma, needed later, is of independent interest.

Lemma 4.1. *Let $e \in E(S)$. Then S is e-solid if and only if there exists a greatest completely simple subsemigroup of S containing e.*

Proof. Necessity. Let

$C_e = \bigcup \{H_g | g \in E(S)$ and there exist $f, h \in E(S)$ such that $[e, f, g, h]$ is an idempotent square in $S\}$.

To see that C_e is closed under multiplication, we let $[e, f, g, h]$ and $[e, f', g', h']$ be idempotent squares in S and $a \in H_g$, $a' \in H_{g'}$. We have the following situation

$$\begin{array}{ccccc} e & — & f & — & f' \\ | & & | & & | \\ h & — & g & — & u \\ | & & | & & | \\ h' & — & v & — & g' \end{array}$$

where the hypothesis guarantees the existence of $u, v \in E(S)$ with the above location. Hence $[e, f', u, h]$ is an idempotent square and the existence of v by Lemma 2.2 implies that $aa' \in H_u$. Therefore $aa' \in C_e$, as required and C_e is a semigroup.

Since C_e is saturated by $\mathcal{H}$, it must be closed for taking of twosided inverses and thus C_e is completely regular. Every element of C_e is $\mathcal{D}$-related to e and hence C_e is bisimple. Consequently C_e is a completely simple subsemigroup of S.

Let C be any completely simple subsemigroup of S which contains e and let $a \in C$. Then there exist $f, g, h \in E(S)$ such that $[e, f, g, h]$ is an idempotent square in C and $a \in H_g$ whence clearly $a \in C_e$. Therefore $C \subseteq C_e$ which establishes the desired maximality of C_e.

Sufficiency. Let $f, g \in E(S)$ be such that $f \;\mathcal{R}\; e \;\mathcal{L}\; g$ and denote by C the greatest completely simple subsemigroup of S containing e. Both $\{e, f\}$ and $\{e, g\}$ are completely simple subsemigroups of S containing e and thus $\{e, f\}$, $\{e, g\} \subseteq C$ by hypothesis. Hence $f, g \in E(C)$ and clearly $f \;\mathcal{R}\; e \;\mathcal{L}\; g$ also in C. Since C is completely simple, there exists $h \in E(C)$ such that $[e, f, g, h]$ is an idempotent square in C and thus also in S. Therefore S is e-solid. □

We are now ready for a multiple characterization of E-solid semigroups. For regular semigroups, further characterizations can be found in Hall [3, Theorem 3].

Theorem 4.2. *The following conditions on S are equivalent.*

(i) *S is E-solid.*
(ii) *If $a, b, c \in G(S)$ are such that $a \;\mathcal{R}\; b \;\mathcal{L}\; c$, then $ca \in G(S)$.*
(iii) *If $a \;\tau\; c$, then $c \;\tau\; x$ for some $x \in H_a$.*
(iv) $\operatorname{tr} \tau$ *is symmetric.*
(v) $\operatorname{tr} \tau$ *is transitive.*
(vi) $\operatorname{tr} \tau$ *is an equivalence relation.*
(vii) $(\operatorname{tr} \gamma)(\operatorname{tr} \delta) = (\operatorname{tr} \delta)(\operatorname{tr} \gamma)$.
(viii) *For every $e \in E(S)$, S is e-solid.*
(ix) *For every $e \in E(S)$, there exists a greatest completely simple subsemigroup C_e of S containing e.*
(x) *$G(S)$ is a pairwise disjoint union of maximal completely simple subsemigroups of S saturated by $\mathcal{L}$ and $\mathcal{R}$ within $G(S)$.*
(xi) *$G(S)$ is a pairwise disjoint union of maximal completely simple subsemigroups M_α of S and every completely simple subsemigroup of S is contained in some M_α.*

Moreover, the collections in parts (ix), (x) and (xi) coincide and consist precisely of all maximal completely simple subsemigroups of S.

Proof. (i) *implies* (ii). Let $a, b, c \in G(S)$ be such that $a \; \mathcal{R} \; b \; \mathcal{L} \; c$. Then $a^0 \; \mathcal{R} \; b^0 \; \mathcal{L} \; c^0$ and by hypothesis there exists $h \in E(S)$ such that $a^0 \; \mathcal{L} \; h \; \mathcal{R} \; c^0$. By Lemma 2.2, we have that $ca \; \mathcal{H} \; e$ and hence $ca \in G(S)$.

(ii) *implies* (iii). Let $a \; \delta \; b \; \gamma \; c$. Then $a, b, c \in G(S)$ and $a \; \mathcal{R} \; b \; \mathcal{L} \; c$. By hypothesis, we have $ca \in G(S)$ and by Lemma 2.2 that $a \; \mathcal{L} \; ca \; \mathcal{R} \; c$. Let $h = (ca)^0$. By Lemma 3.3, $c^{-1}h \; \delta \; c$ and $a^0(c^{-1}h)^{-1}\gamma \; c^{-1}h$ so that for $x = a^0(c^{-1}h)^{-1}$, we get $c \; \tau \; x \; \mathcal{H} \; a$, as required.

(iii) *implies* (iv). Let $e, f \in E(S)$ be such that $e \; \tau \; f$. By hypothesis, we have $f \; \tau \; x$ for some $x \in H_e$. By Corollary 3.4, we must have $x \in E(S)$ and thus $x = e$.

(iv) *implies* (v). Let $e, f, g \in E(S)$ be such that $e \; \tau \; f$ and $f \; \tau \; g$. Then $e \; \delta \; x \; \gamma \; f$ and $f \; \delta \; y \; \gamma \; g$ for some $x, y \in E(S)$ by Corollary 3.4. It follows that $y \; \delta \; f \; \gamma \; x$ so that $y \; \tau \; x$. The hypothesis implies that $x \; \tau \; y$ so, again by the same reference, there exists $z \in E(S)$ such that $x \; \delta \; z \; \gamma \; y$. Hence $e \; \mathcal{R} \; x \; \mathcal{R} \; z$ and $z \; \mathcal{L} \; y \; \mathcal{L} \; g$ which evidently implies that $e \; \delta \; z \; \gamma \; g$ and thus $e \; \tau \; g$.

