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SOME GRÜSS TYPE INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL AND APPLICATIONS

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Abstract. In this paper a Grüss type inequalities for Riemann-Stieltjes integral are proved. Applications to the approximation problem of the Riemann-Stieltjes are also pointed out.

1. Introduction

In 1935 G. Grüss proved the following famous inequality regarding the integral of the product of two functions and the product of the integrals

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x-\left(\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right)\cdot\left(\frac{1}{b-a}\int_a^b g(x)\mathrm{d}x\right)\right|\\&\le\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma)\end{aligned}\tag{1.1}$$

provided that f and g are two integrable functions on $[a,b]$ satisfying the condition $\phi\le f(x)\le\Phi$ and $\gamma\le g(x)\le\Gamma$ for all $x\in[a,b]$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller one.

In [**15**] Dragomir and Fedotov have established the following functional

$$\mathcal{D}(f;u):=\int_a^b f(x)\mathrm{d}u(x)-\frac{u\,(b)-u(a)}{b-a}\int_a^b f(t)\mathrm{d}t\tag{1.2}$$

provided that the Stieltjes integral $\int_a^b f(x)\mathrm{d}u(x)$ and the Riemann integral $\int_a^b f(t)\mathrm{d}t$ exist.

In the same paper [**15**] the authors proved the following inequality.

Theorem 1. *Let $f,u\colon[a,b]\to\mathbb{R}$ be such that u is of bounded variation on $[a,b]$ and f is Lipschitzian with the constant $K>0$. Then we have*

$$|\mathcal{D}(f;u)|\le\frac{1}{2}K(b-a)\bigvee_a^b(u)\tag{1.3}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Also, in [**7**], Dragomir obtained the following inequality

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Theorem 2. *Let $f, u\colon [a,b] \to \mathbb{R}$ be such that u is Lipschitzian on $[a,b]$, i.e.,*

$$|u(y) - u(x)| \leq L|x - y| \qquad \forall x, y \in [a,b], \quad (L > 0)$$

and f is Riemann integrable on $[a,b]$.

If $m, M \in \mathbb{R}$ are such that $m \leq f(x) \leq M$ for any $x \in [a,b]$, then the inequality

$$|\mathcal{D}(f;u)| \leq \frac{1}{2} L(M - m)(b - a) \tag{1.4}$$

holds. The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

For other recent inequalities for the Riemann-Stieltjes integral see [**1**]–[**16**] and the references therein.

The aim of this paper is to obtain several new bounds for $\mathcal{D}(f;u)$. More specifically, the integrand f is assumed to be monotonic nondecreasing on both $[a,x]$ and $[x,b]$, and the integrator u is to be of bounded variation, Lipschitzian and monotonic on $[a,b]$.

2. The case of bounded variation integrators

Theorem 3. *Let $x \in [a,b]$. Let $u\colon [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. Assume that f is monotonic nondecreasing on both $[a,x]$ and $[x,b]$. Then we have the inequality*

$$|\mathcal{D}(f;u)| \leq [f(b) - f(a)] \cdot \bigvee_a^b (u). \tag{2.1}$$

Proof. It is well-known that for a continuous function $p\colon [a,b] \to \mathbb{R}$ and a function $\nu\colon [a,b] \to \mathbb{R}$ of bounded variation, one has the inequality

$$\left| \int_a^b p(t) \mathrm{d}\nu(t) \right| \leq \sup_{t \in [a,b]} |p(t)| \bigvee_a^b (\nu).$$

Therefore, as u is of bounded variation on $[a,b]$, we have

$$\begin{aligned} |\mathcal{D}(f;u)| &= \left| \int_a^b \left[f(x) - \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t \right] \mathrm{d}u(x) \right| \\ &\leq \sup_{x \in [a,b]} \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t \right| \cdot \bigvee_a^b (u) \\ &= \frac{1}{b-a} \sup_{x \in [a,b]} \left| \int_a^b [f(x) - f(t)] \, \mathrm{d}t \right| \cdot \bigvee_a^b (u) \\ &= \frac{1}{b-a} \sup_{x \in [a,b]} \int_a^b |f(x) - f(t)| \, \mathrm{d}t \cdot \bigvee_a^b (u). \end{aligned} \tag{2.2}$$

As f is monotonic nondecreasing on $[a,x]$ and monotonic nondecreasing on $[x,b]$, we get

$$\begin{aligned}\int_a^b |f(x)-f(t)|\,\mathrm{d}t &\le \int_a^x |f(x)-f(t)|\,\mathrm{d}t + \int_x^b |f(x)-f(t)|\,\mathrm{d}t \\ &= (x-a)f(x) - \int_a^x f(t)\mathrm{d}t + \int_x^b f(t)\mathrm{d}t - (b-x)f(x) \\ &= (2x-a-b)f(x) + \int_x^b f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t.\end{aligned}$$

