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DOUBLE COVERS AND VECTOR BUNDLES

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Abstract

Let $\pi : X \to Y$ be a double (ramified) cover of complex manifolds. The purpose of this note is to study the properties of holomorphic rank-2 vector bundles on Y arising as push forward of line bundles on X, extending some results of [S], [Fr] to the non-projective case.

1 Introduction

There are essentially only two methods to construct holomorphic vector bundles over complex manifolds. One of them, the extensions method of Serre used succesfully in the projective case, gives only filtrable holomorphic vector bundles in the non-projective case (see, for example [B-F], [B-L], [B]). The second method, the push forward of line bundles by finite covers, was used in both cases (see, for example [S], [Fr] in the projective case and [A-B-T], [T] in the non-projective case). Let $\pi : X \to Y$ be a double cover of complex manifolds. The purpose of this note is to study the properties of holomorphic rank-2 vector bundles on Y arising as push forward of line bundles on X. For any line bundle $\mathcal{M} \in \mathrm{Pic}(X)$ one defines the *norm* line bundle $\mathrm{Nm}(\mathcal{M}) \in \mathrm{Pic}(Y)$ and one computes the Chern classes of holomorphic rank-2 vector bundles obtained as push forward of line bundles by the double cover.

2 The *norm* line bundle

Let Y be a complex manifold, $B \subset Y$ a reduced smooth effective divisor on Y or zero. Suppose we have a line bundle $\mathcal{L}$ on Y such that

$$\mathcal{O}_Y(B) = \mathcal{L}^{\otimes 2},$$

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and a section $s \in \Gamma(Y, \mathcal{O}_Y(B))$ vanishing exactly along B (if $B = 0$ we take for s the constant function 1). We denote by L the total space of $\mathcal{L}$ and we let $p : L \to Y$ be the bundle projection. If $t \in \Gamma(L, p^*\mathcal{L})$ is the tautological section, then the zero divisor of $p^*s - t^2$ defines an analytic subspace X in L. If $B \neq 0$, since B is reduced and smooth, then also X is smooth and $\pi = p|_X$ exhibits X as a 2-fold ramified covering of Y with branch-locus B. We call $\pi : X \to Y$ the 2-cyclic covering (or ramified double cover) of Y branched along B, determined by $\mathcal{L}$. If $B = 0$, we take $\mathcal{L} \ncong \mathcal{O}_Y$; in this case $\pi : X \to Y$ is called the 2-cyclic unramified covering (or unramified double cover) of Y determined by $\mathcal{L}$.

Conversely, given $\pi : X \to Y$ a finite morphism of degree two between complex manifolds, we can recover B and $\mathcal{L}$ as follows. Let $\tau : X \to X$ be the sheet interchange involution, i.e. $\tau^2 = id$, $\pi \circ \tau = \tau$. Then B is the image under π of the fixed set of τ and $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L}^{-1}$, where the direct sum decomposition corresponds to taking the $+1$ and -1 eigenspaces of τ acting on $\pi_*\mathcal{O}_X$ (see, for example [B-P-V], [M], [Fr]).

The morphism $\pi : X \to Y$ induces the natural homomorphism

$$\pi^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X).$$

In the projective case, by using the natural map $\pi_* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ between the group of divisors on X and similarly on Y, one obtains a homomorphism

$$\pi_* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y).$$

In the non-projective case, since the group of divisors could be very small, we have to find some other substitute for the map π_* by using line bundles.

Let $G = \{id, \tau\}$ be the group of the double cover $\pi : X \to Y$. Let $\mathcal{M} \in \mathrm{Pic}(X)$ be a line bundle. Since

$$\tau^*(\mathcal{M} \otimes \tau^*\mathcal{M}) \cong \mathcal{M} \otimes \tau^*\mathcal{M},$$

then

$$\mathcal{M} \otimes \tau^*\mathcal{M} \in \mathrm{Pic}(X)^G.$$

It follows that $\pi_*(\mathcal{M} \otimes \tau^*\mathcal{M})$ is a G-sheaf and, as a rank 2-vector bundle on Y, splits as a direct sum of line bundles corresponding to taking $+1$ and -1 eigenspaces of τ. Thus, we have

$$\pi_*(\mathcal{M} \otimes \tau^*\mathcal{M}) \cong (\pi_*(\mathcal{M} \otimes \tau^*\mathcal{M}))^G \oplus Z,$$

where the first factor is the invariant part of the G-sheaf $\pi_*(\mathcal{M} \otimes \tau^*\mathcal{M})$.

We have

Definition 1 Let $\mathcal{M} \in \mathrm{Pic}(X)$ be a line bundle. The line bundle

$$\mathrm{Nm}\mathcal{M} := (\pi_*(\mathcal{M} \otimes \tau^*\mathcal{M}))^G$$

is called the *norm* line bundle of $\mathcal{M}$.