(v) *implies* (viii). Let $e, f, g \in E(S)$ be such that $e \; \mathcal{R} \; f \; \mathcal{L} \; g$. Then $g \; \delta \; g \; \gamma \; f$ which yields that $g \; \tau \; f$; also $f \; \delta \; e \; \tau \; e$ which implies that $f \; \tau \; e$. Now the hypothesis implies that $g \; \tau \; e$. Again by Corollary 3.4, we deduce the existence of $h \in E(S)$ such that $g \; \delta \; h \; \gamma \; e$ and hence $e \; \mathcal{L} \; h \; \mathcal{R} \; g$. Therefore S is f-solid where $f \in E(S)$ is arbitrary.

(viii) *implies* (ix). This is a direct consequence of Lemma 4.1.

(ix) *implies* (x). Recall the definition of C_e in Lemma 4.1. We shall show that the family $\mathcal{C} = \{C_e | e \in E(S)\}$ has the requisite properties. First let $e, f \in E(S)$ and assume that $x \in C_e \cap C_f$. From the proof of Lemma 4.1, we know that both C_e and C_f are saturated by $\mathcal{H}$. Hence $H_x \subseteq C_e \cap C_f$. Since C_e is completely simple, H_x must be a group, say with identity g. By maximality of C_g, we obtain $C_e \subseteq C_g$ whence $e \in C_g$. But then by maximality of C_e, we also have $C_g \subseteq C_e$. Therefore $C_e = C_g$ and analogously $C_f = C_g$ so that $C_e = C_f$. Hence the collection $\mathcal{C}$ is pairwise disjoint.

Let C be a completely simple subsemigroup of S containing C_e. Then $e \in C$ which implies that $C \subseteq C_e$. Thus $C_e = C$ and C_e is a maximal completely simple subsemigroup of S.

Let $e, f \in E(S)$ be such that $e \mathcal{L} f$. Then $\{e, f\}$ is a completely simple subsemigroup of S containing e and thus $\{e, f\} \subseteq C_e$ whence $f \in C_e$. We have observed above that C_e is saturated by $\mathcal{H}$. Hence C_e is saturated by $\mathcal{L}$ within $G(S)$ and the same argument is valid for $\mathcal{R}$.

(x) *implies* (xi). Let C be a completely simple subsemigroup of S and $x \in C$. Then $x \in G(S)$ and hence $x \in M_\alpha$ for some α. Now let $y \in C$. Then $x \mathcal{R} z \mathcal{L} y$ in C for some $z \in C$. The hypothesis implies that first $z \in M_\alpha$ and then $y \in M_\alpha$. This shows that $C \subseteq M_\alpha$, as required.

(xi) *implies* (i). Let $e, f, g \in E(S)$ be such that $e \mathcal{R} f \mathcal{L} g$. Denote by M_α, M_β and M_γ the maximal completely simple subsemigroups of S containing e, f and g, respectively, from the collection in the hypothesis. Since $\{e, f\}$ is a completely simple subsemigroup of S, it must be contained in some M_δ. By pairwise disjointness of the collection, we get that $M_\alpha = M_\delta = M_\beta$. Similarly $M_\beta = M_\gamma$. But then $e, f, g \in M_\delta$ where $e \mathcal{R} f \mathcal{L} g$. Since M_δ is completely simple, there exists $h \in E(M_\alpha)$ such that $e \mathcal{L} h \mathcal{R} g$. It follows that S is E-solid.

(i) *is equivalent to* (vi). We have proved above that parts (i), (iv) and (v) are equivalent. This together with the fact that τ is always reflexive on $G(S)$ proves the contention.

(iv) *is equivalent to* (vii). By Corollary 3.4, $\mathrm{tr}\ \tau = (\mathrm{tr}\ \delta)(\mathrm{tr}\ \gamma)$ whence follows the assertion.

The above arguments contain the proof of the remaining assertions of the theorem. $\square$

In Theorem 4.2(x), one cannot omit the condition that M_α be saturated by $\mathcal{L}$ and $\mathcal{R}$ within $G(S)$ or the corresponding condition in Theorem 4.2(xi). Indeed, let

$$\mathcal{S} = \mathcal{M}^0(3, \{1\}, 2; P)$$

with $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Then $G(S)$ is a pairwise disjoint union of maximal completely simple subsemigroups of S, namely

$$\{(1,1,1),(2,1,1)\},\ \{(2,1,2),(3,1,2)\},\ \{0\}.$$

Neither of the first two semigroups is saturated by $\mathcal{R}$ within $G(S)$ since $(2,1,1)\ \mathcal{R}\ (2,1,2)$. Also the completely simple subsemigroup $\{(2,1,1),(2,1,2)\}$ is not contained in any of the above semigroups. Of course, the semigroup S is not E-solid.

The situation with subsemigroups of an arbitrary semigroup S which are left or right groups is much simpler. Indeed, for any $e \in E(S)$, $L_e \cap G(S)$ (respectively $R_e \cap G(S)$) is the greatest subsemigroup of S which is a left (respectively right) group and contains e. It follows that $G(S)$ is a pairwise disjoint union of maximal subsemigroups of S which are left (respectively right) groups.

5. Central semigroups

We start by defining certain functions among some group $\mathcal{H}$-classes of an arbitrary semigroup which will turn out to be antiisomorphisms. Indeed, Corollary 3.4 makes it possible to define the following mappings.

Lemma 5.1. *Let $e, f \in E(S)$.*

(i) *Let $e\ \mathcal{L}\ f$. Then the mapping*

$$\gamma_{ef} : a \to fa^{-1} \quad (a \in H_e)$$

is an antiisomorphism between H_e and H_f. Moreover, $\gamma_{ef}^{-1} = \gamma_{fe}$ and for any $a \in H_e$, $fa^{-1} = (fa)^{-1}$.

(ii) *Let $e\ \mathcal{R}\ f$. Then the mapping*

$$\delta_{ef} : a \to a^{-1}f \quad (a \in H_e)$$

is an antiisomorphism between H_e and H_f. Moreover, $\delta_{ef}^{-1} = \delta_{fe}$ and for any $a \in H_e$, $a^{-1}f = (af)^{-1}$.

