ON THREE INEQUALITIES SIMILAR
TO HARDY-HILBERT'S INTEGRAL INEQUALITY

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ABSTRACT. The following three inequalities, which are similar to Hardy-Hilbert integral inequality are proved.

$$\int\limits_0^\infty \int\limits_0^\infty \frac{x^{\beta/q} y^{\alpha/p} F(x) G(y)}{\max\{x^\lambda, y^\lambda\}} \mathrm{d}x \mathrm{d}y \leq \frac{\lambda}{(\alpha+1)^{1/p}(\beta+1)^{1/q}} \frac{p^{1-1/p} q^{1-1/q}}{(p-1)(q-1)}$$

$$\cdot \left(\int\limits_0^\infty f^p(t)\mathrm{d}t\right)^{1/p} \left(\int\limits_0^\infty g^q(t)\mathrm{d}t\right)^{1/q}.$$

where

$$F(x) = \int\limits_0^x f(t)\mathrm{d}t, \qquad G(y) = \int\limits_0^y g(t)\mathrm{d}t.$$

$$\int\limits_0^\infty \int\limits_0^\infty \frac{x^{1/p} y^{1/q} F(x) G(y)}{(x+y)^4} \mathrm{d}x \mathrm{d}y < B^{1/p}(p,p) B^{1/q}(q,q) \frac{pq}{(p-1)(q-1)}$$

$$\cdot \left(\int\limits_0^\infty f^p(t)\mathrm{d}t\right)^{1/p} \left(\int\limits_0^\infty g^q(t)\mathrm{d}t\right)^{1/q}.$$

ABSTRACT.

$$\int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \frac{x^{1/p}y^{1/q}z^{1/r}\sqrt{xyzF(x)G(y)H(z)}}{(x+y+z)^8}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$< K_p K_q K_r \left(\int\limits_0^\infty f^{p/2}(t)\mathrm{d}t\right)^{1/p} \left(\int\limits_0^\infty g^{q/2}(t)\mathrm{d}t\right)^{1/q} \left(\int\limits_0^\infty h^{r/2}(t)\mathrm{d}t\right)^{1/r}.$$

where

$$H(z) = \int\limits_0^z h(t)\mathrm{d}t, \qquad K_p = \sqrt{\frac{p/2}{p/2-1}} B^{1/p}\left(\frac{p}{2},\frac{p}{2}\right) B^{1/p}(p,p).$$

1. INTRODUCTION

Let $f, g \geq 0$ satisfy

$$0 < \int\limits_0^\infty f^2(t)\mathrm{d}t < \infty \qquad \text{and} \qquad 0 < \int\limits_0^\infty g^2(t)\mathrm{d}t < \infty,$$

then

(1)
$$\int\limits_0^\infty \int\limits_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\,\mathrm{d}y < \pi \left(\int\limits_0^\infty f^2(t)\mathrm{d}t \int\limits_0^\infty g^2(t)\mathrm{d}t\right)^{1/2},$$

where the constant factor π is the best possible (cf. Hardy et al. [3]). Inequality (1) is well known as Hilbert's integral inequality. This inequality had been extended by Hardy [1] as follows

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t)\mathrm{d}t < \infty \qquad \text{and} \qquad 0 < \int_0^\infty g^q(t)\mathrm{d}t < \infty,$$

then

$$(2) \qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,\mathrm{d}x\,\mathrm{d}y < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(t)\mathrm{d}t\right)^{1/p} \left(\int_0^\infty g^q(t)\mathrm{d}t\right)^{1/q},$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (2) is called Hardy-Hilbert's integral inequality and is important in analysis and application (cf. Mitrinovic et al. [4]).

