ON THE CLASS OF ALL RECIPROCAL BASES FOR INTEGERS

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Abstract. In this paper the structure of the class of all reciprocal bases of $\mathbb{N}$ is investigated from metric and topological point of view. For this purpose the method of dyadic values of infinite subsets of $\mathbb{N}$ will be applied.

A set $A \subseteq \mathbb{N}$ is called the reciprocal basis for integers (of $\mathbb{N}$), shortly R-basis, provided that for each $s \in \mathbb{N}$ there exist $a_1 < a_2 < \cdots < a_k$ from A such that $s = \sum_{j=1}^{k} \frac{1}{a_j}$ (cf. [1], [2], [8]). It is well-known that the set of all positive integers is an R-basis. It is proved in [1] that every arithmetic progression is an R-basis. In the same paper a construction of R-bases of zero density based on the fact that for every integer $a \in \mathbb{N}$ the sequence $S_a = \{a, 2a, 3a, \dots\}$ is an R-basis is given. Obviously this concept is closely related to the concept of egyptian fractions.

Note, that if $A \subseteq \mathbb{N}$ is a reciprocal basis of $\mathbb{N}$ then $\sum_{a\in A} a^{-1} = \infty$ and consequently A is infinite.

Denote by $\mathcal{B}_r$ the class of all reciprocal bases of $\mathbb{N}$ and by $\mathcal{U}$ the class of all infinite subsets of $\mathbb{N}$. Then

$$\mathcal{B}_r \subseteq \mathcal{U}. \tag{1}$$

We will use the concept of dyadic values of sets $A \in \mathcal{U}$ for the study of "the magnitude" of $\mathcal{B}_r$ in $\mathcal{U}$.

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If $A = \{a_1 < a_2 < \cdots < a_k < \cdots\} \in \mathcal{U}$, then we put $\rho(A) = \sum_{k=1}^{\infty} 2^{-a_k} = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}$, where $(\varepsilon_k)_1^{\infty}$ is the characteristic function of the set A (i.e. $\varepsilon_k = 1$ if $k \in A$ and $\varepsilon_k = 0$ if $k \in \mathbb{N} \setminus A$). In this way we get an injective mapping ρ of $\mathcal{U}$ onto the interval $(0,1]$. If $\mathcal{S} \subseteq \mathcal{U}$, then we set $\rho(\mathcal{S}) = \{\rho(A) : A \in \mathcal{S}\}$. The magnitude of the set $\rho(\mathcal{S}) \subseteq (0,1]$ enables us to judge the magnitude of the class $\mathcal{S}$ (cf. [4, p. 17]).

The magnitude of $\rho(\mathcal{S})$ can be investigated from the metric point of view (Lebesgue measure, Hausdorff dimension) and also from the topological point of view (Baire's categories).

We recall the following fact about dyadic expansions of real numbers. Each $x \in (0,1]$ can be uniquely expressed in the form $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k}$, where $\varepsilon_k(x) = 0$ or 1 and for an infinitely many k's we have $\varepsilon_k(x) = 1$.

If $m \in \mathbb{N}$ is fixed then the whole interval $(0,1]$ can be written in the form $(0,1] = \cup_{j=0}^{2^m-1} (\frac{j}{2^m}, \frac{j+1}{2^m}] = \cup_{j=0}^{2^m-1} i_m^{(j)}$. To every interval $i_m^{(j)}$ $(0 \le j \le 2^m - 1)$ corresponds a sequence $\varepsilon_1^0, \varepsilon_2^0, \ldots, \varepsilon_m^0$ of numbers $0, 1$ in such a manner that if $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k} \in i_m^{(j)}$, then $\varepsilon_k(x) = \varepsilon_k^0$ $(k = 1, 2, \ldots, m)$. We say shortly that the interval $i_m^{(j)}$ and the sequence $\varepsilon_1^0, \varepsilon_2^0, \ldots, \varepsilon_m^0$ are associated.

1. Topological properties of the set $\rho(\mathcal{B}_r)$

We will show that from the topological point of view the class $\mathcal{B}_r$ is a very large subclass of $\mathcal{U}$ (see (1)).

Let $s \in \mathbb{N}$. Denote by $H(m,s)$ the union of all intervals $i_m^{(j)}$ (m fixed) with the following property: The interval $i_m^{(j)}$ is associated with a sequence $\varepsilon_1^0, \varepsilon_2^0, \ldots, \varepsilon_m^0$ such that for a suitable set M, $M \subseteq \{k \le m : \varepsilon_k^0 = 1\}$ we have $s = \sum_{k \in M} \frac{1}{k}$.

