Acta Math. Univ. Comenianae
Vol. LXXVII, 1(2008), pp. 123–128

PREDICTABLE REPRESENTATION PROPERTY OF SOME HILBERTIAN MARTINGALES

M. EL KADIRI

Abstract. We prove as for the real case that a martingale with values in a separabale real Hilbert space is extremal if and only if it satisfies the predictable representation property.

1. Introduction.

In this article we shall use the stochastic integral with respect to a local vectorial martingale as it is defined in [**2**].

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions, that is, $(\mathcal{F}_t)$ is complete and right continuous. Let $\mathbf{H}$ be a real separable Hilbert space whose inner product and norm are respectively denoted by $\langle , \rangle_{\mathbf{H}}$ and $|| \, ||_{\mathbf{H}}$. The dual of $\mathbf{H}$ will be denoted by $\mathbf{H}'$. For every process X with values in $\mathbf{H}$, we denote by $(\mathcal{F}_t^X)$ the complete and right continuous filtration generated by X. We also denote by E_P the expectation with respect to the probability law P.

Let M be a continuous local $(\mathcal{F}_t)$-martingale with values in $\mathbf{H}$ defined on $(\Omega, \mathcal{F}, P)$. We say that M is $(\mathcal{F}_t)$-extremal if P is an extreme point of the convex set of probabilities law on $\mathcal{F}_\infty = \sigma(\mathcal{F}_s, s \geq 0)$ for which M is a local $(\mathcal{F}_t)$-martingale.

We say that M has the predictable representation property with respect to the filtration $(\mathcal{F}_t)$ if for every real local $(\mathcal{F}_t)$-martingale N on $(\Omega, \mathcal{F}_\infty, P)$ there exists an $(\mathcal{F}_t)$-predictable process (H_t) with values in $\mathbf{H}'$ such that

$$N_t = N_0 + \int_0^t H_s \mathrm{d}M_s$$

for every $t \geq 0$. As in the real case, this is equivalent to the existence, for every $Y \in L^2(\Omega, \mathcal{F}_\infty)$, of an $(\mathcal{F}_t)$-predictable process (H_t) with values in $\mathbf{H}'$ such that

$$Y = E_P(Y) + \int_0^{+\infty} H_s \mathrm{d}M_s$$

Received November 22, 2006; revised January 24, 2008.
2000 *Mathematics Subject Classification.* Primary 60G44.
Key words and phrases. vectorial martingale; extremal martingale; predictable representation.

and $\int_0^{+\infty} ||H_s||^2 \mathrm{d}\langle M\rangle_s < +\infty$ (this can be proved in the same way as in [**3**, Propsition 3.2].

We say that M is extremal or has the predictable representation property if it has this property with respect to the filtration $(\mathcal{F}_t^M)$.

When $\mathbf{H} = \mathbf{R}$ the notions of extremal martingales and predictable representation property coincide with the same usual ones. We recall here that this case was studied by Strook and Yor in a remarkable paper [**4**].

In this paper we will prove that a local $(\mathcal{F}_t)$-martingale on $(\Omega, \mathcal{F}, P)$ with values in $\mathbf{H}$ is extremal if and only if it has the predictable representation property.

We also give some examples of extremal martingales with values in $\mathbf{H}$, that are defined by stochastic integrals of real predictable processes with respect to a cylindrical Brownian motion in $\mathbf{H}$.

In the whole paper we will fix a hilbertian basis $\{e_n : n \in \mathbf{N}\}$ of $\mathbf{H}$.

2. Preliminary results

We denote by H_0^2 the space of bounded continuous real $(\mathcal{F}_t)$-martingales vanishing at 0. Equipped with the inner product

$$\langle M, N\rangle_{H_0^2} = E_P(\langle M, N\rangle_\infty), \qquad \forall M, N \in H_0^2,$$

H_0^2 is a Hilbert space.

If M is a continuous local $(\mathcal{F}_t)$-martingale of integrable square with values in $\mathbf{H}$, we denote by $\langle M\rangle$ the increasing predictable process such that $\|M\|_{\mathbf{H}}^2 - \langle M\rangle$ is a local $(\mathcal{F}_t)$-martingale. If M and N are two local $(\mathcal{F}_t)$-martingales of integrable squares, we define the process $\langle M, N\rangle$ by standard polarisation.