Proof. (i) For $a \in H_e$, we get

$$(fa^{-1})(fa) = f(a^{-1}f)a = fa^{-1}a = fe = f$$

and thus $fa^{-1} = (fa)^{-1}$. If also $b \in H_e$, then with $\varphi = \gamma_{ef}$, we get

$$(a\varphi)(b\varphi) = (fa^{-1})(fb^{-1}) = f(a^{-1}f)b^{-1} = fa^{-1}b^{-1} = f(ba)^{-1} = (ba)\varphi$$

and φ is an antihomomorphism. Now letting $\psi = \gamma_{fe}$, we obtain

$$a\varphi\psi = e(fa^{-1})^{-1} = e(fa) = (ef)a = ea = a$$

and similarly $c\psi\varphi = c$ for any $c \in H_f$. Therefore φ is a bijection between H_e and H_f and so an antiisomorphism and $\varphi^{-1} = \psi$.

(ii) This is the dual of part (i). $\square$

Corollary 5.2. *Let $e, f, g \in E(S)$.*

$$\text{If } e \ \mathcal{L} \ f \ \mathcal{R} \ g, \ \text{ then } \gamma_{ef}\delta_{fg} : a \to fag,$$
$$\text{if } e \ \mathcal{R} \ f \ \mathcal{L} \ g, \ \text{ then } \delta_{ef}\gamma_{fg} : a \to gaf,$$

are isomorphisms between H_e and H_g.

Proof. This follows directly from Lemma 5.1. $\square$

For G a group and $g \in G$, we define ε_g by

$$\varepsilon_g : x \to g^{-1}xg \qquad (x \in G).$$

We now summarize the most salient features of the compositions of the above functions for a given idempotent square $[e, f, g, h]$ (for the definition, see the end of Section 3).

Theorem 5.3. *Let $[e, f, g, h]$ in S and $a \in H_e$.*

(i) $hf(a\delta_{ef}\gamma_{fg}) = haf = (a\gamma_{eh}\delta_{hg})hf \in V(a^{-1})$.

(ii) $\delta_{ef}\gamma_{fg} = \gamma_{eh}\delta_{hg}\varepsilon_{hf}$.
(iii) $\delta_{gh}\gamma_{he}\delta_{ef}\gamma_{fg} = \varepsilon_{hf}$.

Proof. (i) Indeed, by Corollary 5.2, we have

$$\begin{aligned} hf(a\delta_{ef}\gamma_{fg}) &= hfgaf = hf(af) = haf, \\ (a\gamma_{eh}\delta_{hg})hf &= haghf = (ha)hf = haf, \\ (haf)a^{-1}(haf) &= ha(fa^{-1}h)af = haa^{-1}af = haf, \\ a^{-1}(haf)a^{-1} &= (a^{-1}h)a(fa^{-1}) = a^{-1}aa^{-1} = a^{-1}, \end{aligned}$$

as required.

(ii) This follows directly from part (i).

(iii) This is an immediate consequence of part (i) in view of Lemma 5.1. □

We now extend a well-known concept from completely regular to arbitrary semigroups as follows. A semigroup S is *central* if for any $e, f \in E(S)$ such that $ef \in G(S)$, ef is in the center of the group H_{ef}.

Theorem 5.4. *The following conditions on S are equivalent.*

(i) *S is central.*
(ii) *If $[e, f, g, h]$ in S, then $\delta_{ef}\gamma_{fg} = \gamma_{eh}\delta_{hg}$.*
(iii) *If $[e, f, g, h]$ in S and $a \in H_e$, then $gaf = hag$.*
(iv) *Every completely simple subsemigroup of S is central.*

Proof. (i) *implies* (ii). If $[e, f, g, h]$ is in S, then $hf \in H_g$ by Lemma 2.2 and the hypothesis implies that ε_{hf} is the identity mapping on H_g; the desired conclusion now follows by Theorem 5.3(ii).

(ii) *implies* (iii). This follows directly from Corollary 5.2.

(iii) *implies* (i). First let $e, f, h \in E(S)$ be such that $f \; \mathcal{R} \; e \; \mathcal{L} \; h$ and $fh \in H_e$. By Lemma 2.2, there exists $g \in E(S)$ such that $f \; \mathcal{L} \; g \; \mathcal{R} \; h$. Hence $[e, f, g, h]$ is an idempotent square in S and the hypothesis implies that for

$a \in H_e$, we have $gaf = hag$. Hence

$$\begin{aligned} fha &= fh(ae) = fha(eh) = fhae(gh) = f(hag)h \\ &= f(gaf)h = (fg)afh = fafh = f(ea)fh \\ &= (fe)afh = eafh = afh \end{aligned}$$

and fh is in the center of H_e.

We now consider the general case. Hence let $e, f, h \in E(S)$ be such that $fh \in H_e$. By Lemma 2.4, there exist $f', h' \in E(S)$ such that $fh = f'h'$ and $f' \mathcal{R} f'h' \mathcal{L} h'$. Hence $f' \mathcal{R} e \mathcal{L} h'$ and $f'h' \in H_e$. The case above yields that $f'h'$ is in the center of H_e and hence so is fh. Therefore S is central.

(i) *implies* (iv). This is trivial.

(iv) *implies* (i). Let $e, f, g \in E(S)$ be such that $eg \in H_f$. Letting $x = (ef)^{-1} \in V(ef)$ and applying Lemma 2.4, we deduce that we may assume that $e, f, g \in E(S)$ are such that $e \mathcal{R} f \mathcal{L} g$ and $eg \in H_f$. Thus $eg \in R_e \cap L_g$ which by Lemma 2.2 implies that $L_e \cap R_g$ is a group. Denoting by h the identity element of this group, we obtain the idempotent square $[e, f, g, h]$.

