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LINEARLY NORMAL CURVES IN $\mathbf{P}^4$ AND $\mathbf{P}^5$

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Abstract

We extend a theorem proved by Dolcetti and Pareschi in [DE] concerning the existence of linearly normal curves in $\mathbf{P}^3$ to similar ones for $\mathbf{P}^4$ and $\mathbf{P}^5$ (Theorems A and B). However such a theorem does not seem to hold for $n \geq 6$.

1 Preliminaries

We work over an algebraically closed field k of characteristic zero. We use the standard notations from [Ha].

DEFINITION 1.1. A smooth, irreducible curve $C \subset \mathbf{P}^n$ is called *linearly normal* if it is non-degenerate (i.e. not contained in any hyperplane) and it is not a projection of a curve from a bigger projective space.

REMARK 1.2. *Let $C \subset \mathbf{P}^n$ be a smooth, irreducible curve and let's denote by $\mathcal{I}_C$ its ideals sheaf. Then C is linearly normal $\Leftrightarrow h^j(\mathcal{I}_C(1)) = 0$, $j = 0, 1 \Leftrightarrow C$ is embedded in $\mathbf{P}^n$ using a complete linear system.*

REMARK 1.3. *From the Riemann-Rock theorem we deduce immediately that, if $C \subset \mathbf{P}^n$ is a linearly normal curve of degree d and genus g, then $g \geq d - n$.*

For $n \in \mathbf{Z}$, $n \geq 3$, we consider the following Problems:

$HC(n)$: For which pairs (d, g) of positive integers there is a smooth, irreducible, non-degenerate curve $C \subset \mathbf{P}^n$ of degree d and genus g? (see also [P1], [P2]).

$LN(n)$: For which pairs (d, g) of positive integers there is a (smooth, irreducible) linearly normal curve $C \subset \mathbf{P}^n$ of degree d and genus g?

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DEFINITION 1.4. A pair of positive integers (d, g) is called a *gap for* $HC(n)$ if there is no smooth, irreducible, non-degenerate curve $C \subset \mathbf{P}^n$ of degree d and genus g. Analogously we define a *gap for* $LN(n)$.

We recall now from [H] the Harris-Eisenbud numbers. Let p be an integer so that $0 \leq p \leq n-2$, $n \in \mathbf{Z}$, $n \geq 3$. Then

$$\pi_p = \pi_p(d,n) := \frac{m_p(m_p-1)}{2}(n+p-1) + m_p(\varepsilon_p + p) + \mu_p, \tag{1.1}$$

where

$$m_p = m_p(d,n) := [(d-1)/(n+p-1)]_* \tag{1.2}$$

(we denote by $[x]_*$ the integer part of the real number x)

$$\varepsilon_p = \varepsilon_p(d,n) := d - 1 - m_p(n+p-1) \tag{1.3}$$

$$\mu_p = \mu_p(d,n) := \max(0, [(p-n+2+\varepsilon_p)/2]_*). \tag{1.4}$$

We remark that $\mu_0 = 0$ and $\pi_p = d^2/(2(n+p-1)) + O(d)$.

REMARK 1.5. *If $C \subset \mathbf{P}^n$ is a (smooth, irreducible) non-degenerate curve of degree d and genus g, then $d \geq n$ and $0 \leq g \leq \pi_0(d,n)$ (Castelnuovo Theorem [C]).*

The Problem $HC(3)$ was solved by Gruson and Peskine ([GP1], [GP2]). $LN(3)$ was solved, based on Gruson-Peskine results by Dolcetti and Pareschi ([DP]); they gave a positive answer to a Conjecture of Hartshorne, namely:

THEOREM 1.6 [DP]. *If $d, g \in \mathbf{Z}$, $d \geq 3$, $g \geq d-3$ (see Remarks 1.3 and 1.5), then the pair (d,g) is a gap for $LN(3)$ if and only if is a gap for $HC(3)$.*

In this paper we prove similar results for $\mathbf{P}^4$ and $\mathbf{P}^5$, namely:

THEOREM A. *If $d, g \in \mathbf{Z}$, $d \geq 4$, $g \geq d-4$ (see Remarks 1.3 and 1.5), then the pair (d,g) is a gap for $LN(4)$ if and only if it is a gap for $HC(4)$.*

THEOREM B. *If $d, g \in \mathbf{Z}$, $d \geq 5$, $g \geq d-5$ (see Remarks 1.3 and 1.5), then the pair (d,g) is a gap for $LN(5)$ if and only if it is a gap for $HC(5)$.*

The Theorem A will be proved in §2 and the Theorem B will be proved in §3.

