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ON DUAL OF BANACH SEQUENCE SPACES

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Dedicated to the memory of professor Parviz Azimi

ABSTRACT. J. Hagler and P. Azimi have introduced a class of Banach sequence spaces, the $X_{\alpha,1}$ spaces as a class of hereditarily ℓ_1 Banach spaces. In this paper, we show that (i) $X^*_{\alpha,1}$, the dual of Banach space $X_{\alpha,1}$ contains asymptotically isometric copies of ℓ_∞, (ii) $X^*_{\alpha,1}$ is nonseparable although $X_{\alpha,1}$ is a separable Banach space. Also, we show $X_{\alpha,1}$ is not hereditarily indecomposable.

1. INTRODUCTION

The concept of a Banach space containing an asymptotically isometric copy of ℓ^1 was introduced by Dowling and Lennard who initially used the proof where non-reflexive subspaces of $L^1[0,1]$ fail the fixed point property [**7**]. Later, in [**8**], the concept of a Banach space containing an asymptotically isometric copy of c_0 was introduced and it was proved that Banach spaces containing an asymptotically isometric copy of c_0 fail the fixed point property. In [**9**], the notion of a Banach space containing an asymptotically isometric copy of ℓ^∞ was introduced and an asymptotically isometric version of the classical Bessaga-Pelczynski Theorem was proved, namely the statement that a dual Banach space X^* contains an asymptotically isometric copy of c_0 if and only if X^* contains an asymptotically isometric copy of ℓ_∞. Then, Dowling in [**6**] extended this result and proved that Banach space X^* contains an asymptotically isometric copy of c_0 if and only if X^* contains an isometric copy of ℓ_∞.

J. Hagler and P. Azimi [**3**] introduced a class of dual Banach sequence spaces, the $X_{\alpha,p}$ spaces as a class of hereditarily ℓ_p Banach spaces. For $p = 1$, each of the spaces is hereditarily complementably ℓ_1 and yet fails the Schur property and for $1 < p < \infty$ is hereditarily ℓ_p [**1**]. In [**4**], Azimi and first named author showed that for $1 \le p < \infty$, the Banach spaces $X_{\alpha,p}$ contain asymptotically isometric copy of ℓ_p. Here, using two methods we show that the Banach spaces $X^*_{\alpha,1}$, the dual of Banach spaces $X_{\alpha,1}$, are nonseparable. By the first method, we show $X^*_{\alpha,1}$ contain asymptotically isometric copy of ℓ_∞. A result of [**6**] shows that $X^*_{\alpha,1}$ contain isometric copy of ℓ_∞, and then they are nonseparable. By the second method,

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we give a direct proof to show that $X_{\alpha,1}^*$ are nonseparable. Finally, we show that $X_{\alpha,1}$ contain an unconditional basic sequence and by a result of [**10**], these spaces are not hereditarily indecomposable.

Now we go through the construction of the $X_{\alpha,p}$ spaces.

A block F is an interval (finite or infinite) of integers. For any block F and a finitely non-zero sequence of scalars $x = (t_1, t_2, \dots)$, we let $\langle x, F\rangle = \sum_{j\in F} t_j$. A sequence of blocks $F_1, F_2, \dots$ is admissible if $\max\ F_i < \min\ F_{i+1}$ for each i. Finally, let $1 = \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \dots$ be a sequence of real numbers with $\lim_{i\to\infty} \alpha_i = 0$ and $\sum_{i=1}^{\infty} \alpha_i = \infty$.

We now define a norm which uses the α_i's and an admissible sequence of blocks in its definition. Let $1 \leq p < \infty$ and $x = (t_1, t_2, t_3, \dots)$ be a finitely non-zero sequence of reals. Define

$$\|x\| = \max \left[\sum_{i=1}^{n} \alpha_i |\langle x, F_i\rangle|^p\right]^{1/p},$$

where the max is taken over all n and admissible sequences $F_1, F_2, \dots$. The Banach space $X_{\alpha,p}$ is the completion of the finitely non-zero sequences of scalars in this norm.