Now let $a \in H_f$. Since $[e, f, g, h]$ is an idempotent square, all the products and inverses obtained from the set $\{e, f, g, h, a\}$ are contained in the set $H_e \cup H_f \cup H_g \cup H_h$. Hence the set of all such products and inverses is a completely simple subsemigroup C of S. By hypothesis C is central and thus $aeg = ega$, as required. $\square$

6. Completely regular semigroups

In view of Lemma 2.2, the conclusion in Theorem 4.2(ii) is equivalent to $ac \,\mathcal{H}\, b$. What happens if we take a, b, c in S rather than in $G(S)$? The answer is very simple.

Proposition 6.1. *The following conditions on S are equivalent.*

(i) *S is completely regular.*

(ii) *Every element of S has a left (respectively right, twosided) inverse.*

(iii) *If* $a \ \mathcal{R} \ b \ \mathcal{L} \ c$, *then* $ac \ \mathcal{H} \ b$.
(iv) τ *is a reflexive relation on* S.

Proof. The equivalence of parts (i) and (ii) and of (i) and (iv) follows directly from Lemma 3.2.

(i) *implies* (iii). Let $a \ \mathcal{R} \ b \ \mathcal{L} \ c$. The existence of $h \in E(S)$ such that $a^0 \ \mathcal{L} \ h \ \mathcal{R} \ c^0$ by Lemma 2.2 implies that $ac \ \mathcal{H} \ b$.

(iii) *implies* (i). For any $a \in S$, we have $a \ \mathcal{R} \ a \ \mathcal{L} \ a$ and thus $a^2 \ \mathcal{H} \ a$. Hence a is a completely regular element of S which yields the assertion. ☐

In order to treat central completely regular semigroups we first consider completely simple semigroups.

Lemma 6.2. *On a central completely simple semigroup* τ *is an equivalence relation.*

Proof. In view of the Rees theorem and [8, Proposition III.6.2],we may set $S = \mathcal{M}(I, G, \Lambda; P)$ where P is normalized and all its entries lie in the center of G. Reflexivity of τ follows from Proposition 6.1. We will freely use Lemma 3.3. If

$$\begin{aligned}(1)\qquad &(i, g, \lambda) \ \delta \ (i, g, \lambda)^{-1}(i, p_{\mu i}^{-1}, \mu) \\ &= (i, p_{\lambda i}^{-1} g^{-1} p_{\mu i}^{-1}, \mu) \ \gamma \ (j, p_{\mu j}^{-1}, \mu)(i, p_{\lambda i}^{-1} g^{-1} p_{\mu i}^{-1}, \mu) \\ &= (j, p_{\mu j}^{-1} p_{\mu i} g p_{\lambda i} p_{\mu i}^{-1}, \mu) = (j, p_{\mu j}^{-1} g p_{\lambda i}, \mu)\end{aligned}$$

since $p_{\mu i}$ lies in the center of G, then

$$\begin{aligned}(2)\qquad &(i, g, \lambda) \ \gamma \ (j, p_{\lambda j}^{-1}, \lambda)(i, g, \lambda)^{-1} \\ &= (j, p_{\lambda j}^{-1} g^{-1} p_{\lambda i}^{-1}, \lambda) \ \delta \ (j, p_{\lambda j}^{-1} g^{-1} p_{\lambda i}^{-1}, \lambda)^{-1}(j, p_{\mu j}^{-1}, \mu) \\ &= (j, p_{\lambda j}^{-1} p_{\lambda i} g p_{\lambda j} p_{\mu j}^{-1}, \mu) = (j, p_{\lambda i} g p_{\mu j}^{-1}, \mu)\end{aligned}$$

since $p_{\lambda j}$ lies in the center of G. The hypothesis also implies that the middle entries of (1) and (2) are equal. Hence τ is symmetric.

We now turn to transitivity. On the one hand

$$
\begin{aligned}
(3)\qquad & (i,g,\lambda)\ \delta\ (i,g,\lambda)^{-1}(i,p_{\mu i}^{-1},\mu) \\
&= (i,p_{\lambda i}^{-1}g^{-1}p_{\mu i}^{-1},\mu)\ \gamma\ (j,p_{\mu j}^{-1},\mu)(i,p_{\lambda i}^{-1}g^{-1}p_{\mu i}^{-1},\mu)^{-1} \\
&= (j,p_{\mu j}^{-1}p_{\mu i}gp_{\lambda i}p_{\mu i}^{-1},\mu)\ \delta\ (j,p_{\mu j}^{-1}p_{\mu i}gp_{\lambda i}p_{\mu i}^{-1},\mu)^{-1}(j,p_{\nu j}^{-1},\nu) \\
&= (j,p_{\mu j}^{-1}p_{\mu i}p_{\lambda i}^{-1}g^{-1}p_{\mu i}^{-1}p_{\mu j}p_{\nu j}^{-1},\nu) \\
&\quad \gamma\ (k,p_{\nu k}^{-1},\nu)(j,p_{\mu j}^{-1}p_{\mu i}p_{\lambda i}^{-1}g^{-1}p_{\mu i}^{-1}p_{\mu j}p_{\nu j}^{-1},\nu)^{-1} \\
&= (k,p_{\nu k}^{-1}p_{\nu j}p_{\mu j}^{-1}p_{\mu i}gp_{\lambda i}p_{\mu i}^{-1}p_{\mu j}p_{\nu j}^{-1},\nu)
\end{aligned}
$$

and on the other hand,

$$
\begin{aligned}
(4)\qquad & (i,g,\lambda)\ \delta\ (i,g,\lambda)^{-1}(i,p_{\nu i}^{-1},\nu) \\
&= (i,p_{\lambda i}^{-1}g^{-1}p_{\nu i}^{-1},\nu)\ \gamma\ (k,p_{\nu k}^{-1},\nu)(i,p_{\lambda i}^{-1}g^{-1}p_{\nu i}^{-1},\nu)^{-1} \\
&= (k,p_{\nu k}^{-1}p_{\nu i}gp_{\lambda i}p_{\nu i}^{-1},\nu).
\end{aligned}
$$

Taking into account that all sandwich matrix entries lie in the center of G, we obtain that the elements in (3) and (4) are equal. It follows that τ is transitive and thus is an equivalence relation on S. □

We also need a kind of converse of the above lemma.