Utilizing the monotonicity property of f on both intervals, we have

$$\int_x^b f(t)\mathrm{d}t \le (b-x)\, f(b) \qquad \text{and} \qquad \int_a^x f(t)\mathrm{d}t \ge (x-a)f(a)$$

which imply that

$$\int_a^b |f(x)-f(t)|\,\mathrm{d}t \le (2x-a-b)f(x) + (b-x))f(b) - (x-a)f(a).$$

Taking 'sup' for both sides, we get

$$\begin{aligned}\sup_{x\in[a,b]} \int_a^b |f(x)-f(t)|\mathrm{d}t &\le \sup_{x\in[a,b]} \{(2x-a-b)f(x) + (b-x)f(b) - (x-a)f(a)\} \\ &= (b-a)\,[f(b)-f(a)]\,. \qquad (2.3)\end{aligned}$$

Combining (2.2) and (2.3), we get

$$\begin{aligned}|\mathcal{D}(f;u)| &\le \frac{1}{b-a} \sup_{x\in[a,b]} \int_a^b |f(x)-f(t)|\,\mathrm{d}t \cdot \bigvee_a^b (u) \\ &\le [f(b)-f(a)] \cdot \bigvee_a^b (u),\end{aligned}$$

and the theorem is proved. □

Corollary 1. *Let f be as in Theorem 3. Let $u \in C^{(1)}[a,b]$. Then we have the inequality*

$$|\mathcal{D}(f;u)| \le [f(b)-f(a)] \cdot \|u'\|_{1,[a,b]} \qquad (2.4)$$

where $\|\cdot\|_1$ is the L_1 norm, namely $\|u'\|_{1,[a,b]} := \int_a^b |u'(t)|\,\mathrm{d}t$.

Corollary 2. *Let f be as in Theorem 3. Let $u\colon [a,b] \to \mathbb{R}$ be a Lipschitzian mapping with the constant $L > 0$. Then we have the inequality*

$$|\mathcal{D}(f;u)| \le L(b-a)[f(b)-f(a)]. \qquad (2.5)$$

Corollary 3. *Let f be as in Theorem 3. Let $u\colon [a,b] \to \mathbb{R}$ be a monotonic mapping. Then we have the inequality*

$$|\mathcal{D}(f;u)| \le [f(b)-f(a)] \cdot |u\,(b) - u(a)|\,. \qquad (2.6)$$

3. The case of Lipschitzian integrators

Theorem 4. *Let $x \in [a,b]$. Let $u, f\colon [a,b] \to \mathbb{R}$ be such that u is L-Lipschitzian on $[a,b]$ and f is monotonic nondecreasing on both $[a,x]$ and $[x,b]$. Then we have the inequality*

$$|\mathcal{D}(f;u)| \le L\left[\frac{1}{2}(b-a)(f(b)-f(a)) + \int_a^b f(x)\mathrm{d}x\right]. \tag{3.1}$$

Proof. It is well-known that for a Riemann integrable function $p\colon [a,b] \to \mathbb{R}$ and L-Lipschitzian function $\nu\colon [a,b] \to \mathbb{R}$, one has the inequality

$$\left|\int_a^b p(t)d\nu(t)\right| \le L\int_a^b |p(t)|\,\mathrm{d}t.$$

Therefore, as u is L-Lipschitzian on $[a,b]$, we have

$$\begin{aligned} |\mathcal{D}(f;u)| &= \left|\int_a^b \left[f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right]\mathrm{d}u(x)\right| \\ &\le L\int_a^b \left|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|\mathrm{d}x = \frac{L}{b-a}\int_a^b \left|\int_a^b [f(x)-f(t)]\,\mathrm{d}t\right|\mathrm{d}x. \end{aligned} \tag{3.2}$$

As f is monotonic nondecreasing on $[a,x]$ and monotonic nondecreasing on $[x,b]$, we get

$$\begin{aligned} \left|\int_a^b [f(x)-f(t)]\,\mathrm{d}t\right| &\le \int_a^b |f(x)-f(t)|\,\mathrm{d}t \\ &= \int_a^x |f(x)-f(t)|\,\mathrm{d}t + \int_x^b |f(x)-f(t)|\,\mathrm{d}t \\ &= (x-a)f(x) - \int_a^x f(t)\mathrm{d}t + \int_x^b f(t)\mathrm{d}t - (b-x)\,f(x) \\ &= (2x-a-b)\,f(x) + \int_x^b f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t. \end{aligned}$$