Lemma 2 $\pi^*(\mathrm{Nm}\mathcal{M}) \cong \mathcal{M} \otimes \tau^*\mathcal{M}$.

Proof: There exists a natural morphism of line bundles

$$\pi^*(\pi_*(\mathcal{M} \otimes \tau^*\mathcal{M})^G) \to \mathcal{M} \otimes \tau^*\mathcal{M}.$$

For any $x \in X$ let $y = \pi(x) \in Y$ and choose a Stein neighbourhood V of y; then $\pi^{-1}(V)$ is a Stein neighbourhood of $\pi^{-1}(y)$ and one finds a section $s \in \mathcal{M}(\pi^{-1}(V))$ with $s(x_i) \neq 0$ for all $x_i \in \pi^{-1}(y)$. Then $s.s^\tau$ is a G-invariant section of $\mathcal{M} \otimes \tau^*\mathcal{M}(\pi^{-1}(V))$, which generates $\mathcal{M} \otimes \tau^*\mathcal{M}$ in a neigbourhood of $\pi^{-1}(y)$. It follows that the above natural morphism is an isomorphism.

3 Rank-2 vector bundles

Let $\pi : X \to Y$ be a double cover as above. We have that $\pi_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{L}^{-1}$ is an $\mathcal{O}_Y$- algebra, which is coherent as an $\mathcal{O}_Y$-module. Multiplication is given by

$$(a, l).(b, m) = (ab + \Phi(l \otimes m), am + bl),$$

where a, b are (local) sections of $\mathcal{O}_Y$, l, m are (local) sections of $\mathcal{L}^{-1}$, for some isomorphism

$$\Phi : \mathcal{L}^{-2} \xrightarrow{\sim} \mathcal{O}_Y(-B) \subset \mathcal{O}_Y.$$

In fact, $X = \mathrm{Specan}(\mathcal{A})$, where $\mathcal{A} = \pi_*\mathcal{O}_X$ is a sheaf of $\mathcal{O}_Y$-algebras and for any $\mathcal{O}_X$-module $\mathcal{F}$ the $\mathcal{O}_Y$-module $\pi_*\mathcal{F}$ is an $\mathcal{A}$-module (see [Fi]).

Let $\mathcal{M} \in \mathrm{Pic}(X)$ be a line bundle. For the rank-2 vector bundle $\pi_*\mathcal{M}$ we have the following result:

Theorem 3 $\det(\pi_*\mathcal{M}) \cong \mathrm{Nm}\mathcal{M} \otimes \mathcal{L}^{-1}$.

Proof: Since $\pi : X \to Y$ is a finite morphism we have the isomorphism:

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{\sim} H^1(Y, (\pi_*\mathcal{O}_X)^*). \tag{1}$$

(see [H], [M]). By choosing a suitable open covering $\mathcal{U} = (U_i)$ of Y we can suppose that all the vector bundles $\mathcal{L}$, $\pi_*\mathcal{O}_X$ and $\pi_*\mathcal{M}$ have local trivializations with respect to this open covering. If (f_{ij}) is an 1-cocycle of the covering $\mathcal{U}$ with coefficients in $\mathcal{O}_Y^*$, which defines the line bundle $\mathcal{L}$, then the rank-2 vector bundle $\mathcal{A} = \pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{L}^{-1}$ is defined with respect to this covering by the 1-cocycle

$$\begin{pmatrix} 1 & 0 \\ 0 & f_{ij}^{-1} \end{pmatrix}$$

In fact, we have the local trivializations

$$\mathcal{O}_{U_i} \stackrel{\varphi_i}{\to} \mathcal{L}^{-1}|_{U_i}$$

and the isomorphisms composition

$$\mathcal{O}_{U_i \cap U_j} \stackrel{\varphi_j}{\to} \mathcal{L}^{-1}|_{U_i \cap U_j} \stackrel{\varphi_i^{-1}}{\to} \mathcal{O}_{U_i \cap U_j},$$

such that $f_{ij}^{-1} = (\varphi_i^{-1} \circ \varphi_j)(1)$. Then, we get the isomorphisms composition

$$\mathcal{O}^2_{U_i \cap U_j} \to \mathcal{A}|_{U_i \cap U_j} \to \mathcal{O}^2_{U_i \cap U_j},$$

given by the above matrix.

The morphism $\Phi : \mathcal{L}^{-2} \to \mathcal{O}_Y$ gives the morphisms

$$\Phi_{ij} : \mathcal{O}^{\otimes 2}_{U_i \cap U_j} \to \mathcal{L}^{-2}|_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j},$$

where $\Phi_{ij}(l \otimes m) = lmt_{ij},\ t_{ij} \in \mathcal{O}(U_i \cap U_j)$.