Lemma 6.3. *A completely simple semigroup on which τ is symmetric is central.*

Proof. By the Rees theorem, we may set $S = \mathcal{M}(I, G, \Lambda; P)$ where P is normalized. We will freely use Lemma 3.3. First

$$
\begin{aligned}
&(i, g, \lambda)\ \delta\ (i, g, \lambda)^{-1}(i, 1, 1) \\
&\qquad = (i, p_{\lambda i}^{-1} g^{-1}, 1)\gamma\ (1, 1, 1)(i, p_{\lambda i}^{-1} g^{-1}, 1)^{-1} \\
&\qquad = (1, g p_{\lambda i}, 1).
\end{aligned} \tag{5}
$$

By hypothesis, there exists $j \in I$ such that

$$
\begin{aligned}
&(i, g, \lambda)\ \gamma\ (j, p_{\lambda j}^{-1}, \lambda)(i, g, \lambda)^{-1} \\
&\qquad = (j, p_{\lambda j}^{-1} g^{-1} p_{\lambda i}^{-1}, \lambda)\ \delta\ (j, p_{\lambda j}^{-1} g^{-1} p_{\lambda i}^{-1}, \lambda)^{-1}(j, 1, 1) \\
&\qquad = (j, p_{\lambda j}^{-1} p_{\lambda i} g p_{\lambda j}, 1).
\end{aligned} \tag{6}
$$

Since (5) and (6) must be equal, we get $j = 1$ which implies that $p_{\lambda j} = 1$ and thus $g p_{\lambda i} = p_{\lambda i} g$. It follows that each $p_{\lambda i}$ belongs to the center of G. Now [8, Proposition III.6.2], implies that S is central. □

We are now able to prove the desired result.

Theorem 6.4. *The following conditions on S are equivalent.*

(i) *S is completely regular and central.*
(ii) *τ is an equivalence relation on S.*
(iii) *τ is reflexive on S and symmetric.*

Proof. (i) *implies* (ii). This follows directly from Lemma 6.2 since clearly $\tau \subseteq \mathcal{D}$.

(ii) *implies* (iii). This is trivial.

(iii) *implies* (i). Since τ is reflexive on S, Proposition 6.1 yields that S is completely regular. Hence S is a semilattice Y of completely simple semigroups S_α. For each $\alpha \in Y$, $\tau|_{S_\alpha}$ is the τ-relation on S_α and is symmetric, so by Lemma 6.3, S_α is central. Now [8, Theorem II.6.4], implies that S is central. □

In the next theorem we encounter *orthogroups*, that is completely regular semigroups whose idempotents form a subsemigroup.

Theorem 6.5. *The semigroup S is an orthogroup if and only if τ is a congruence on S.*

Proof. Necessity. The semigroup S is a semilattice Y of rectangular groups S_α. For each $\alpha \in Y$, we may take $S_\alpha = L_\alpha \times G_\alpha \times R_\alpha$ where L_α, G_α and R_α are a left zero semigroup, a group and a right zero semigroup, respectively. Let $i, j \in L_\alpha$, $g \in G_\alpha$, 1 be the identity of G_α and $\lambda, \mu \in R_\alpha$. By Lemma 3.3, we get

$$(i, g, \lambda)\ \delta\ (i, g^{-1}, \lambda)(i, 1, \mu) = (i, g^{-1}, \mu)\ \gamma\ (j, 1, \mu)(i, g, \mu) = (j, g, \mu).$$

It follows that for $a = (i, g, \lambda) \in S_\alpha$, $b = (j, h, \mu) \in S_\beta$, we have

$$a\ \tau\ b \Leftrightarrow \alpha = \beta,\ g = h,$$

Thus $\tau|_{S_\alpha} = \sigma_{S_\alpha}$, the least group congruence on S_α and τ is an equivalence relation contained in $\mathcal{D}$. Hence $\tau = \bigcup_{\alpha \in Y} \tau|_{S_\alpha} = \bigcup_{\alpha \in Y} \sigma_{S_\alpha}$; but $\bigcup_{\alpha \in Y} \sigma_{S_\alpha} = \nu_S$, the least Clifford congruence on S, by [4, Theorem (ii)]. Therefore $\tau = \nu_S$ and thus τ is a congruence on S.

Sufficiency. Proposition 6.1 implies that S is completely regular. Let $e, f \in E(S)$; we wish to show that $ef \in E(S)$. In view of Lemma 2.4 we may suppose that $e\ \mathcal{R}\ ef\ \mathcal{L}\ f$. Hence $e\ \delta\ (ef)^0\ \gamma\ f$ so that $e\ \tau\ f$. The hypothesis implies that $ef\ \tau\ f$ which by Corollary 3.4 yields that $ef \in E(S)$. Therefore S is an orthogroup. □

Proposition 6.6. *The following conditions on S are equivalent:*

(i) *S is a band.*
(ii) $\gamma = \mathcal{L}$.
(iii) $\delta = \mathcal{R}$.
(iv) $\tau = \mathcal{D}$.

Proof. That part (i) implies the remaining parts is obvious.

(ii) *implies* (i). First by Lemma 3.2, S is completely regular. Let $a \in S$ and $x \in V(a)$. Then $a \,\mathcal{L}\, xa$ and hence $a \,\gamma\, xa$. But then $a = a(xa)a = a^2$.

(iii) *implies* (i). This is the dual of the preceding case.

(iv) *implies* (i). By Lemma 3.2, S is completely regular. Let $a \in S$ and $x \in V(a)$. Then $a \,\mathcal{D}\, xa$ and by hypothesis, there exists $b \in S$ such that $a \,\delta\, b \,\gamma\, xa$. Hence $b = sxa$ for some $s \in S$ and thus

$$a = aba = a(bxab)a = abxa = a(sxa)xa = asxa = ab \in E(S)$$

and S is a band. □

7. Almost $\mathcal{L}$-unipotent semigroups

Let θ be an equivalence relation on S. Then S is called *θ-unipotent* if every θ-class contains exactly one idempotent. We modify this concept by saying that S is *almost θ-unipotent* if every θ-class contains at most one idempotent.

