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GRÖBNER BASIS AND DEPTH OF REES ALGEBRAS

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Introduction

Let $B = K[X_1, \ldots, X_n]$ be a polynomial ring over a field K and $A = B/J$ a quotient ring of B by a homogeneous ideal J. Let m denote the maximal graded ideal of A. Then the Rees algebra $R = A[mt]$ may be considered a standard graded K-algebra and has a presentation $B[Y_1, \ldots, Y_n]/I_J$. For instance, if $J = 0$ then $R \cong K[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/(H)$, where $H := \{X_iY_j - X_jY_i | 1 \leq i < j \leq n\}$.

The generators of I_J can be easily described as follows. For any homogeneous form $f = \sum_{1\leq i_1\leq \ldots \leq i_d \leq n} a_{i_1\ldots i_d} X_{i_1} \ldots X_{i_d} \in B$ of degree d we set

$f^{(k)} = \sum_{1\leq i_1\leq \ldots \leq i_d \leq n} a_{i_1\ldots i_d} X_{i_1} \ldots X_{i_{d-k}} Y_{i_{d-k+1}} \ldots Y_{i_d}$

for $k = 0, \ldots, d$. For any subset $L \subset B$ of homogeneous polynomials in B we set

$L' := \{f^{(k)} | f \in L, k = 0, \ldots, \deg f\}$. If L is a minimal system of generators of J, then $L' \cup H$ is a minimal system of generators of I_J (see Proposition 1.1) and if L is a Gröbner basis of J for the reverse lexicographic order induced by $X_1 > \ldots > X_n > Y_1 > \ldots > Y_n$ then $L' \cup H$ is a Gröbner basis of I_J (see Theorem 1.3). This procedure is described in [HPT1]. However it is not included in the new version [HPT2] even it has its own value (it is used in [HOP]). Our Section 1 is an attempt to give a printed presentation.

The purpose of [HPT2] is to compare the homological properties of A and R. In particular the Castelnuovo-Mumford regularity of R, reg R, is $\leq$ reg $A+1$ (see also [E]). Unfortunately, depth R could be $>$ depth $A+1$ as shows an example of Goto [G], but if A is a polynomial algebra in one variable over a standard graded K-algebra then it holds depth $R \leq$ depth $A+1$ (see [HPT2]). The proof from [HPT2] uses a description of the local cohomology of R in

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terms of the local cohomology of A. Our Section 2 contains a direct proof of the above inequality which does not use the local cohomology. This is part of the joint work with J.Herzog and N.V.Trung which was not inclosed in [HPT1], [HPT2].

1. Gröbner basis of Rees algebras

Let A be a standard graded K-algebra with maximal graded ideal $m = (x_1, \dots, x_n)$, $A = B/J$ where $B = K[X_1, \dots, X_n]$ is a polynomial ring over a field K and J is a homogeneous ideal of B. Then the Rees algebra $R = A[mt]$ may be considered as a bigraded module over the bigraded polynomial ring $S = K[X_1, \dots, X_n, Y_1, \dots, Y_n]$ (where deg $X_i = (1,0)$, deg $Y_j = (1,1)$) and has a presentation S/I_J via the bigraded canonical surjection $\phi : S \to R$ given by $\phi(X_i) = x_i$ and $\phi(Y_j) = x_j t$.

Let $f = \sum_{1 \le i_1 \le \dots \le i_d \le n} a_{i_1 \dots i_d} X_{i_1} \dots X_{i_d} \in B$ be a homogeneous form of degree d. For $k = 0, \dots, d$ we set

$f^{(k)} = \sum_{1 \le i_1 \le \dots \le i_d \le n} a_{i_1 \dots i_d} X_{i_1} \dots X_{i_{d-k}} Y_{i_{d-k+1}} \dots Y_{i_d}$.

Notice that $f^{(k)}$ is bihomogeneous of degree (d, k). For any subset $L \subset B$ of homogeneous polynomials in B we set

$L' := \{f^{(k)} | f \in L, k = 0, \dots, \deg f\}$.