Let us recall the main properties of $X_{\alpha,1}$ spaces [**3**].

Theorem 1.1.

(1) *$X_{\alpha,1}$ is hereditarily l_1.*

(2) *The sequence (e_i) is a normalized boundedly complete basis for $X_{\alpha,1}$. Thus, $X_{\alpha,1}$ is a dual space.*

(3) (i) *The sequence (e_i) is a weak Cauchy sequence in $X_{\alpha,1}$ with no weak limit in $X_{\alpha,1}$. In particular, $X_{\alpha,1}$ fails the Schur property,*

(ii) *There is a subspace X_0 of $X_{\alpha,1}$ which fails the Schur property, yet it is weakly sequentially complete.*

(4) *Let $B_1(X_{\alpha,1})$ denote the first Baire class of $X_{\alpha,1}$ in its second dual, i.e., $B_1(X_{\alpha,1}) = \{x^{**} \in X_{\alpha,1}^{**} : x^{**}$ is a weak* limit of a sequence (x_n) in $X_{\alpha,1}\}$ Then $\dim B_1(X_{\alpha,1})/X_{\alpha,1} = 1$.*

Also, a result in [**2**] shows the following.

Theorem 1.2.

(1) *The Banach space $X_{\alpha,1}$ contains asymptotically isometric copies of ℓ_1.*

(2) *The predual of $X_{\alpha,1}$ contains asymptotically isometric copies of c_0.*

(3) *Any $X_{\alpha,1}$ fails the Dunford-Pettis property.*

2. The Results

Here, using two methods, we show $X_{\alpha,1}^*$, the dual of Banach space $X_{\alpha,1}$, is a nonseparable Banach space. By the first method we show that $X_{\alpha,1}^*$ contains an isometric copy of ℓ_∞ and by the second method, a direct one, we show that $X_{\alpha,1}^*$ contains no countable dense subset. Using the first method, we show that the Banach space $X_{\alpha,1}^*$ contains asymptotically isometric copies of ℓ^∞. A result of

[**6**] shows this that Banach space contains isometric copies of ℓ^∞, and then is non separable.

Definition 2.1. A Banach space X is said to contain asymptotically isometric copies of ℓ^∞ if there is a null sequence $(\varepsilon_n)_n$ in $(0,1]$ and a bounded linear operator $T\colon \ell^\infty \to X$ such

$$\sup_{n\in\mathbb{N}}(1-\varepsilon_n)|t_n| \le \|T((t_n)_n)\| \le \sup_{n\in\mathbb{N}}|t_n|$$

for all sequence $(t_n)_n \in \ell^\infty$.

Theorem 2.2. *$X^*_{\alpha,1}$ contains asymptotically isometric copies of ℓ^∞.*

Proof. Let (ε_i) be a decreasing sequence in (0,1] and V be an infinite dimensional subspace of $X_{\alpha,1}$. The proof of [**3**, Lemma 5] together with a trivial modification ($1-\varepsilon_i$, $i=1,2,\dots$ instead of 1/2) shows that we may assume the following.

There exist two sequences – (v_i) in V and (n_i) of integers such that

(1) For integers n_i ($> n_{i-1}$) put $N_i = n_1 + n_2 + \ldots + n_{i-1}$, $i > 1$ and $N_0 = 0$.
(2) For each i, there is a sequence of admissible blocks $F^i_1, F^i_2, \dots, F^i_{n_i}$ with
  (a) $\max F^i_{n_i} < \min F^{i+1}_1$ for each i;
  (b) $\sum_{j=1}^{n_i} \alpha_j \mid \langle v_i, F^i_j\rangle \mid = \|v_i\| = 1$
  (c) $\langle v_k, F^i_j\rangle = 0$ for $k \ne i$.
  (d) $\sum_{j=1}^{n_i} \alpha_{j+N_i} \mid \langle v_i, F^i_j\rangle| > 1-\varepsilon_i$

and

$$\left\|\sum_{j=1}^{n} t_j v_j\right\| \ge \sum_{j=1}^{n} (1-\varepsilon_j) \mid t_j \mid .$$