It is a well-known fact that (every) integer s can be represented as a sum of reciprocal values of some distinct integers (cf. [1]). So for s there exists an m such that $H(m,s) \neq \emptyset$. Put $H(s) = \cup_{m=1}^{\infty} H(m,s)$. Hence $H(s) \neq \emptyset$.

The following auxiliary result will be used in what follows.

Lemma 1.1. *We have*

$$\rho(\mathcal{B}_r) = \bigcap_{s=1}^{\infty} H(s) = \bigcap_{s=1}^{\infty} \bigcup_{m=1}^{\infty} H(m,s). \tag{2}$$

Proof. 1) Let $x \in \rho(\mathcal{B}_r)$. We show that x belongs to the right-hand side of (2).

Since $x \in \rho(\mathcal{B}_r)$, we have $x = \rho(A)$, where $A = \{a_1 < a_2 < \cdots < a_k < \cdots\} \subseteq \mathbb{N}$, A being an R-basis. Hence there are numbers $a_{j_1} < a_{j_2} < \cdots < a_{j_t}$ from A such that $s = \frac{1}{a_{j_1}} + \cdots + \frac{1}{a_{j_t}}$. Put $m = a_{j_t} \in \mathbb{N}$. The sequences $\varepsilon_1^0, \varepsilon_2^0, \ldots, \varepsilon_m^0$ of 0's and 1's satisfying the conditions $\varepsilon_{a_{j_i}}^0 = 1$ $(i = 1, 2, \ldots, t)$ are associated with some intervals $i_m^{(l)}$ (m fixed) and these intervals are subsets of the set $H(m,s)$. Hence x belongs to $H(s)$. This is true for an arbitrary $s \in \mathbb{N}$, therefore x belongs to the right-hand side of (2).

2) Let $x = \sum_{j=1}^{\infty} \varepsilon_j 2^{-j}$ belong to the right-hand side of (2). Put $A = \{j : \varepsilon_j = 1\}$. Then $x = \rho(A)$. We will show that x belongs to $\rho(\mathcal{B}_r)$. For this it suffices to show that $A \in \mathcal{B}_r$.

Let $v \in \mathbb{N}$. We show that v can be expressed as a sum of reciprocal values of a finite number of distinct elements of A.

Since $x \in H(v)$, there is an $m \in \mathbb{N}$ such that $x \in H(m,v)$. By the definition of the set $H(m,v)$ there exists an interval $i_m^{(l)}$ $(l \in \{0, 1, \ldots, 2^m - 1\})$ such that $x \in i_m^{(l)}$ and $i_m^{(l)}$ is associated with a sequence $\varepsilon_1^0, \varepsilon_2^0, \ldots, \varepsilon_m^0$ of 0's and 1's such that for a set $M \subseteq \{k \le m : \varepsilon_k^0 = 1\}$ we have

$$v = \sum_{k \in M} \frac{1}{k}. \tag{3}$$

For the dyadic expansion $x = \sum_{j=1}^{\infty} \varepsilon_j 2^{-j}$ we have $\varepsilon_j = \varepsilon_j^0$ $(j = 1, 2, \ldots, m)$ and so the set M consists of some k's, $k \le m$ such that $\varepsilon_k = 1$. Hence these k's belong to the set A and the number v can be expressed by (3) as a sum of reciprocal values of some distinct elements of A. Since v is an arbitrary positive integer, we see that $A \in \mathcal{B}_r$. □

Let $\mathcal{S} \subseteq \mathcal{U}$. Denote by $c\mathcal{S}$ the class $\mathcal{U} \setminus \mathcal{S}$ (complement of $\mathcal{S}$ in $\mathcal{U}$). Hence $c\mathcal{B}_r = \mathcal{U} \setminus \mathcal{B}_r$. The class $c\mathcal{B}_r$ is the class of all infinite sets $A \subseteq \mathbb{N}$ that are not R-bases. Hence for each $A \in c\mathcal{B}_r$ there exists at least one $s \in \mathbb{N}$ such that s cannot be expressed as a finite sum of reciprocal values of distinct elements of A.

In what follows the interval $(0, 1]$ will be considered as a metric space with the Euclidean metric.