The line bundle $\mathcal{M} \in \operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$ defines by the isomorphism (1) an 1-cocycle $(\tilde{\gamma}_{ij})$, $\tilde{\gamma}_{ij} \in \mathcal{A}^*(U_i \cap U_j)$. Let γ_{ij} be the composition

$$\mathcal{O}^2(U_i \cap U_j) \stackrel{1 \oplus \varphi_j}{\to} \mathcal{A}(U_i \cap U_j) \stackrel{\tilde{\gamma}_{ij}}{\to} \mathcal{A}(U_i \cap U_j) \stackrel{1 \oplus \varphi_i^{-1}}{\to} \mathcal{O}^2(U_i \cap U_j);$$

then (γ_{ij}) is an 1-cocycle which defines the rank-2 vector bundle $\pi_* \mathcal{M}$. If we write $\gamma_{ij} = (c'_{ij}, c''_{ij})$, for any $(\alpha, \beta) \in \mathcal{O}^2(U_i \cap U_j)$, we get

$$\gamma_{ij}(\alpha, \beta) = (c'_{ij}\alpha + c''_{ij} t_{ij} \beta, f_{ij}^{-1}(c'_{ij}\beta + c''_{ij}\alpha)).$$

It follows that an 1-cocycle which defines the rank-2 vector bundle $\pi_* \mathcal{M}$ is given by the following matrix

$$\begin{pmatrix} c'_{ij} & c''_{ij} t_{ij} \\ f_{ij}^{-1} c''_{ij} & c'_{ij} \end{pmatrix}$$

Then, the line bundle $\det(\pi_* \mathcal{M})$ is given by the 1-cocycle

$$f_{ij}^{-1}((c'_{ij})^2 - t_{ij}(c''_{ij})^2).$$

In the direct sum decomposition $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L}^{-1}$, the second factor corresponds to the -1 eigenvalue of τ. It follows that an 1-cocycle defined by the line bundle $\mathcal{M} \otimes \tau^*\mathcal{M}$ in $H^1(Y, (\pi_*\mathcal{O}_X)^*)$ is given by

$$\gamma_{ij}.\tau^*\gamma_{ij} = (c'_{ij}, c''_{ij}).(c'_{ij}, -c''_{ij}) = ((c'_{ij})^2 - t_{ij}(c''_{ij})^2, 0).$$

For the rank-2 vector bundle $\pi_*(\mathcal{M} \otimes \tau^*\mathcal{M})$ we get the 1-cocycle given by the following matrix

$$\begin{pmatrix} (c'_{ij})^2 - t_{ij}(c''_{ij})^2 & 0 \\ 0 & f_{ij}^{-1}((c'_{ij})^2 - t_{ij}(c''_{ij})^2) \end{pmatrix}$$

It follows that the G-invariant part of $\pi_*(\mathcal{M} \otimes \tau^*\mathcal{M})$, i.e. $\mathrm{Nm}\mathcal{M}$ is given by an 1-cocycle $(c'_{ij})^2 - t_{ij}(c''_{ij})^2$. Finally, we get

$$\det(\pi_*\mathcal{M}) \cong \mathrm{Nm}\mathcal{M} \otimes \mathcal{L}^{-1}.$$

Lemma 4 *We have the exact sequence of vector bundles on X:*

$$0 \to \tau^*\mathcal{M} \otimes \pi^*(\mathcal{L}^{-1}) \to \pi^*(\pi_*\mathcal{M}) \to \mathcal{M} \to 0.$$

Proof: (see [Fr]) The natural morphism

$$\pi^*(\pi_*\mathcal{M}) \to \mathcal{M}$$

is surjective; then the kernel is the line bundle

$$\det(\pi^*(\pi_*\mathcal{M})) \otimes \mathcal{M}^{-1} = \pi^*(\det(\pi_*\mathcal{M})) \otimes \mathcal{M}^{-1}.$$

By applying Theorem 3 and Lemma 2 we get the desired result.

Corollary 5

$$c_2(\pi_*\mathcal{M}) = \frac{1}{2}(c_1^2(\mathrm{Nm}\mathcal{M}) - \pi_*(c_1^2(\mathcal{M})) - \pi_*(c_1(\mathcal{M})).c_1(\mathcal{L})).$$

Proof: (see [Fr]) By applying all the results above and the projection formula one gets the conclusion.

Remark By using push forward of line bundles by finite covers in [A-B-T] one obtained the complete answer to the existence problem of holomorphic vector bundles of any rank over primary Kodaira surfaces.

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