If M is a bounded continuous local $(\mathcal{F}_t)$-martingale with values in $\mathbf{H}$, we denote by $\mathcal{L}_p^2(\mathbf{H}', M)$ the space of $(\mathcal{F}_t)$-predictable processes $h = (H_t)$ with values in $\mathbf{H}'$ such that $E_P(\int_0^{+\infty} \|H_s\|^2 \mathrm{d}\langle M\rangle_s) < \infty$. We denote by $H \cdot M$ the matringale $(\int_0^t H_s \mathrm{d}M_s)$.

For any stopping time T of the filtration $(\mathcal{F}_t)$ and any process $X = (X_t)$, X^T denotes the process $(X_{t\wedge T})$. We say that a stopping time T reduces a local martingale Z if Z^T is a bounded martingale.

Proposition 2.1. *Let M be a continuous local $(\mathcal{F}_t)$-martingale with values in $\mathbf{H}$, then every real continuous local $(\mathcal{F}_t^M)$-martingale $X = (X_t)$, vanishing at 0, can be uniquely written as*

$$X = H \cdot M + L$$

where H is a $(\mathcal{F}_t^M)$-predictable process with values in $\mathbf{H}'$ and L is a real local $(\mathcal{F}_t^M)$-martingale such that, for any stopping time T such that the martingales M^T and X^T are bounded, L^T is orthogonal in H_0^2 to the subspace

$$G = \{H \cdot M^T : H \in \mathcal{L}_p^2(\mathbf{H}', M^T)\}.$$

Proof. The unicity of the decomposition is easy, let us prove the existence. Since M and X are local martingales, there exists a sequence (T_n) of stopping times reducing X and M; let T be one of them. Put

$$G = \{H \cdot M^T : H \in \mathcal{L}_p^2(\mathbf{H}', M^T)\}.$$

It is then clear that G is a closed subspace of H_0^2 and that we can write $X^T = \bar{H}.M^T + \bar{L}$, where $\bar{L} \in G^\perp$. For any bounded stopping time S, we have

$$\begin{aligned} E_P(M_S^T L_S) &= E_P(M_S^T E_P(L_\infty|\mathcal{F}_s)) \\ &= E_P(M_s^T L_\infty) = 0 \end{aligned}$$

since $M^{T\wedge S} \in G$. Because of the unicity, H and L extend to processes satisfying the desired conditions. □

We will also need a vectorial version of a theorem in the measure theory due to Douglas. Let us consider the set $\mathcal{P}$ of sequences $\pi = (P_n)$, $n \geq 1$, of probability measures on $(\Omega, \mathcal{F})$. For any probability measure P on $(\Omega, \mathcal{F})$, denote by $\pi(P)$ the element of $\mathcal{P}$ defined on E by $P_n = P$ for every $n \geq 1$.

For any function f with values in $\mathbf{H}$ defined on a set E, let f_n be the functions defined by $f_n(x) = \langle f(x), e_n\rangle_{\mathbf{H}}$ for every $x \in E$.

Let $\mathcal{L}$ be a set of $\mathcal{F}$-mesurable functions with values in $\mathbf{H}$, we denote by $\mathcal{K}_{\mathcal{L}}$ the set of sequences $\pi = (P_n) \in \mathcal{P}$ such that $f_n \in L^1_{P_n}(\Omega, \mathcal{F})$ and $\int f_n dP_n = 0$ for any $f \in \mathcal{L}$ and any $n \geq 1$. It is easy to see that the set $\mathcal{K}_{\mathcal{L}}$ is convex.