Theorem 1.1. *The set $\rho(\mathcal{B}_r)$ is an $F_{\sigma\delta}$-set in $(0, 1]$.*

Proof. We use Lemma 1.1. Recall that the set $H(m, s)$ is a union of a finite number of intervals $i_m^{(l)}$ (m fixed). Therefore $H(m, s)$ is an F_σ-set in $(0, 1]$. Then the right-hand side of (2) is an $F_{\sigma\delta}$-set in $(0, 1]$. The same holds for $\rho(\mathcal{B}_r)$. □

Remark. By the definition of $c\mathcal{B}_r$ and injectivity of the mapping $\rho : \mathcal{U} \to (0, 1]$ we get

$$\rho(c\mathcal{B}_r) = (0, 1] \setminus \rho(\mathcal{B}_r). \tag{4}$$

From this and from Theorem 1.1 follows that the set $\rho(c\mathcal{B}_r)$ is a $G_{\delta\sigma}$-set in $(0, 1]$.

We have shown that the both sets $\rho(\mathcal{B}_r)$, $\rho(c\mathcal{B}_r)$ belong to the second Borel class. We will determine their Baire's categories.

Theorem 1.2. *The set $\rho(\mathcal{B}_r)$ is a residual set in $(0, 1]$.*

Proof. It suffices to prove that the set $\rho(c\mathcal{B}_r)$ is a dense set of the first Baire category in $(0, 1]$. The density of the set $\rho(c\mathcal{B}_r)$ follows from the fact that the set $\rho(\mathcal{K})$, $\mathcal{K}$ being the class of all $A \subseteq \mathbb{N}$ with $\sum_{a \in A} a^{-1} < \infty$, is dense in $(0, 1]$ (cf. [6, Theorem 3]). We have $\mathcal{K} \subseteq c\mathcal{B}_r$ so that $\rho(\mathcal{K}) \subseteq \rho(c\mathcal{B}_r)$ and the density of $\rho(c\mathcal{B}_r)$ follows.

We prove that the set $\rho(c\mathcal{B}_r)$ is a set of the first category in $(0, 1]$.

By (4) and (2) we get

$$\rho(c\mathcal{B}_r) = (0, 1] \setminus \bigcap_{s=1}^{\infty} \bigcup_{m=1}^{\infty} H(m, s) = \bigcup_{s=1}^{\infty} \bigcap_{m=1}^{\infty} cH(m, s). \tag{5}$$

(Where $cH(m,s) = (0,1] \setminus H(m,s)$.)

In virtue of (5) it suffices to prove that each of the sets $\cap_{m=1}^{\infty} cH(m,s) (s = 1,2,\dots)$ is nowhere dense in $(0,1]$.

Fix $s \in \mathbb{N}$. On account of the well-known criterion of nowheredensity of a set in metric space (cf. [3, p. 37]) it suffices to show that the following statement holds:

Every non-empty interval $I \subset (0,1]$ contains an interval $J \subseteq I$ such that $J \cap cH(m', s) = \emptyset$ for an $m' \in \mathbb{N}$.

Let $I \subset (0,1]$ be an interval. Choose the numbers m, d, $m \in \mathbb{N}$, $0 \le d \le 2^m - 1$ in such a way that $i_m^{(d)} \subset I$. We show that there is a subinterval $i_{m+v}^{(t)}$ of $i_m^{(d)}$ such that

$$i_{m+v}^{(t)} \cap cH(m+v, s) = \emptyset. \tag{6}$$

holds.

If $i_m^{(d)} \subseteq H(m,s)$ then we put $v = 0$ and $t = d$.

Let $i_m^{(d)} \subseteq H(m,s)$ does not hold. Since the set $\{m+1, m+2, \dots, m+k, \dots\}$ is an R-basis (cf. [2]) there exist n_k $(k = 1,2,\dots,j)$, $m+1 \le n_1 < n_2 < \dots < n_j$, such that

$$s = \sum_{k=1}^{j} \frac{1}{n_k}. \tag{7}$$

Put $n_j = m + v$ (i.e. $v = n_j - m$) and $\varepsilon_k^0 = 1$ for $k = m+1, m+2, \dots, m+v$. Let the interval $i_m^{(d)}$ be associated with the sequence $\varepsilon_1^0, \varepsilon_2^0, \dots, \varepsilon_m^0$ of 0's and 1's. Construct the interval $i_{m+v}^{(t)}$ which is associated with the sequence $\varepsilon_1^0, \varepsilon_2^0, \dots, \varepsilon_m^0, \varepsilon_{m+1}^0, \dots, \varepsilon_{m+v}^0$ of numbers 0, 1. Then $i_{m+v}^{(t)} \subseteq i_m^{(d)}$ and from (7) we get (6). $\square$