Utilizing the monotonicity property of f on both intervals, we have

$$\int_x^b f(t)\mathrm{d}t \le (b-x)\,f(b) \qquad \text{and} \qquad \int_a^x f(t)\mathrm{d}t \ge (x-a)f(a),$$

which imply that

$$\int_a^b |f(x)-f(t)|\,\mathrm{d}t \le (2x-a-b)\,f(x) + (b-x)\,f(b) - (x-a)f(a). \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\begin{aligned}
|\mathcal{D}(f;u)| &\le \frac{L}{b-a}\int_a^b \left|\int_a^b [f(x)-f(t)]\,\mathrm{d}t\right| \mathrm{d}x \\
&\le \frac{L}{b-a}\int_a^b [(2x-a-b)\,f(x)+(b-x)\,f(b)-(x-a)f(a)]\,\mathrm{d}x \\
&= \frac{1}{2}L(b-a)\,[f(b)-f(a)] + \frac{L}{(b-a)}\int_a^b (2x-a-b)\,f(x)\mathrm{d}x \\
&\le \frac{1}{2}L(b-a)\,[f(b)-f(a)] + \frac{L}{(b-a)}\cdot \max_{x\in[a,b]}\{2x-a-b\}\cdot \int_a^b f(x)\mathrm{d}x \\
&= L\left[\frac{1}{2}(b-a)\,(f(b)-f(a)) + \int_a^b f(x)\mathrm{d}x\right]
\end{aligned}$$

and the theorem is proved. □

4. The case of monotonic integrators

Theorem 5. *Let $x \in [a,b]$. Let $u, f\colon [a,b]\to\mathbb{R}$ be a continuous mappings on $[a,b]$. Assume that u is monotonic nondecreasing mapping on $[a,b]$ and $f\colon [a,b]\to\mathbb{R}$ is monotonic nondecreasing on both intervals $[a,x]$ and $[x,b]$. Then we have the inequality*

$$|\mathcal{D}(f;u)| \le 2u(b)\cdot\left[f(b)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]. \tag{4.1}$$

Proof. It is well-known that for a monotonic non-decreasing function $\nu\colon[a,b]\to\mathbb{R}$ and continuous function $p\colon [a,b]\to\mathbb{R}$, one has the inequality

$$\left|\int_a^b p(t)d\nu(t)\right| \le \int_a^b |p(t)|\,d\nu(t).$$

Therefore, as u is monotonic non-decreasing on $[a,b]$, we have

$$\begin{aligned}
|\mathcal{D}(f;u)| &= \left|\int_a^b \left[f(x)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right]\mathrm{d}u(x)\right| \le \int_a^b \left|f(x)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|\mathrm{d}u(x) \\
&= \frac{1}{b-a}\int_a^b \left|\int_a^b [f(x)-f(t)]\,\mathrm{d}t\right| \mathrm{d}u(x). \qquad (4.2)
\end{aligned}$$

As f is monotonic nondecreasing on $[a,x]$ and monotonic nondecreasing on $[x,b]$, we get

$$\begin{aligned}\left|\int_a^b [f(x)-f(t)]\,\mathrm{d}t\right| &\le \int_a^b |f(x)-f(t)|\,\mathrm{d}t\\ &\le \int_a^x |f(x)-f(t)|\,\mathrm{d}t+\int_x^b |f(x)-f(t)|\,\mathrm{d}t\\ &= (x-a)f(x)-\int_a^x f(t)\mathrm{d}t+\int_x^b f(t)\mathrm{d}t-(b-x)\,f(x)\\ &= (2x-a-b)\,f(x)+\int_x^b f(t)\mathrm{d}t-\int_a^x f(t)\mathrm{d}t.\end{aligned}$$

Utilizing the monotonicity property of f on both intervals, we have

$$\int_x^b f(t)\mathrm{d}t \le (b-x)\,f(b) \qquad \text{and} \qquad \int_a^x f(t)\mathrm{d}t \ge (x-a)f(a)$$

which imply that

$$\int_a^b |f(x)-f(t)|\,\mathrm{d}t \le (2x-a-b)\,f(x)+(b-x)\,f(b)-(x-a)f(a). \tag{4.3}$$

Using (4.2) and (4.3), we get

$$\begin{aligned}&|\mathcal{D}(f;u)|\\ &\le \frac{1}{b-a}\int_a^b [(2x-a-b)\,f(x)+(b-x)\,f(b)-(x-a)f(a)]\,\mathrm{d}u(x)\end{aligned} \tag{4.4}$$