Theorem 7.1. *The following conditions on S are equivalent.*

(i) *For any $a \in S$ and $a', a'' \in V(a)$, we have $a'a = a''a$.*
(ii) *For any $a \in S$ and $a', a'' \in V(a)$, we have $a' \,\mathcal{R}\, a''$.*
(iii) *S is almost $\mathcal{L}$-unipotent.*
(iv) *Every element of S has at most one left inverse.*
(v) *Every left inverse of any element of S is twosided.*
(vi) *Every completely simple subsemigroup of S is a right group.*

Proof. (i) *implies* (ii). Let $a \in S$ and $a', a'' \in V(a)$. By hypothesis, we have $a'a = a''a$ whence $a' \,\mathcal{R}\, a'a = a''a \,\mathcal{R}\, a''$.

(ii) *implies* (iii). Let $e, f \in E(S)$ be such that $e \,\mathcal{L}\, f$. Then $e, f \in V(e)$ and the hypothesis implies that $e \,\mathcal{R}\, f$. But then $e = f$.

(iii) *implies* (iv). Let $a \in S$ and $b, c \in V_l(a)$. In view of Lemma 3.2, we have $b, c \in G(S)$ and thus $b^0 \mathcal{L} c^0$. But then the hypothesis implies that $b^0 = c^0$ which by Lemma 2.1 yields $b = c$.

(iv) *implies* (v). Let $a \in S$ and $b \in V_l(a)$. By hypothesis, b is the unique left inverse of a and hence $b = a^{-1}$. Thus b is a twosided inverse of a.

(v) *implies* (i). Let $a', a'' \in V(a)$. Then $a'a \mathcal{L} a^0 \mathcal{L} a''a$ and thus $a'a, a''a \in V_l(a^0)$ and the hypothesis implies that $a'a = a''a = a^0$.

(iii) *implies* (vi). If C is a completely simple subsemigroup of S, then C is $\mathcal{L}$-unipotent and is thus a right group.

(vi) *implies* (iii). Let $e, f \in E(S)$ be such that $e \mathcal{L} f$. Then $\{e, f\}$ is a completely simple subsemigroup of S so is a right group whence $e = f$. □

Corollary 7.2. *The following conditions on S are equivalent.*

(i) *S is almost $\mathcal{L}$- and $\mathcal{R}$-unipotent.*
(ii) *Every element of S has at most one inverse.*
(iii) $\tau = \varepsilon_{G(S)}$.
(iv) $\operatorname{tr} \tau = \varepsilon_{E(S)}$.
(v) *Every completely simple subsemigroup of S is a group.*

Proof. (i) *implies* (ii). Let $a \in S$ and $a', a'' \in V(a)$. By Theorem 7.1 and its dual, we get $a'a = a''a$ and $aa' = aa''$ whence $a' = a'aa' = a''aa'' = a''$.

(ii) *implies* (iii). Let $a, b \in G(S)$ be such that $a \; \tau \; b$. Then $a \; \delta \; c \; \gamma \; b$ for some $c \in G(S)$ which implies that $a, b \in V(c)$. The hypothesis then yields that $a = b$.

(iii) *implies* (iv). This is trivial.

(iv) *implies* (i). Let $e, f \in E(S)$. If $e \mathcal{L} f$, then $e \; \delta \; e \; \gamma \; f$ and if $e \mathcal{R} f$, then $e \; \delta \; f \; \gamma \; f$ so that in either case $e = f$.

(i) *is equivalent to* (v). This follows directly from Theorem 7.1 and its dual. □

We are somewhat more explicit in the following result. Recall that a band satisfying the identity $xa = axa$ is *right regular.*

Proposition 7.3. *The semigroup S satisfies any of the conditions in Theorem 7.1 and the product of any two idempotents of S is a regular element if and only if $E(S)$ is either empty or is a right regular band. In the latter case, the set R of regular elements of S is a subsemigroup of S.*

Proof. Necessity. Let $e, f \in E(S)$. We show first that $ef \in E(S)$. To this end, let $x \in V(ef)$. Then $xef, fxef \in E(S)$ and $xef \ \mathcal{L} \ fxef$. The hypothesis that S is almost $\mathcal{L}$-unipotent implies that $xef = fxef$. Multiplying on the right by x, we get $x = fx$. But then $x = xefx = xex$ which implies that $xe \in E(S)$. Since also $efxe \in E(S)$ and $xe \ \mathcal{L} \ efxe$, the hypothesis implies that $xe = efxe$. This gives $xef = efxef = ef$ so that $ef \in E(S)$. In particular $ef, fef \in E(S)$ and also $ef \ \mathcal{L} \ fef$ which by hypothesis yields that $ef = fef$. Therefore $E(S)$ is a right regular band.

Sufficiency. This is obvious since a right regular band is trivially $\mathcal{L}$-unipotent.

For the last assertion of the proposition, let $a, b \in R$, $x \in V(a)$, $y \in V(b)$. Then $xa, by \in E(S)$ and the hypothesis implies that $xaby = byxaby$ and thus

$$ab = a(xaby)b = a(byxaby)b = abyxa(byb) = abyxab.$$

$\square$

Proposition 7.3 was proved by Venkatesan ([9, Theorem 1] under the covering hypothesis that S be a regular semigroup. For these semigroups, we adopt the label *right inverse semigroups.* This terminology may be justified in view of parts (i) and (ii) of Theorem 7.1. Because of part (iii) of the same theorem, they are also referred to as $\mathcal{L}$-unipotent. In view of Proposition 7.3, they are also known as right regular orthodox semigroups. In particular, the semigroup R in Proposition 7.3 is a right inverse semigroup.

We also have the dual concept of a *left inverse semigroup.* Clearly S is both a left and a right inverse semigroup if and only if S is an inverse semigroup. The structure of left inverse semigroups was elucidated by Yamada [10].