We recall that $HC(4)$ and $HC(5)$ were solved by Rathmann [Ra].
The Theorems A and B says that a Dolcetti-Pareschi type results hold for $\mathbf{P}^4$ and $\mathbf{P}^5$ also. However, we do not expect a similar result for $n \geq 6$.
We will use in §2, §3 the following

LEMMA 1.7 [DP]. *Let $C \subset \mathbf{P}^n$ be a smooth, irreducible curve and H a hyperplane intersecting C transversally. Let $Z \subset H$ a curve. We assume:* i) $h^i(\mathcal{I}_{Z,H}(1)) = 0$, $j = 0,1$ *(here $\mathcal{I}_{Z,H}$ is the ideals-sheaf of Z in H)*; ii) $C \cap Z = C \cap H$ *(as schemes). Then $X := C \cup Z$ satisfies: $h^j(\mathcal{I}_X(1)) = 0$, $j = 0,1$ ($\mathcal{I}_X$ is the ideals-sheaf of X).*

LEMMA 1.8. *Let $X \subset \mathbf{P}^n$ be a regular (i.e. $q = h^1(\mathcal{O}_X) = 0$) smooth, irreducible, projective surface. Let $H \subset \mathbf{P}^n$ be a general hyperplane so that $C := H \cap X$ is a smooth, irreducible curve. Then $h^j(\mathcal{I}_{C,H}(1)) = 0$, $j = 0,1$.*

Proof. Let S be a smooth, irreducible, projective abstract surface and $\mathcal{L} \in Pic(S)$ a very ample invertible sheaf so that $X = \varphi_{[\mathcal{L}]}(S)$. Let $H_0 \in [\mathcal{L}]$ and $C_0 := \varphi^{-1}_{[\mathcal{L}]}(C) \subset S$. We consider the standard exact sequence

$$\begin{array}{ccccccc} \mathcal{O} \to \mathcal{O}_S \to & \mathcal{O}_S(C_0) & \to & \mathcal{O}_{C_0} & \to 0 \\ & || & & || & \\ & \mathcal{O}_S(H_0) & & \mathcal{O}_{C_0}(H_0) & \end{array}$$

From $h^1(\mathcal{O}_S) = 0$, we get the surjection $H^0(\mathcal{O}_S(H_0)) \twoheadrightarrow H^0(\mathcal{O}_{C_0}(H_0))$. So

$$|H_0|\Big|_{C_0} = |C_0 \cap H_0|. \tag{1.5}$$

Because the embedding of C_0 in $H \simeq \mathbf{P}^{n-1}$ is given by $[\mathcal{L}]|_{C_0}$ (i.e. $[H_0]|_{C_0}$), from (1.5) we deduce that $C = \varphi_{|C_0 \cap H_0|}(C_0)$, so the embedding of C in H is given by a complete linear system, so $h^j(\mathcal{I}_{C,H}(1)) = 0$, $j = 0,1$, by the Remark 1.2.□

REMARK 1.9. *Lemma* 1.8 *applies to smooth, irreducible, rational, projective surfaces, since these are regular surfaces.*

LEMMA 1.10. $\pi_1(d,n) \geq \pi_0(d,n+1)$, $(\forall)d \geq 1$, $(\forall)n \geq 3$ (see (1.1)-(1.4)).

Proof. It is easy to see that the functions (1.1)-(1.4) satisfy:

$$\pi_p(d + (n+p-1), n) = \pi_p(d,n) + (d+p+1) \tag{1.6}$$

(1.7) $$m_p(d+(n+p-1),n) = m_p(d,n)+1$$

(1.8) $$\varepsilon_p(d+(n+p-1),n) = \varepsilon_p(d,n);\ \mu_p(d+(n+p-1),n) = \mu_p(d,n).$$

Put $u(d,n) := \pi_0(d,n+1)$; $v(d,n) := \pi_1(d,n) - m_1(d,n)$. Then, from (1.6), (1.7), (1.8) for π_0, π_1, m_1, we deduce

(1.9) $$u(d+n,n) = u(d,n)+(d-1);\ v(d+n,n) = v(d,n)+(d-1).$$

We can see that, if $v(d,n) \geq u(d,n)$, $(\forall)d \geq 1$, then the Lemma follows. From (1.9) we deduce that it is enough to prove the previous inequality for $m_1(d,n) = m_0(d,n+1) = 0$, the general case resulting then by induction on $m_1(d,n)$. But $m_1(d,n) = 0 \Rightarrow u(d,n) = 0$ and $v(d,n) = \mu_1(d,n) \geq 0.\square$