The following theorem is a vectorial version of a classical theorem in the measure theory due to Douglas ([**3**, Chap. V, Theorem 4.4])

Theorem 2.2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{L}$ a set of $\mathcal{F}$-measurable functions with values in $\mathbf{H}$ and $\mathcal{L}^*$ the vector space generated by $\mathcal{L}$ and the constants in $\mathbf{H}$. Then $\mathcal{L}^*$ is dense in $L^1_P(\Omega, \mathbf{H})$ if and only if, $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$.*

Proof. The idea of the proof is the same as for the classical theorem of Douglas. Assume that $\mathcal{L}^*$ is dense in $L^1(\Omega, \mathbf{H})$, let $\pi_1 = (P_n^1)$ and let $\pi_2 = (P_n^2)$ in $\mathcal{K}_{\mathcal{L}}$ such that $\pi(P) = \alpha\pi_1 + (1-\alpha)\pi_2$, with $0 \leq \alpha \leq 1$. If $\alpha \neq 0, 1$, then π_1, π_2 and $\pi(P)$ would be identical on $\mathcal{L}^*$, and therefore on $L^1_P(\Omega, \mathbf{H})$ by density, hence $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$.

Conversely, assume that $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$. Then if $\mathcal{L}^*$ is not dense in $L^1_P(\Omega, \mathbf{H})$, there exists, following Hahn-Banach theorem, a continuous linear form φ not identically 0 on $L^1_P(\Omega, \mathbf{H})$ which vanishes on $\mathcal{L}^*$. But such a form can be identified with an element of $L^\infty(\Omega, \mathbf{H})$, hence we can find functions $g_n \in L^\infty_P(\Omega), n \in \mathbf{N}$, such that

$$\phi(f) = \sum_{n\geq 1} \int g_n f_n \mathrm{d}P$$

for any $f = \sum_n f_n e_n \in L^1(\Omega, \mathbf{H})$. We can assume that $\|g_n\|_\infty \leq 1$ for any n. Put, for any $n \geq 1$, $P_n^1 = (1-g_n)P$ and $P_n^2 = (1+g_n)P$, then $\pi_1 = (P_n^1) \in \mathcal{K}_{\mathcal{L}}$, $\pi_2 = (P_n^2) \in \mathcal{K}_{\mathcal{L}}$ and $\pi(P) = \alpha\pi_1 + (1-\alpha)\pi_2$, but $\pi_1 \neq \pi_2$, which is a contradiction with the fact that $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$. □

Remark. If $\mathbf{H} = \mathbf{R}$, then $\mathcal{K}_{\mathcal{L}}$ is simply the convex set of probability laws Q on $(\Omega, \mathcal{F})$ such that $\mathcal{L} \subset L^1_Q(\Omega)$ and $\int f \mathrm{d}Q = 0$ for any $f \in \mathcal{L}$. In this case, Theorem 2.2 is reduced to the classical Douglas Theorem.

Let $X = (X_t)$ be an integrable process with values in $\mathbf{H}$ (i.e. X_t is integrable for every t). Put

$$\mathcal{L} = \{1_A(X_t - X_s) : A \in \mathcal{F}^X_s, s \leq t\}.$$

Then X is a martingale under the law P, with values in $\mathbf{H}$, if and only if $\pi(P) \in \mathcal{L}^*$. If $\mathcal{F} = \mathcal{F}^X_\infty$, this is equivalent to $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$ or to P is an extreme point of the set of probability laws Q on $(\Omega, \mathcal{F}^X_\infty)$, X being a local martingale of $(\mathcal{F}^X)$ under the law Q.

Proposition 2.3. *Assume that $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$. Then every $\mathbf{H}$-valued local $(\mathcal{F}^X_t)$-martingale has a continuous version.*

Proof. As in the real case it suffices to prove that for any $Y \in L^1(\Omega, \mathbf{H})$, the martingale N defined by

$$N_t = E_P(Y|\mathcal{F}^X_t)$$

is continuous. It is not hard to see that this result is true if $Y \in \mathcal{L}^*$. Now, by Theorem 2.2, if $Y \in L^1(\Omega, \mathbf{H})$, one can find a sequence (Y_n) in $\mathcal{L}^*$ which converges to Y in $L^1(\Omega, \mathbf{H})$- norm. For any $\varepsilon > 0$, one has

$$P[\sup_{s \leq t} \|E_P(Y_n|\mathcal{F}^X_s) - E_P(Y|\mathcal{F}^X_s)\|_{\mathbf{H}} \geq \varepsilon] \leq \varepsilon^{-1} E_P(\|Y_n - Y\|_{\mathbf{H}}).$$

Hence, by reasoning as for the real martingales, we obtain the desired result. □

It follows easily from the above proposition that if $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$, then every local $(\mathcal{F}^X_t)$-martingale with values in a real separable Hilbert space $\mathbf{K}$ has a continuous version.