B. Yang gave the following extension of (2) as follows:

Theorem ([5]). *If $\lambda > 2 - \min\{p, q\}$, $f, g \geq 0$, satisfy*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)\mathrm{d}t < \infty \qquad \text{and} \qquad 0 < \int_0^\infty t^{1-\lambda} g^q(t)\mathrm{d}t < \infty,$$

then

$$(3) \qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,\mathrm{d}x\,\mathrm{d}y < k_\lambda(p) \left(\int_0^\infty t^{1-\lambda} f^p(t)\mathrm{d}t\right)^{1/p} \left(\int_0^\infty t^{1-\lambda} g^q(t)\mathrm{d}t\right)^{1/q},$$

where the constant factor $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible, B is the beta function.

The object of this paper is that to give some new inequalities similar to that of Hardy-Hilbert's inequality.

We need the following result for our aim

Theorem A ([2]). *Let f be a nonnegative integrable function. Define*

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

Then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p \mathrm{d}x < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\mathrm{d}x, \qquad p > 1.$$

2. NEW RESULTS

We state and prove the following:

Theorem 1. *Let $f, g \geq 0$, $F(x) = \int_0^x f(t)\mathrm{d}t$, $G(y) = \int_0^y g(t)\mathrm{d}t$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $p = \lambda - \alpha - 1 > 1$, $q = \lambda - \beta - 1 > 1$, $\alpha, \beta > -1$ then*

$$\int_0^\infty \int_0^\infty \frac{x^{\beta/q} y^{\alpha/p} F(x) G(y)}{\max\{x^\lambda, y^\lambda\}}\mathrm{d}x\,\mathrm{d}y \leq \frac{\lambda}{(\alpha+1)^{1/p}(\beta+1)^{1/q}} \frac{p^{1-1/p} q^{1-1/q}}{(p-1)(q-1)}$$

$$\cdot \left(\int_0^\infty f^p(t)\mathrm{d}t\right)^{1/p} \left(\int_0^\infty g^q(t)\mathrm{d}t\right)^{1/q}.$$

Proof. We have

$$\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\frac{x^{\beta/q}y^{\alpha/p}F(x)G(y)}{\max\{x^{\lambda},y^{\lambda}\}}\mathrm{d}x\,\mathrm{d}y$$

$$\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\frac{y^{\alpha/q}F(x)}{\left(\max\{x^{\lambda},y^{\lambda}\}\right)^{1/p}}\cdot\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\frac{x^{\beta/q}G(y)}{\left(\max\{x^{\lambda},y^{\lambda}\}\right)^{1/q}}\mathrm{d}x\,\mathrm{d}y$$

$$\leq\left(\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\frac{y^{\alpha}F^{p}(x)}{\max\{x^{\lambda},y^{\lambda}\}}\mathrm{d}x\,\mathrm{d}y\right)^{1/p}\left(\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\frac{x^{\beta}G^{q}(y)}{\max\{x^{\lambda},y^{\lambda}\}}\mathrm{d}x\,\mathrm{d}y\right)^{1/q}$$

$$=M^{1/p}N^{1/q}.$$

Observe that

$$\begin{aligned}
M &= \int\limits_{0}^{\infty}F^{p}(x)\mathrm{d}x\int\limits_{0}^{\infty}\frac{y^{\alpha}}{\max\{x^{\lambda},y^{\lambda}\}}\mathrm{d}x\,\mathrm{d}y = \int\limits_{0}^{\infty}F^{p}(x)\mathrm{d}x\left(\int\limits_{0}^{x}\frac{y^{\alpha}}{x^{\lambda}}\mathrm{d}y+\int\limits_{x}^{\infty}\frac{y^{\alpha}}{y^{\lambda}}\mathrm{d}y\right) \\
&= \frac{\lambda}{(\alpha+1)(\lambda-\alpha-1)}\int\limits_{0}^{\infty}\frac{F^{p}(x)}{x^{\lambda-\alpha-1}}\mathrm{d}x = \frac{\lambda}{(\alpha+1)p}\int\limits_{0}^{\infty}\left(\frac{F(x)}{x}\right)^{p}\mathrm{d}x \\
&< \frac{\lambda}{(\alpha+1)}\frac{p^{p-1}}{(p-1)^{p}}\int\limits_{0}^{\infty}f^{p}(x)\mathrm{d}x,
\end{aligned}$$

in view of Theorem A.