Now, using Riemann-Stieltjes integral, we have

$$\begin{aligned}\int_a^b (2x-a-b)\,f(x)\mathrm{d}u(x) &= (b-a)\,[f(b)u(b)+f(a)u(a)]\\ &\quad -2\int_a^b u(x)f(x)\mathrm{d}x-\int_a^b (2x-a-b)\,u(x)\mathrm{d}f(x)\\ \int_a^b (b-x)\,f(b)\mathrm{d}u(x) &= f(b)\int_a^b (b-x)\,\mathrm{d}u(x)\\ &= -(b-a)u(a)f(b)+f(b)\int_a^b u(x)\mathrm{d}x\end{aligned}$$

and

$$\int_a^b (x-a)f(a)\mathrm{d}u(x) = f(a)\int_a^b (x-a)\mathrm{d}u(x) = (b-a)u(b)f(a)-f(a)\int_a^b u(x)\mathrm{d}x.$$

Therefore, by (4.4), we get

$$\begin{aligned}
&\int_a^b [(2x-a-b)\,f(x)+(b-x)\,f(b)-(x-a)f(a)]\,\mathrm{d}u(x)\\
&=(b-a)\,[f(b)u(b)+f(a)u(a)]-2\int_a^b u(x)f(x)\mathrm{d}x\\
&\quad-\int_a^b (2x-a-b)\,u(x)\mathrm{d}f(x)-(b-a)u(a)f(b)+f(b)\int_a^b u(x)\mathrm{d}x\\
&\quad-(b-a)u(b)f(a)+f(a)\int_a^b u(x)\mathrm{d}x\\
&=(b-a)\,(f(b)-f(a))\,(u(b)-u(a))+(f(a)+f(b))\int_a^b u(x)\mathrm{d}x\\
&\quad-2\int_a^b u(x)f(x)\mathrm{d}x-\int_a^b (2x-a-b)\,u(x)\mathrm{d}f(x).
\end{aligned} \tag{4.5}$$

Now, by the monotonicity property of u, we have $\int_a^b u(x)\mathrm{d}x \le (b-a)\,u(b)$,

$$\int_a^b u(x)f(x)\mathrm{d}x \ge u\,(a)\int_a^b f(x)\mathrm{d}x$$

and

$$\int_a^b (2x-a-b)\,u(x)\mathrm{d}f(x) \ge (a-b)\,u(a)\int_a^b \mathrm{d}f(x)=(a-b)\,u(a)\cdot(f(b)-f(a))$$

which by (4.5) give

$$\begin{aligned}
&\int_a^b [(2x-a-b)\,f(x)+(b-x)\,f(b)-(x-a)f(a)]\,\mathrm{d}u(x)\\
&=(b-a)\,(f(b)-f(a))\,(u(b)-u(a))+(f(a)+f(b))\int_a^b u(x)\mathrm{d}x\\
&\quad-2\int_a^b u(x)f(x)\mathrm{d}x-\int_a^b (2x-a-b)\,u(x)\mathrm{d}f(x)\\
&\le (b-a)\,(f(b)-f(a))\,(u(b)-u(a))+(b-a)\,(f(a)+f(b))\,u(b)\\
&\quad-2u\,(a)\int_a^b f(x)\mathrm{d}x-(a-b)\,u(a)\cdot(f(b)-f(a))\\
&=(b-a)\,[(f(b)-f(a))\,u(b)+(f(a)+f(b))\,u(b)]-2u\,(a)\int_a^b f(x)\mathrm{d}x\\
&=2(b-a)f(b)u(b)-2u\,(a)\int_a^b f(x)\mathrm{d}x.
\end{aligned}$$

Therefore, by (4.4) we get

$$\begin{aligned}|\mathcal{D}(f;u)| &\le \frac{1}{b-a}\int_a^b \left[(2x-a-b)\,f(x)+(b-x)\,f(b)-(x-a)f(a)\right]\mathrm{d}u(x)\\ &\le 2f(b)u(b)-2\frac{u\,(a)}{b-a}\int_a^b f(x)\mathrm{d}x.\end{aligned}$$

Now, using the properties of 'max' function and the monotonicity of u, we get

$$\begin{aligned}|\mathcal{D}(f;u)| &\le 2f(b)u(b)-2\frac{u\,(a)}{b-a}\int_a^b f(x)\mathrm{d}x\\ &\le 2\cdot\max\{u(a),u(b)\}\cdot\left[f(b)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]\\ &= 2u(b)\cdot\left[f(b)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]\end{aligned}$$

which proves the inequality (4.1). □

5. A Numerical quadrature formula for the Riemann-Stieltjes integral

In this section, an approximation for the Riemann-Stieltjes integral $\int_a^b f(x)\mathrm{d}u(x)$, is given in terms of the Riemann integral $\int_a^b f(t)\mathrm{d}t$.