Theorem 2.4. *Let M be a continuous local $(\mathcal{F}_t)$-martingale defined on $(\Omega, \mathcal{F}, P)$ with values in $\mathbf{H}$. The following statements are equivalent:*

i) *M is extremal.*
ii) *M has the predictable representation property with respect to $(\mathcal{F}^M_t)$, and the σ-algebra $\mathcal{F}^M_0$ is P-a.s. trivial.*

Proof. Put

$$\mathcal{L} = \{1_A(M_t - M_s) : A \in \mathcal{F}^M_s, s \leq t\}.$$

Assume first that M is extremal, that is $\pi(P)$ is an extremal point of $\mathcal{K}_{\mathcal{L}}$, and let $Y \in L^\infty_P(\Omega, \mathcal{F}^M_\infty)$. Then by Proposition 2.1, there exist a predictable process $H = (H_t)$ with values in $\mathbf{H}'$ and a real continuous martingale $L = (L_t)$ such that

$$E_P(Y|\mathcal{F}^M_t) = E_P(Y) + \int_0^t H_s \mathrm{d}M_s + L_t, \qquad \forall t \geq 0,$$

with $\langle H \cdot M^T, L\rangle = 0$ for any $(\mathcal{F}^M_t)$-stopping time T and any $K \in \mathcal{L}^2_P(\mathbf{H}, M^T)$. By stopping and by virtue of the relation $\langle M, L^T\rangle = \langle M, Y\rangle^T$, we can assume that $|L|$ is bounded by a constant $k > 0$. Put $P^1_n = (1 + \frac{L_\infty}{2k})P$ and $P^2_n = (1 - \frac{L_\infty}{2k})P$. Let $\pi_1 = (P^1_n)$ and $\pi^2_n = (P^2_n)$; then we have $\pi(P) = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. Hence $\pi_1 = \pi_2 = \pi(P)$

because of the extremality of $\pi(P)$, then $L_\infty = 0$. We then deduce that $L = 0$, furthermore

$$E_P(Y|\mathcal{F}_t^M) = E_P(Y) + \int_0^t H_s \mathrm{d}M_s,$$

or

$$Y = E_P(Y) + \int_0^{+\infty} H_s \mathrm{d}M_s.$$

Conversely, if $P = \alpha P_1 + (1-\alpha)P_2$, where $\pi(P_1), \pi(P_2) \in \mathcal{K}_{\mathcal{L}}$, then the real martingale $(\frac{\mathrm{d}P_{1|\mathcal{F}_t^M}}{\mathrm{d}P})$ admits a continuous version L because P has the predictable representation property, and $L \cdot M$ is a martingale under the probability P, hence $\langle M, L\rangle = 0$. But we have $L_t = L_0 + \int_0^t H_s \mathrm{d}M_s$, henceforth $\langle X, L\rangle_t = \int_0^t H_s \mathrm{d}\langle X\rangle_s$. It then follows that P-a.s. we have $H_s = \mathrm{d}\langle M\rangle_s$-p.s., hence $\int_0^t H_s \mathrm{d}M_s = 0$, for every $t \geq 0$. Then L is constant, i.e. $L_t = L_0$ for every $t \geq 0$. Since $\mathcal{F}_0^M$ is trivial, we have $L_0 = 1$ and then $P = P_1 = P_2$. This proves that P is extremal. □

3. Examples

Proposition 3.1. *The cylindrical Brownian motion in* $\mathbf{H}$ *(defined on a probability space* $(\Omega, \mathcal{F}, P)$ *is an extremal martingale.*

Proof. Let us remark that since $\mathbf{H}$ is separable, then the σ-algebra generated by the continuous linear forms on $\mathbf{H}$ is identical to the Borel σ-algebra of $\mathbf{H}$.