Similarly, It can be shown that

$$N < \frac{\lambda}{(\beta+1)} \frac{q^{q-1}}{(q-1)^q} \int\limits_0^\infty g^q(x)\mathrm{d}x.$$

Combining these inequalities to obtain

$$\int\limits_0^\infty \int\limits_0^\infty \frac{x^{\beta/q}y^{\alpha/p}F(x)G(y)}{\max\{x^\lambda, y^\lambda\}}\mathrm{d}x\,\mathrm{d}y \leq \frac{\lambda}{(\alpha+1)^{1/p}(\beta+1)^{1/q}} \frac{p^{1-1/p}q^{1-1/q}}{(p-1)(q-1)}$$

$$\cdot \left(\int\limits_0^\infty f^p(t)\mathrm{d}t\right)^{1/p} \left(\int\limits_0^\infty g^q(t)\mathrm{d}t\right)^{1/q}.$$

$\square$

Theorem 2. *Let* $f, g \geq 0$, $F(x) = \int_0^x f(t)\mathrm{d}t$, $G(y) = \int_0^y g(t)\mathrm{d}t$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then, we have*

$$\int\limits_0^\infty \int\limits_0^\infty \frac{x^{1/p}y^{1/q}F(x)G(y)}{(x+y)^4}\mathrm{d}x\mathrm{d}y < B^{1/p}(p,p)B^{1/q}(q,q)\frac{pq}{(p-1)(q-1)}$$

$$\cdot \left(\int\limits_0^\infty f^p(t)\mathrm{d}t\right)^{1/p} \left(\int\limits_0^\infty g^q(t)\mathrm{d}t\right)^{1/q}.$$

Proof.

$$\int\limits_0^\infty \int\limits_0^\infty \frac{x^{1/p}y^{1/q}F(x)G(y)}{(x+y)^4}\mathrm{d}x\mathrm{d}y$$

$$= \int\limits_0^\infty \int\limits_0^\infty \frac{y^{1/q}F(x)}{(x+y)^2} \cdot \frac{x^{1/p}G(y)}{(x+y)^2}\mathrm{d}x\,\mathrm{d}y$$

$$\leq \left(\int\limits_0^\infty \int\limits_0^\infty \frac{y^{p/q}F^p(x)}{(x+y)^{2p}}\mathrm{d}x\,\mathrm{d}y\right)^{1/p} \left(\int\limits_0^\infty \int\limits_0^\infty \frac{x^{q/p}G^q(y)}{(x+y)^{2q}}\mathrm{d}x\,\mathrm{d}y\right)^{1/q}$$

$$= P^{1/p}Q^{1/q}.$$

Now, we consider

$$P = \int\limits_0^\infty \left(\frac{F(x)}{x}\right)^p \mathrm{d}x \int\limits_0^\infty \frac{\left(\frac{y}{x}\right)^{p-1}\frac{1}{x}}{\left(1+\frac{y}{x}\right)^{2p}}\mathrm{d}y = \int\limits_0^\infty \left(\frac{F(x)}{x}\right)^p \mathrm{d}x \int\limits_0^\infty \frac{u^{p-1}}{(1+u)^{2p}}\mathrm{d}u$$

$$< B(p,p)\left(\frac{p}{p-1}\right)^p \int\limits_0^\infty f^p(x)\mathrm{d}x,$$

by Theorem A.