Let $\varphi_i \in X^*_{a,1}$ be defined by

$$\varphi_i(x) = \sum_{j=1}^{n_i} \varepsilon^i_j \alpha_{j+N_i} \langle x, F^i_j\rangle.$$

where $\varepsilon^i_j = \operatorname{sgn}\langle v_i, F^i_j\rangle$,

Properties (1)–(4) imply that $\phi_i(v_i) > 1-\varepsilon_i$ and $\phi_i(v_j) = 0$ for $i \ne j$. Let scalars $t_1, \dots, t_n$ be given. Since $\|v_i\| = 1$ for all i, we see that

$$\left|\sum_{i=1}^{n} t_i \phi_i(v_j)\right| \ge (1-\varepsilon_j)\,|t_j|\,.$$

This implies that

$$\left\|\sum_{i=1}^{n} t_i \phi_i\right\| \ge \max_j (1-\varepsilon_j)|t_j|. \tag{1}$$

Now by definition of ϕ_i for $x \in X$, $\sum_i |\phi_i(x)| \le \|x\|$. This implies that

$$\left\|\sum_{i=1}^{n} t_i \phi_i\right\| \le \max_i |t_i|.$$

Define $T\colon \ell^\infty \to X^*_{\alpha,1}$ by

$$T((t_n)_n)(x) = \sum_{n=1}^{\infty} t_n \varphi_n(x)$$

for all $(t_n) \in \ell^\infty$ and all $x \in X_{\alpha,1}$. For each $(t_n) \in \ell^\infty$ and each $x \in X_{\alpha,1}$, we have

$$\begin{aligned}\|T(t_n)_n)(x)\| &= \|\sum_{n=1}^{\infty} t_n \varphi_n(x)\| \\ &\leq \sup_n |t_n| \sum_{n=1}^{\infty} |\varphi_n(x)| \\ &\leq \sup_n |t_n| \|x\|.\end{aligned}$$

Thus

$$\|T(t_n)_n)\| \leq \sup_n |t_n|.$$

On the other hand, by using (1),

$$\begin{aligned}\|T(t_n)_n)\| &= \sup\{|T(t_n)_n)(x)| : \|x\| \leq 1\} \\ &= \sup\left\{\left|\sum t_n \varphi_n(x)\right| : \|x\| \leq 1\right\} \\ &= \left\|\sum_{i=1}^{\infty} t_i \phi_i\right\| \\ &\geq \sup_i (1-\varepsilon_i)|t_i|.\end{aligned}$$

This implies that $X^*_{\alpha,1}$ contains asymptotically isometric copies of ℓ^∞. □

In [**6**], dowling showed that if a Banach space contains asymptotically isometric copies of ℓ^∞, it contains isometric copies of ℓ^∞. Then, the previous theorem shows the following

Theorem 2.3. *$X^*_{\alpha,1}$ contains isometric copies of ℓ^∞ and is nonseparable.*

A theorem of [**6**] with together the previous theorem imply that

Proposition 2.4.

1. *$X^*_{\alpha,1}$ contains an asymptotically isometric copy of c_0.*
2. *$X^*_{\alpha,1}$ contains an isometric copy of c_0.*
3. *ℓ^1 is isometric to a quotient of $X_{\alpha,1}$.*
4. *There is a sequence (x_n) in the unit ball of $X_{\alpha,1}$ and a bounded linear operator $S\colon X_{\alpha,1} \to \ell^1$ with $\|S\| \leq 1$ and $\lim \|Sx_n - e_n\| = 0$, where (e_n) is the standard unit vector basis of ℓ^1.*

Here, by a direct method, we show that $X^*_{\alpha,1}$ is nonseparable.

Theorem 2.5. *$X^*_{\alpha,1}$, the dual of the space $X_{\alpha,1}$, is nonseparable.*

Proof. Let $\{F_i\}$ be a sequence of blocks of integers such that $\max F_i < \min F_{i+1}$ and $F = (F_1, F_2, \ldots)$. Now, we define the linear functional

$$f_F(x) = \sum_{i=1}^{\infty} \langle x, F_i \rangle$$

on $X_{\alpha,1}$.