Corollary 7.4. *The semigroup S satisfies any of the conditions in Corollary 7.2 and the product of any two idempotents of S is a regular element if and only if $E(S)$ is either empty or is a semilattice. In the latter case, the set R of all regular elements of S is an inverse subsemigroup of S.*

Proof. This follows easily from Proposition 7.3 and its dual. □

We are now able to characterize the congruences on a regular semigroup generated by $\operatorname{tr}\gamma$, $\operatorname{tr}\delta$, $\operatorname{tr}\tau$ and τ. If θ is a relation on S, θ^* denotes the congruence on S generated by θ. A regular semigroup whose idempotents form a regular band (that is satisfies the identity $axya = axaya$) is called a *regular orthodox semigroup*.

Theorem 7.5. *In any regular semigroup S,*

(i) *$(\operatorname{tr}\gamma)^*$ is the least right inverse congruence,*
(ii) *$(\operatorname{tr}\delta)^*$ is the least left inverse congruence,*
(iii) *$(\operatorname{tr}\gamma)^* \cap (\operatorname{tr}\delta)^*$ is the least regular orthodox semigroup congruence,*
(iv) *$\tau^* = (\operatorname{tr}\tau)^* = (\operatorname{tr}\gamma)^*(\operatorname{tr}\delta)^* = (\operatorname{tr}\delta)^*(\operatorname{tr}\gamma)^* = (\operatorname{tr}\gamma)^* \vee (\operatorname{tr}\delta)^*$ is the*

least inverse congruence.

Proof. (i) Obviously $\operatorname{tr}\gamma = \operatorname{tr}\mathcal{L}$. By [7, Theorem 1(ii)], $(\operatorname{tr}\mathcal{L})^*$ is the least right inverse congruence on S.

(ii) This is the dual of part (i).

(iii) We note first that by [7, Theorem 1(ix)] in view of parts (i) and (ii), the relation $(\operatorname{tr}\gamma)^* \cap (\operatorname{tr}\delta)^*$ is the least regular orthodox semigroup congruence on S.

(iv) Let $\lambda = \tau^*$ and $\rho = (\operatorname{tr}\tau)^*$.

Let $e, f \in E(S/\rho)$ be such that $e\ \mathcal{L}\ f$. By [2, Theorem 5], there exist $e', f' \in E(S)$ such that $e'\rho = e$, $f'\rho = f$ and $e'\ \mathcal{L}\ f'$. But then $e'\ \delta\ e'\ \gamma\ f'$ and thus $e'\ \tau\ f'$ and $e = e'\rho = f'\rho = f$. Hence S/ρ is $\mathcal{L}$-unipotent. By duality, we conclude that S/ρ is also $\mathcal{R}$-unipotent and is thus an inverse semigroup. It follows that ρ is an inverse congruence and since $\lambda \supseteq \rho$, also λ is an inverse congruence.

Let θ be an inverse congruence on S and let $a \;\delta\; b \;\gamma\; c$. By Lemma 3.3, we have

$$a^0 \;\mathcal{R}\; b^0 \;\mathcal{L}\; c^0, \quad b = a^{-1}b^0, \quad c = c^0 b^{-1}$$

so that

$$(a\theta)^0 \;\mathcal{R}\; (b\theta)^0 \;\mathcal{L}\; (c\theta)^0, \quad b\theta = (a\theta)^{-1}(b\theta)^0, \quad c\theta = (c\theta)^0(b\theta)^{-1}.$$

But then

$$a^0 \;\theta\; b^0 \;\theta\; c^0, \quad b \;\theta\; a^{-1}, \quad c \;\theta\; b^{-1}.$$

Hence $a \;\theta\; b^{-1} \;\theta\; c$ whence $a \;\theta\; c$. It follows that $\tau \subseteq \theta$ and thus $\lambda \subseteq \theta$, proving the minimality of λ. For $\theta = \rho$, we obtain $\lambda \subseteq \rho$ and equality prevails. Therefore $\lambda = \rho$ is the least inverse congruence on S.

The remaining equalities follow from [7, Corollary (ii) to Theorem 1] in view of parts (i) and (ii). □

In particular, the relation τ in Theorem 6.4 is the least Clifford congruence on S.

8. Locally almost $\mathcal{L}$-unipotent semigroups

In order to treat this case, we introduce the following concept. Elements a and b of a semigroup S are *bounded inverses* of each other if $a, b \in G(S)$, $b \in V(a)$ and there exists $e \in E(S)$ such that $e \geq a^0$ and $e \geq b^0$. Obviously the twosided inverse of an element a is bounded. We shall use onesided bounded inverses, so we let $B(a)$ be the set of all bounded inverses of a and

$$B_l(a) = V_l(a) \cap B(a), \quad B_r(a) = V_r(a) \cap B(a).$$

Let θ be an equivalence relation on S. Then S satisfies *θ-majorization* if for any $e, f, g \in E(S)$, $e \geq f$, $e \geq g$ and $f \;\theta\; g$ imply that $f = g$. For a property $\mathcal{P}$ of semigroups, S is *locally* $\mathcal{P}$ if for any $e \in E(S)$, the semigroup eSe has property $\mathcal{P}$.

Theorem 8.1. *The following conditions on S are equivalent.*

(i) *S is locally an almost $\mathcal{L}$- unipotent semigroup.*

(ii) *S satisfies $\mathcal{L}$-majorization.*
(iii) *Every element of S has at most one bounded left inverse.*
(iv) *Every bounded left inverse of any element of S is twosided.*
(v) *Every subsemigroup of S which is a completely simple semigroup with an identity adjoined is a right group with an identity adjoined.*

Proof. (i) *implies* (ii). Let $e, f, g \in E(S)$ be such that $e \geq f$, $e \geq g$ and $f \,\mathcal{L}\, g$. Then $f, g \in E(eSe)$ and $f \,\mathcal{L}\, g$ in eSe which by hypothesis implies that $f = g$.

(ii) *implies* (iii). Let $a \in S$ and $b \in B_l(a)$. Then there exists $e \in E(S)$ such that $e \geq a^0$ and $e \geq b^0$. Since $a \,\mathcal{L}\, b$, we have $a^0 \,\mathcal{L}\, b^0$ and hence the hypothesis implies that $a^0 = b^0$. But then $b = a^{-1}$ which proves its uniqueness.