Similarly, $\quad Q < B(q,q)\left(\frac{q}{q-1}\right)^q \int\limits_0^\infty g^q(y)\mathrm{d}y.$

Therefore, we have

$$\int_0^\infty \int_0^\infty \frac{x^{1/p}y^{1/q}F(x)G(y)}{(x+y)^4}\mathrm{d}x\mathrm{d}y < B^{1/p}(p,p)B^{1/q}(q,q)\frac{pq}{(p-1)(q-1)}$$

$$\cdot \left(\int_0^\infty f^p(t)\mathrm{d}t\right)^{1/p} \left(\int_0^\infty g^q(t)\mathrm{d}t\right)^{1/q}.$$

This completes the proof of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Theorem 3. *Let* $f,g,h \geq 0$, $p,q,r > 2$ $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$,

$$F(x) = \int_0^x f(t)\mathrm{d}t, \qquad G(y) = \int_1^y g(t)\mathrm{d}t, \qquad H(z) = \int_0^z h(t)\mathrm{d}t.$$

Then

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{1/p}y^{1/q}z^{1/r}\sqrt{xyzF(x)G(y)H(z)}}{(x+y+z)^8}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$< K_pK_qK_r \left(\int_0^\infty f^{p/2}(x)\mathrm{d}x\right)^{1/p} \left(\int_0^\infty g^{q/2}(y)\mathrm{d}y\right)^{1/q} \left(\int_0^\infty h^{r/2}(z)\mathrm{d}z\right)^{1/r}.$$

where

$$K_p = \sqrt{\frac{p/2}{p/2-1}}B^{1/p}\left(\frac{p}{2},\frac{p}{2}\right)B^{1/p}(p,p).$$

Proof.

$$\int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \frac{x^{1/p}y^{1/q}z^{1/r}\sqrt{xyzF(x)G(y)H(z)}}{(x+y+z)^8}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \left(\frac{y^{(1/2-1/p)}z^{(1/q+1/r)}\sqrt{F(x)}}{(x+y+z)^2} \cdot \frac{z^{(1/2-1/q)}x^{(1/p+1/r)}\sqrt{G(y)}}{(x+y+z)^2} \right.$$

$$\left. \cdot \frac{x^{(1/2-1/r)}y^{(1/p+1/q)}\sqrt{H(z)}}{(x+y+z)^2} \right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$\leq \left(\int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \frac{y^{(p/2-1)}z^{p-1}F^{p/2}(x)}{(x+y+z)^{2p}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \right)^{1/p}$$

$$\cdot \left(\int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \frac{z^{(q/2-1)}x^{q-1}G^{q/2}(y)}{(x+y+z)^{2q}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \right)^{1/q}$$

$$\cdot \left(\int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \frac{x^{(r/2-1)}y^{r-1}H^{r/2}(z)}{(x+y+z)^{2r}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \right)^{1/r}$$

$$= A^{1/p}B^{1/q}C^{1/r}.$$

Observe that

$$A = \int_0^\infty \frac{F^{p/2}(x)}{x^{p/2}}\mathrm{d}x \int_0^\infty \frac{\left(\frac{y}{x}\right)^{p/2-1}\frac{1}{x}}{\left(1+\frac{y}{x}\right)^p}\mathrm{d}y \int_0^\infty \frac{\left(\frac{z}{x+y}\right)^{p-1}\frac{1}{x+y}}{\left(1+\frac{z}{x+y}\right)^{2p}}\mathrm{d}z$$

$$= \int_0^\infty \left(\frac{F(x)}{x}\right)^{p/2}\mathrm{d}x \int_0^\infty \frac{u^{p/2-1}}{(1+u)^p}\mathrm{d}u \int_0^\infty \frac{v^{p-1}}{(1+v)^{2p}}\mathrm{d}v$$

$$= B\left(\frac{p}{2},\frac{p}{2}\right) B(p,p) \int_0^\infty \left(\frac{F(x)}{x}\right)^{p/2}\mathrm{d}x < K_p^p \int_0^\infty f^{p/2}(x)\mathrm{d}x,$$

in view of Theorem A.

Similarly,

$$B < K_q^q \int_0^\infty g^{p/2}(y)\mathrm{d}y, \qquad C < K_r^r \int_0^\infty h^{r/2}(z)\mathrm{d}z.$$

The proof is complete. □

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