2 The proof of Theorem A

By Lemma 1.10 and Castelnuovo Theorem (Remark 1.5) it is enough to prove that (d,g) is not a gap for $LN(4)$ if and only if (d,g) is not a gap for $HC(4)$ in the domain of the (d,g)-plane given by

(2.1) $$d \geq 4,\ d-4 \leq g \leq \pi_1(d,4).$$

NOTATIONS. *If $X \subset \mathbf{P}^n$ is a rational surface obtained by blowing up $s+1$ points from $\mathbf{P}^2$ and embedding the abstract surface obtained in such a way using the very ample linear system $[a;b_0,b_1,\ldots,b_s]$ we say that X is of $[a;b_0,b_1,\ldots,b_s]$-type. A surface of $[p+2;p,1^{3p-n+5}]$-type is denoted by X_p^n (as in* [P2]*).*

We shall use the following known

THEOREM 2.1.
1) (Păsărescu [P1], Rathmann [Ra]): *For any $d_0, g_0 \in \mathbf{Z}$, $d_0 \geq 4$ and $(d_0 + 12\sqrt{d_0+9} - \frac{11}{2}d_0 - 35 \leq g \leq \frac{1}{8}d_0(d_0-4)+1$ there is a smooth, irreducible, non-degenerate curve $C_0 \subset \mathbf{P}^4$ of degree d_0 and genus g_0, lying on a smooth Del Pezzo surface in $\mathbf{P}^4$ (surface X_1^4);*
2) (Kleppe [Kl]): *For any $d_0, g_0 \in \mathbf{Z}$, $d_0 \geq 6$ and $\frac{1}{\sqrt{5}}d_0^{3/2} - d_0 - 1 \leq g \leq \frac{1}{10}d_0(d_0-3)+1$ there is a smooth, irreducible, non-degenerate curve $C_0 \subset \mathbf{P}^4$ of degree d_0 and genus g_0, lying on a Castelnuovo surface in $\mathbf{P}^4$ (surface X_2^4);*
3) (Păsărescu [P1], Rathmann [Ra]): *For any $d_0, g_0 \in \mathbf{Z}$, $d_0 \geq 7$ and $0 \leq g_0 \leq \frac{1}{12}d_0(d_0-2)+1$ there is a smooth, irreducible, non-degenerate curve $C_0 \subset \mathbf{P}^4$*

of degree d_0 and genus g_0, lying on a surface $X_3'^4$ of degree 6 in $\mathbf{P}^4$ (this surface may be the surface used in [P1], §1 *for $n = 4$, or a Bordiga surface, as in* [Ra], *which is a surface of* $[4; 1^{10}]$*-type).*

We continue the proof of the Theorem A. This will be done in 5 steps:
A. If $g = d - 4$, $d \geq 4$ the existence of the linearly normal curves of degree d and genus g in $\mathbf{P}^4$ follows from the existence of the non-special curves. Since $\pi_0(4, 4) = 0$, it follows from (2.1) that we need to construct linearly normal curves in the range

$$d - 3 \leq g \leq \pi_1(d, 4),\ d \geq 5. \tag{2.2}$$

Further, we use the Theorem 2.1.
B. **Claim**. *If $d, g \in \mathbf{Z}$, $d \geq 8$ and $(d+8)\sqrt{d+5} - \frac{9}{2}d - 17 \leq g \leq \frac{1}{8}d(d-4)+1$ there is a linearly normal curve $C_0 \subset \mathbf{P}^4$ of degree d and genus g lying on a Del Pezzo surface from $\mathbf{P}^4$ (surface X_1^4).*
C. **Claim**. *If $d, g \in \mathbf{Z}$, $d \geq 11$ and $\frac{1}{\sqrt{5}}(d-5)^{3/2} \leq g \leq \frac{1}{10}d(d-3)+1$ there is a linearly normal curve $C \subset \mathbf{P}^4$ of degree d and genus g lying on a Castelnuovo surface from $\mathbf{P}^4$ (surface X_2^4).*
D. **Claim**. *If $d, g \in \mathbf{Z}$, $d \geq 13$ and $d-4 \leq g \leq \frac{1}{12}d(d-2)+1$ there is a linearly normal curve $C \subset \mathbf{P}^4$ of degree d and genus g lying on a Bordiga surface from $\mathbf{P}^4$ (surface of* $[4; 1^{10}]$*-type).*
The proofs of the Claims C. D. are similar to the proof of the Claim B.
E. It easy to check that

$$d \geq 11 \Rightarrow \frac{1}{12}d(d-2) + 1 \geq \frac{1}{\sqrt{5}}(d-5)^{3/2} \tag{2.4}$$

$$d \geq 10 \Rightarrow \frac{1}{10}d(d-3) + 1 \geq (d+8)\sqrt{d+5} - \frac{9}{2}d - 17. \tag{2.5}$$

