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A TWO-DIMENSIONAL DOMAIN WHOSE INTEGRAL CLOSURE IS NOT T-LINKED

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Abstract

We construct a two-dimensional domain D having two nonzero v-coprime elements a, b such that a, b are not v-coprime in the integral closure of D.

Let D be an integral domain with quotient field K and D' the integral closure of D. By an overring of D we mean a ring between D and K. Recall that for a nonzero fractional ideal I of D, $I_v = (I^{-1})^{-1} = (D : I) : I = \cap\{xD;\ xD \supseteq I,\ x \in K\}$. It is well known that for $x, y \in D \setminus \{0\}$, $xD \cap yD$ is a principal ideal if and only if so is $((x, y)D)_v$. According to [3], an overring E of D is *t-linked* over D, if whenever $x_1, ..., x_n \in D \setminus \{0\}$ with $((x_1, ..., x_n)D)_v = D$, we have $((x_1, ..., x_n)E)_v = E$.

In [3], it was asked whether D' is always t-linked over D. While this is true if $\dim(D) \leq 1$ [3, Corollary 2.7], in [4, Example 4.1] there were constructed examples of domains D of every dimension ≥ 3 such that D' is not t-linked over D (see also [5, Proposition 3] for a generalization). As noted in [4, page 1482], the two-dimensional case remained open.

The aim of this note is to construct a two-dimensional domain D such that D' is not t-linked over D. Call two nonzero elements $x, y \in D$ *v-coprime*, if $((x, y)D)_v = D$, equivalently, if $xD \cap yD = xyD$. Our plan is to construct a two-dimensional domain D having two nonzero v-coprime elements a, b such that a, b are not v-coprime in the integral closure of D (hence D' is not t-linked over D). For that, we use a composite domain construction of type $A + XB[X]$. More precisely, whenever $A \subseteq B$ is an extension of domains, we can consider the subring $A + XB[X]$ of $B[X]$ consisting of all polynomials in $B[X]$ with constant term in A (see [8] and its references). Any unexplained

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material is standard, as in [6], [7].

We begin with the following simple lemma.

Lemma 1. *Let $A \subseteq B$ be an extension of domains, $D = A + XB[X]$ and $0 \neq a \in A$. Then a, X are v-coprime in D if and only if $aB \cap A = aA$.*

Proof. Assume that $aB \cap A = aA$ and let $h \in aD \cap XD$. There exist $f, g \in D$, say $f = \sum f_i X^i$ and $g = \sum g_j X^j$, such that $h = af = Xg$. Obviously, $u = g/a$ lies in $B[X]$. Also, $af_1 = g_0 \in aB \cap A = aA$, so $f_1 \in A$. As $u(0) = g_0/a = f_1 \in A$, $u \in D$. So $h = aXu \in aXD$. Hence a, X are v-coprime in D.

Conversely, assume that $aB \cap A \neq aA$. Then $ab \in A$ for some $b \in B \setminus A$. Hence $abX \in aD \cap XD$, but $abX \notin aXD$ because $b \notin A$. So a, X are not v-coprime in D. $\square$

We present our construction followed by a specific example.

Theorem 2. *Let $A \subseteq B$ be an integral extension of PIDs and $0 \neq p \in A$ a prime element. Assume there exist two distinct prime elements q and r of B which divide p in B (i.e., p decomposes in B) and let $D = A + XB_{qB}[X]$. Then D is two-dimensional and p, X are v-coprime in D but not v-coprime in D'. In particular, D is a two-dimensional domain such that D' is not t-linked over D.*

Proof. By [1, Theorem 2.7], the integral closure of D is $D' = B + XB_{qB}[X]$, because B is integrally closed, so the integral closure of A in B_{qB} is B. By [2, Example 2.11], D' is two-dimensional, hence so is D. As pA is a maximal ideal of A and pA survives in B_{qB}, $pA = pB_{qB} \cap A$. So p, X are v-coprime in D, cf. Lemma 1. Since r is a unit of B_{qB}, r divides X in D'. So r is a non-invertible common factor of p and X in D'. Consequently, p, X are not v-coprime in D'. The 'in particular' statement is clear. $\square$

Example 3. *As a specific example, we may take $A = \mathbf{Z}$, $B = \mathbf{Z}[i]$, $p = 5$, $q = 2 + i$ and $r = 2 - i$. So $\mathbf{Z} + X\mathbf{Z}[i]_{(2+i)}[X]$ is a two-dimensional domain with D' not t-linked over D.*

Remark 4. *Let D be the domain in Theorem 2 and $D_n = D[Y_1, ..., Y_n]$ where $Y_1, ..., Y_n$ are indeterminates over D and $n \geq 0$. It is easy to see that p, X are v-coprime in D_n but not v-coprime in D'_n. Moreover, dim(D_n) = dim$(D'_n) = n + 2$. Indeed, $D' = B + XB_{qB}[X]$ is the directed union (inductive limit) of its subrings $B[X/s]$ for $s \in S$, where $S = B \setminus qB$. Consequently, $D'_n = \cup_{s \in S} B[X/s, Y_1, ..., Y_n]$. Since dim$(B[X/s, Y_1, ..., Y_n]) = n + 2$ [6, Theorem 30.5], a direct limit argument shows that dim$(D'_n) = n + 2$. So we get such examples in each dimension ≥ 2.*

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