Theorem 6. *Let f,u be as in Theorem 3 and consider*

$$I_h := \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$$

a partition of $[a,b]$. Denote $h_i = x_{i+1}-x_i$, $i = 1,2,\cdots n-1$. Then we have

$$\int_a^b f(x)\mathrm{d}u(x) = A_n\,(f,u,I_h) + R_n\,(f,u,I_h)\,, \tag{5.1}$$

where

$$A_n\,(f,u,I_h) = \sum_{i=0}^{n-1}\frac{u\,(x_{i+1})-u\,(x_i)}{h_i}\times\int_{x_i}^{x_{i+1}} f(t)\mathrm{d}t \tag{5.2}$$

and the Remainder $R_n\,(f,u,I_h)$ satisfies the estimation

$$|R_n\,(f,u,I_h)| \le [f(b)-f(a)]\cdot\bigvee_a^b(u). \tag{5.3}$$

Proof. Applying Theorem 3 on the intervals $[x_i,x_{i+1}]$, $i = 1,2,\cdots n-1$, we get

$$\left|\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}u(x) - \frac{u\,(x_{i+1})-u\,(x_i)}{h_i}\int_{x_i}^{x_{i+1}} f(t)\mathrm{d}t\right| \le [f\,(x_{i+1})-f\,(x_i)]\bigvee_{x_i}^{x_{i+1}}(u).$$

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce that

$$\begin{aligned}\left|\int_a^b f(x)\mathrm{d}u(x) - A_n(f,u,I_h)\right| &\le \sum_{i=0}^{n-1}\left[f(x_{i+1}) - f(x_i)\right]\bigvee_{x_i}^{x_{i+1}}(u) \\ &\le \max_{i=\overline{0,n-1}}\{f(x_{i+1}) - f(x_i)\}\cdot\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(u) \\ &= \left[f(b) - f(a)\right]\cdot\bigvee_a^b(u),\end{aligned}$$

and the theorem is proved. □

Theorem 7. *Let f,u be as in Theorem 4. Let I_h be as above. Then we have*

$$\int_a^b f(x)\mathrm{d}u(x) = A_n(f,u,I_h) + R_n(f,u,I_h), \tag{5.4}$$

where $A_n(f,u,I_h)$ is defined in (5.2) *and the Remainder $R_n(f,u,I_h)$ satisfies the estimation*

$$|R_n(f,u,I_h)| \le L\left[\frac{1}{2}\nu(h)(f(b)-f(a)) + \int_a^b f(x)\mathrm{d}x\right], \tag{5.5}$$

where $\nu(h) = \max_{i=\overline{0,n-1}}\{h_i\}$.

Proof. Applying Theorem 4 on the intervals $[x_i, x_{i+1}]$, $i = 1,2,\cdots n-1$, we get

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}u(x) - \frac{u(x_{i+1}) - u(x_i)}{h_i}\int_{x_i}^{x_{i+1}} f(t)\mathrm{d}t\right| \\ &\qquad\le L\left[\frac{h_i}{2}(f(x_{i+1}) - f(x_i)) + \int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x\right].\end{aligned}$$

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce that

$$\begin{aligned}&\left|\int_a^b f(x)\mathrm{d}u(x) - A_n(f,u,I_h)\right| \\ &\qquad\le L\sum_{i=0}^{n-1}\left[\frac{h_i}{2}(f(x_{i+1}) - f(x_i)) + \int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x\right] \\ &\qquad\le L\left[\frac{1}{2}\max_{i=\overline{0,n-1}}\{h_i\}\cdot\sum_{i=0}^{n-1}(f(x_{i+1}) - f(x_i)) + \int_a^b f(x)\mathrm{d}x\right] \\ &\qquad\le L\left[\frac{1}{2}\nu(h)(f(b)-f(a)) + \int_a^b f(x)\mathrm{d}x\right],\end{aligned}$$

and the theorem is proved. □

Remark 1. Similarly, one may apply Theorem 5 to approximate $\int_a^b f(x)\mathrm{d}u(x)$ in terms of $\int_a^b f(t)\mathrm{d}t$. We shall omit the details to the interested reader.

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