From B, C, D, (2.4) and (2.5) we deduce that for any $d, g \in \mathbf{Z}$, $d \geq 13$, $d - 4 \leq g \leq \frac{1}{8}d(d-4) + 1$ there is a linearly normal curve $C \subset \mathbf{P}^4$ of degree d and genus g. Because, for $d \geq 5$ we have $g \leq \frac{1}{8}d(d-4)$, $g \in \mathbf{Z} \Leftrightarrow g \leq \pi_1(d, 4)$, $g \in \mathbf{Z}$, it follows that, in order to cover with linearly normal curves the domain (2.2), we need (use A. also) linearly normal curves for $d-3 \leq g \leq \frac{1}{8}d(d-4)+1$, $5 \leq d \leq 12$.

The necessary list follows (the curves from the list are linearly normal by Lemma 1.7 and semicontinuity, because $C_0 \in |C - H|$ is smooth); the curves belong to linear systems on X_1^4:

$$(d,g) = (6,2) : C \in [4;2,1^4];\ (d,g) = (7,3) : C \in [4;1^5]$$
$$(d,g) = (8,5) : C \in [6;2^5];\ (d,g) = (9,6) : C \in [6;2^4,1]$$
$$(d,g) = (10,7) : C \in [6;2^3,1^2];\ (d,g) = (10,8) : C \in [7;3,2^4]$$
$$(d,g) = (11,8) : C \in [6;2^2,1^3];\ (d,g) = (11,9) : C \in [7;3,2^3,1]$$
$$(d,g) = (11,10) : C \in [7;2^5];\ (d,g) = (12,9) : C \in [6;2,1^4]$$
$$(d,g) = (12,10) : C \in [7;3,2^2,1^2];\ (d,g) = (12,11) : C \in [7;2^4,1]$$
$$(d,g) = (12,12) : C \in [8;3^2,2^3];\ (d,g) = (12,13) : C \in [9;3^5].$$

Now, the proof of the Theorem A is completed.$\square$

3 The proof of Theorem B

This is quite similar to the proof of the Theorem A, by using instead Theorem 2.1 the following known Theorem 3.1. We omit the details. However, the list of the linearly normal curves constructed directly like in step E from the proof of the Theorem A is much longer.

THEOREM 3.1.
1) (Păsărescu [P1], Rathmann [Ra]): *For any* $d_0, g_0 \in \mathbf{Z}$, $d_0 \geq 20$ *and* $(d_0 + 30)\sqrt{2d_0+40} - \frac{23}{2}d_0 - 188 \leq g_0 \leq \frac{1}{10}d_0(d_0-5)+1$ *there is a smooth, irreducible, non-degenerate curve* $C_0 \subset \mathbf{P}^5$ *of degree* d_0 *and genus* g_0, *lying on a smooth Del Pezzo surface* $\mathbf{P}^5$ *(surface* X_1^5*)*;
2) (Kleppe [Kl]): *For any* $d_0, g_0 \in \mathbf{Z}$, $d_0 \geq 7$ *and* $\frac{1}{2}d_0^{3/2} - \frac{3}{2}d_0 + 3 \leq g_0 \leq \frac{1}{12}d_0(d_0-4)+1$ *there is a smooth, irreducible, non-degenerate curve* $C_0 \subset \mathbf{P}^5$ *of degree* d_0 *and genus* g_0, *lying on a Castelnuovo surface in* $\mathbf{P}^5$ *(surface* X_2^5*)*;
3) (Ciliberto-Sernesi [CS], $\delta = 3$, $r = 5$): *For any* $d_0, g_0 \in \mathbf{Z}$, $d_0 \geq 8$ *and* $0 \leq g_0 \leq \frac{1}{14}(d_0-5)^2$ *there is a smooth, irreducible, non-degenerate curve* $C_0 \subset \mathbf{P}^5$ *of degree* d_0 *and genus* g_0, *lying on a rational surface (surface* $X_3'^5$*) of degree 7 in* $\mathbf{P}^5$.$\square$

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