Let Q be a probability measure on $\mathcal{F}_\infty^B$ for which B is a local martingale. Then for any non identically 0 continuous linear form ϕ on $\mathbf{H}$, the process $\phi(B) = (\phi(B_t))$ is a real Brownian motion under the probabilities measures P and Q; then $Q = P$ on $\mathcal{F}_\infty^{\phi(B)}$. Since ϕ is arbitrary, it follows that $Q = P$ on $\sigma(\{\phi(B_t) : t \geq 0,\ \phi \in \mathbf{H}'\} = \mathcal{F}_\infty^B$. Hence P is the unique probability measure on $\mathcal{F}_\infty^B$ for which B is a local martingale. Then the martingale B is extremal. □

Proposition 3.2. *Let* (H_t) *be a real process* $(\mathcal{F}_t^B)$*-predictable such that* P*-almost every* $\omega \in \Omega$*, the set* $\{s \geq 0 : H_s(\omega) \neq 0\}$ *is of null Lebesgue measure, and such that* $E^P(\int_0^{+\infty} H_s^2 \mathrm{d}s) < +\infty$*. Then the martingale* M *defined by* $M_t = \int_0^t H_s \mathrm{d}B_s$ *is extremal.*

Proof. By replacing if necessary the Brownian motion (B_t) by the Brownian motion $(\int_0^t \mathrm{sgn}(H_s)\mathrm{d}B_t)$, we may assume that the process (H_t) is non-negative. We have, up to a multiplicative constant,

$$\langle M\rangle_t = \int_0^t H_s^2 \mathrm{d}s \cdot I, \qquad \forall t \geq 0,$$

where I is the identity operator on $\mathbf{H}$ (see [**2**]), hence the process (H_t) is $(\mathcal{F}_t^B)$-adapted. On the other hand, we have

$$B_t = \int_0^t \frac{1}{H_s} \mathrm{d}M_s,$$

hence B is $(\mathcal{F}_t^M)$-adapted. Then $(\mathcal{F}_t^M) = (\mathcal{F}_t^B)$. If $N = (N_t)$ is a real $\mathcal{F}_t^M$-martingale, we have, following the above proposition,

$$N_t = c + \int_0^t X_s \mathrm{d}B_s, \qquad \forall t \geq 0,$$

where $X = (X_t)$ is a $(\mathcal{F}_t^B)$-predictable process with values in $\mathbf{H}$ and c is a constant, hence

$$N_t = c + \int_0^t \frac{X_s}{H_s} \mathrm{d}M_s, \qquad \forall t \geq 0.$$

Then M is extremal by Theorem 2.4. □

Let (H_t) be as in the above proposition. Then for P-almost every $\omega \in \Omega$, the mapping from $\mathbf{H}$ in $\mathbf{H}$ defined by $x \mapsto H_t(\omega)x$ is for almost every $t \geq 0$ (in the Lebesgue measure sense) an isomorphism from $\mathbf{H}$ into itself. This suggests the following problem:

Problem: Let $\mathbf{H}$ and $\mathbf{K}$ be two real separable Hilbert spaces, (B_t) a cylindrical Brownian motion in $\mathbf{H}$, and let (H_t) be a predictable process with values in $\mathcal{L}(\mathbf{H}, \mathbf{K})$ such that the stochastic integral $\int_0^t H_s \mathrm{d}B_s$ is well defined and that for any $t \geq 0$, and P-almost any $\omega \in \Omega$, H_t is for almost every $t \geq 0$ an isomorphism from $\mathbf{H}$ in $\mathbf{K}$. Is $H \cdot B$ extremal?

References

1. Dellacherie C. and Meyer P. A., *Probabilités et Potentiel*, Chap. IV and V, Hermann, Paris, 1998.

2. Metivier M., and Pellaumail J., *Stochastic Integration*, Academic Press, 1980.

3. Revuz D., and Yor M., *Continuous martingales and brownian motion*, Springer-Verlag, 1992.

4. Strook D., and Yor M., *On extremal solutions of martingales problems*, Ann. Sci. Ecole Norm. Sup., **13**(1) (1980), 95–164.

M. El Kadiri, B.P. 726, Salé-Tabriquet, Salé, Morroco, *e-mail*: `elkadiri@fsr.ac.ma`