Let F_ϕ be a finite block of integers and x_ϕ be a corresponding unit vector in $X_{\alpha,1}$ such that

$$1 = \|x_\phi\| = \langle x_\phi, F_\phi \rangle.$$

Now, we select blocks F_0 and F_1 disjoint from each other and from F_ϕ such that $\max F_\phi < \min F_0$ and $\max F_\phi < \min F_1$. Moreover, we can select x_0 and x_1 in $X_{\alpha,1}$ such that

$$1 = \|x_0\| = \langle x_0, F_0 \rangle, \qquad 1 = \|x_1\| = \langle x_1, F_1 \rangle.$$

Next, we can find the sets F_{00} and F_{01} disjoint from each other and from F_0 such that

$$\max F_0 < \min F_{00}, \qquad \max F_0 < \min F_{01}.$$

Let us select x_{00} and x_{01} such that

$$1 = \|x_{00}\| = \langle x_{00}, F_{00} \rangle, \qquad 1 = \|x_{01}\| = \langle x_{01}, F_{01} \rangle.$$

We select F_{10} and F_{11} disjoint from each other and from F_1 such that

$$\max F_1 < \min F_{10}, \qquad \max F_1 < \min F_{11}.$$

Let us select x_{10} and x_{11} such that

$$1 = \|x_{10}\| = \langle x_{10}, F_{10} \rangle, \qquad 1 = \|x_{11}\| = \langle x_{11}, F_{11} \rangle.$$

In the obvious method we correspond to the dyadic tree, $T = \bigcup_{n=0}^{\infty} \{0, 1\}^n$ disjoint sets F_{10}, F_{11}, F_{000}, F_{001}, F_{010}, $F_{011}, \ldots$ of integers and corresponding sequences $x_{10}, x_{11}, x_{000}, x_{001}, x_{010}, x_{011}, \ldots$ as above.

Since for any two branches $F^1 = (F_\phi, F_0, F_{00}, \ldots)$ and $F^2 = (F_\phi, F_0, F_{01}, \ldots)$, we have

$$f_{F^1}(x_{00}) = 1, \qquad f_{F^2}(x_{00}) = 0,$$

hence $\|f_{F^1} - f_{F^2}\| \geq 1$.

Assertion of the theorem follows from the fact that the set of all branches is uncountable. So $X^*_{\alpha,1}$ is not separable. □

Definition 2.6. A Banach space X is said to be hereditarily indecomposable (HI) provided it does not contain an infinitely dimensional closed subspace which can be divided into the direct sum of its two infinite dimensional closed subspaces, i.e., for every pair of infinite dimensional closed subspaces $X_1, X_2 \subset X$ and for every $\varepsilon > 0$, there exist $x_1 \in X_1$ and $x_2 \in X_2$ with $\|x_1\| = 1 = \|x_2\|$ such that $\|x_1 + x_2\| < \varepsilon$ (see [5]).

By the result of Gowers and Maurey [**10**], every Banach space with an unconditional basic sequence is not hereditarily indecomposable. We show that $X_{\alpha,p}$ is not hereditarily indecomposable. Indeed, we show $X_{\alpha,p}$ has an unconditional basic sequence.

Theorem 2.7. *The Banach space $X_{\alpha,1}$ has an unconditional basic sequence. In particular, $X_{\alpha,1}$ is not hereditarily indecomposable.*

Proof. Let (e_i) denote the sequence of usual unit vectors in X (i.e., $e_i(j) = \delta_{ij}$ for integers i and j). Let $u_i = e_{2i} - e_{2i-1}$ and X_0 be the closed subspace of X generated by the sequence (u_i). It is obvious that for any scalars (t_i) and any j, $\| \sum_{i \neq j} t_i u_i \| \leq \| \sum_i t_i u_i \|$. [**11**, Proposition 1.c.6] shows that the sequence (u_i) is an unconditional basic sequence. The result of Gowers and Maurey [**10**] shows that the Banach space X_0 and then $X_{\alpha,1}$ is not hereditarily indecomposable. $\square$

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