(iii) *implies* (iv). Let $a \in S$ and $b \in B_l(a)$. Then $a^{-1} \in B_l(a)$ by hypothesis implies that $b = a^{-1}$. Hence b is a twosided inverse of a.

(iv) *implies* (v). Let $C \cup \{e\}$ be a subsemigroup of S where C is a completely simple semigroup with e adjoined to it as an identity. If $f, g \in E(S)$ are such that $f \,\mathcal{L}\, g$, then both f and g are bounded left inverses of f and the hypothesis implies that $f = g$. Thus C is a right group.

(v) *implies* (i). Let $e \in E(S)$ and C be a completely simple subsemigroup of eSe. If $e \in C$, then C is a (right) group. Otherwise $C \cup \{e\}$ is a completely simple semigroup with an identity adjoined so by hypothesis C is a right group. By Theorem 7.1, eSe is almost $\mathcal{L}$-unipotent. □

Corollary 8.2. *The following conditions on S are equivalent.*

(i) *S is locally an almost $\mathcal{L}$- and $\mathcal{R}$-unipotent semigroup.*
(ii) *S satisfies $\mathcal{L}$- and $\mathcal{R}$-majorization.*
(iii) *Every element of S has at most one bounded inverse.*
(iv) *Every bounded inverse of any element of S is twosided.*
(v) *Every subsemigroup of S which is a completely simple semigroup with an identity adjoined is a group with an identity adjoined.*

Proof. (i), (ii) *and* (v) *are equivalent.* This follows directly from Theorem 8.1 and its dual.

(i) *implies* (iii). Let $a \in S$ and $b \in B(a)$. Then there exists $e \in E(S)$ such that $e \geq a^0$ and $e \geq b^0$. By hypothesis, eSe is an almost $\mathcal{L}$- and $\mathcal{R}$-unipotent semigroup; also $a, b \in eSe$ and $b \in V(a)$ in eSe. Now Corollary 7.2 implies that $b = a^{-1}$. Therefore a^{-1} is the only bounded inverse of a.

(iii) *implies* (iv). This is obvious.

(iv) *implies* (ii). Let $e, f, g \in E(S)$ be such that $e \geq f$, $e \geq g$ and $f \mathcal{L} g$. Then g is a bounded inverse of f and by hypothesis, must be twosided. It follows that $f = g$. Hence S satisfies $\mathcal{L}$-majorization; analogously it also satisfies $\mathcal{R}$-majorization. □

Recall first that a band satisfying the identity $xya = yxa$ is *right normal.* Compare the next result with Proposition 7.3.

Proposition 8.3. *The semigroup S satisfies any of the conditions in Theorem 7.1, any of the conditions in the dual of Theorem 8.1 and the product of any two idempotents of S is a regular element if and only if $E(S)$ is either empty or is a right normal band. In the latter case, the set R of all regular elements of S is a right inverse locally inverse semigroup.*

Proof. Necessity. Assume that $E(S)$ is not empty. By Proposition 7.3, $E(S)$ is a right regular band and by the dual of Theorem 8.1, S satisfies $\mathcal{R}$-majorization. Hence $E(S)$ is a right regular band which satisfies $\mathcal{R}$-majorization and it is well known that then $E(S)$ must be a right normal band.

Sufficiency. By Proposition 7.3, S satisfies the conditions in Theorem 7.1. It is well known that a right normal band satisfies $\mathcal{R}$-majorization. It follows that also S satisfies $\mathcal{R}$-majorization. Hence S satisfies the dual of the conditions in Theorem 8.1. The product of any two idempotents is not only a regular element but it is an idempotent.

Assume that $E(S)$ is a right normal band. By Proposition 7.3, the set R is a right inverse subsemigroup of S. Since $E(S) = E(R)$ is a right normal band, it is locally a semilattice and thus R is locally inverse. □

Regular semigroups in Proposition 8.3 are called right normal right inverse semigroups by Madhavan [5]. In that paper he constructed for them a representation by means of partial transformations on a set which generalizes that of Wagner for inverse semigroups. They are also called right normal orthodox semigroups.

9. Remarks

When S is regular and $e, f \in E(S)$, Nambooripad [6] introduced the *sandwich set* $S(e, f)$ of e and f, one of whose formulations is

$$S(e, f) = fV(ef)e.$$

It is easy to see that $S(e, f) = V(ef) \cap fSe$ so that $S(e, f)$ is the set of all inverses of ef in fSe. In fact, $S(e, f)$ is always a rectangular band. Here we encounter special kinds of inverses of elements of S of the form ef for $e, f \in E(S)$. Of course, one can consider $S(e, f)$ in an arbitrary semigroup. Under the hypothesis that the product of any two idempotents is a regular element, which we used in Proposition 7.3, Corollary 7.4 and Proposition 8.3, all sandwich sets $S(e, f)$ are nonempty and are thus rectangular bands.

1. Hall T. E., *On regular semigroups whose idempotents form a subsemigroup*, Bull. Austral. Math. Soc. **1** (1969), 195–208.
2. ______, *Congruences and Green's relations on regular semigroups*, Glasgow Math. J. **13** (1972), 167–175.
3. ______, *On regular semigroups*, J. Algebra **24** (1973), 1–24.
4. Jones P. R., *The least inverse and orthodox congruences on a completely regular semigroup*, Semigroup Forum **27** (1983), 390–392.
5. Madhavan S., *On right normal right inverse semigroups*, Semigroup Forum **12** (1976), 333–339.
6. Nambooripad K. S. S., *The natural partial order on a regular semigroup*, Proc. Edinburgh Math. Soc. **23** (1980), 249–260.
7. Pastijn F. and Petrich M., *Congruences on regular semigroups associated with Green's relations*, Boll. Un. Mat. Ital **1**(B) (1987), 591–603.
8. Petrich M. and Reilly N. R., *Completely regular semigroups*, Wiley, New York, 1999.
9. Venkatesan P. S., *Right (left) inverse semigroups*, J. Algebra **31** (1974), 209–217.

10. Yamada M., *Orthodox semigroups whose idempotents satisfy a certain identity*, Semigroup Forum **6** (1973), 180–188.

Mario Petrich, 21420 Bol, Brač, Croatia