2. Metric properties of the set $\rho(\mathcal{B}_r)$

We have seen that the set $\rho(\mathcal{B}_r)$ belongs to the second Borel class (Theorem 1.1). So it is Lebesgue measurable and it would be desirable to determine $\lambda(\rho(\mathcal{B}_r))$ – the Lebesgue measure of $\rho(\mathcal{B}_r)$. Unfortunately we are not able to do this and therefore it remains as an open problem. It is interesting that we can determine the Hausdorff dimension (cf. [5]) of the set $\rho(\mathcal{B}_r)$. But unfortunately from this result we cannot derive the magnitude of $\lambda(\rho(\mathcal{B}_r))$.

Theorem 2.1. *We have* $\dim \rho(\mathcal{B}_r)) = 1$.

Proof. It is well-known that there exists a set $A \in \mathcal{B}_r$ with $d(A) = 0$ (cf. [2]), where $d(A)$ denotes the asymptotic density of A, i.e. $d(A) = \lim_{n\to\infty} \frac{A(n)}{n}$, $A(n) = |A \cap \{1, 2, \dots, n\}|$.

Obviously every set $D \supseteq A$, $D \subseteq \mathbb{N}$ belongs again to $\mathcal{B}_r$. Denote by $\mathcal{S}(A)$ the set $\{D \subseteq \mathbb{N} : A \subseteq D\}$. Then we have $\mathcal{S}(A) \subseteq \mathcal{B}_r$ and so

$$\dim \rho(\mathcal{S}(A)) \le \dim \rho(\mathcal{B}_r). \tag{8}$$

In virtue of (8) it suffices to show that

$$\dim \rho(\mathcal{S}(A)) = 1. \tag{9}$$

We will prove it using the following result which is an easy consequence of [7, Theorem 2.7]:

(S) Let M be a set of positive integers and (ε_j^0), $j \in M$ be a fixed sequence of 0's and 1's. Denote by $Z = Z(M, (\varepsilon_j^0), j \in M)$ the set of all $x = \sum_{j=1}^{\infty} \varepsilon_j(x) 2^{-j} \in (0, 1]$ for which $\varepsilon_j(x) = \varepsilon_j^0$ if $j \in M$ and $\varepsilon_j(x) = 0$ or 1 if $j \in \mathbb{N} \setminus M$. Then

$$\begin{aligned} \dim(Z) &= \liminf_{n\to\infty} \frac{\log \prod\limits_{j\le n,\, j\in\mathbb{N}\setminus M} 2}{n \log 2} \\ &= \liminf_{n\to\infty} \frac{\mathbb{N} \setminus M\ (n)}{n} = 1 - \overline{d}(M), \end{aligned} \tag{10}$$

where $\overline{d}(M) = \limsup\limits_{n\to\infty} \frac{M(n)}{n}$.

Put in (S) (see (10)): $M = A$, $\varepsilon_j^0 = 1$ for $j \in A$. Then $Z(M, (\varepsilon_j^0), j \in M) = \rho(\mathcal{S}(A))$ and from (10) we obtain $\rho(\mathcal{S}(A)) = 1 - \overline{d}(A) = 1 - d(A) = 1$. Hence (9) holds. $\square$

Remark. There are infinite sets $A \subseteq \mathbb{N}$ with $\sum_{a \in A} a^{-1} = \infty$ and zero asymptotic density that do not belong to $\mathcal{B}_r$. If $A = \{a_1 < a_2 < \cdots < a_k < \cdots\} \subseteq \mathbb{N}$, $\gcd(a_i, a_j) = 1$ for $i \neq j$, then the set A does not belong to $\mathcal{B}_r$, because it is easy to show that the number 1 cannot be expressed in the form $1 = \frac{1}{a_{i_1}} + \frac{1}{a_{i_2}} + \cdots + \frac{1}{a_{i_m}}$, $i_1 < i_2 < \cdots < i_m$. Taking the set of all prime numbers for A we get a set of zero density with $\sum_{a \in A} a^{-1} = \infty$ which does not belong to $\mathcal{B}_r$.

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