Proposition 1.1 *Let L be a (minimal) system of generators of J, then $\{L' \cup H\}$ is a (minimal) system of generators of I_J, where $H := \{X_i Y_j - X_j Y_i | 1 \le i < j \le n\}$.*

Proof. Let $P = B[X_1 t, \dots, X_n t] \subset B[t]$, $\phi_1 : S \to P$, $\phi_2 : P \to R$ be the maps given by $(X, Y) \to (X, Xt)$, respectively $(X, Xt) \to (x, xt)$. We have $\phi = \phi_2 \phi_1$. Since ϕ is bigraded I_J is bigraded too. Clearly we have $L' \cup H \subset I_J$. Conversely, let $f \in I_J$, we may choose f bigraded with deg $f = (a, b)$. Then $\phi_1(f) = f(X, Xt) = f(X, X)t^b$, and so $0 = \phi(f) = f(x, x)t^b$, that is $f(x, x) = 0$. Therefore, there exist homogeneous elements $g_i \in B$ and $f_i \in L$ such that $f(X, X) = \sum_{i=1}^r g_i f_i$. We may suppose $L = \{f_1, \dots, f_r\}$. Let $b_i = \min\{\deg f_i, b\}$. Then

$\phi_1(f) = f(X, X)t^b = \sum_{i=1}^r (g_i t^{b-b_i})(f_i t^{b_i}) = \phi_1(\sum_{i=1}^r g_i^{(b-b_i)} f_i^{(b_i)})$,

and so $f \in L' \cup H$, since Ker ϕ_1 is generated by H.

Now let L be a minimal system of generators of J. We first show that $\phi_1(L')$ is a minimal system of generators of the ideal $J_1 := \phi_1(I_J)$ in P. Indeed, $\phi_1(L') = \{f_i t^b | 1 \le i \le r, 0 \le b \le \deg f_i\}$. Suppose this is not a minimal system of generators of J_1. Then there exists an equation

$f_i t^b = \sum_j \sum_k (f_j t^{b_{jk}})(g_{jk} t^{c_{jk}})$, where $b_{jk} \leq$deg f_j, $b_{jk} + c_{jk} = b$ and $f_j t^{b_{jk}} \neq f_i t^b$ for all j, k, and where all summands are bihomogeneous of degree (d, b) with d =deg f_i. Notice that the right hand sum contains no summand of the form $(f_i t^{b_{ik}})(g_{ik} t^{c_{ik}})$. In fact, otherwise we would have deg $g_{ik} t^{c_{ik}} = (0, b - b_{ik})$, and so $b_{ik} = b$ which is impossible. It follows that $f_i = \sum_{j \neq i} (\sum_k g_{jk}) f_j$, a contradiction to the minimality of L.

Now suppose that $L' \cup H$ is not a minimal system of generators of I_J. If one of the $f_i^{(k)}$ is a linear combination of the other elements of $L' \cup H$, then $\phi(L')$ is not a minimal system of generators of J_1, a contradiction. Next suppose one of the elements of H, say, $h = X_1 Y_2 - X_2 Y_1$, is a linear combination of the other elements of $L' \cup H$. Only the elements of bidegree $(2, 1)$ can be involved in such a linear combination. In other words,

$h = \sum \lambda_f f^{(1)} + \tilde{h}$ with $\lambda_f \in K$.

Here the sum is taken over all $f \in L$ of degree 2, and $\tilde{h}$ is a K-linear combination of the polynomials $X_i Y_j - X_j Y_i$ different from h. Since the monomial $X_2 Y_1$ does not appear in any polynomial on the right hand side of the equation, we get a contradiction.

Now we present an elementary Lemma useful in the next theorem.

Lemma 1.2 *The Hilbert function $H(R, -) : \mathbf{N} \to \mathbf{N}$ of R is given by $H(R, i) = (i+1) H(A, i)$, $i \in \mathbf{N}$, $H(A, -)$ being the Hilbert function of A. In particular, $e(R) = dim\ A\ \ e(A)$.*

Proof. We have $R_i = \oplus_{|u|+|v|=i} K X^u (Xt)^v = \oplus_{|u|+|v|=i} K X^{u+v} t^{|v|} = \oplus_{s=0}^{i} (\oplus_{|w|=i} K X^w) t^s$. Thus $\mathrm{H}(R, i) = (i+1)\mathrm{H}(A, i)$. Let $\mathrm{P}_A(z) = e(A) z^{d-1}/(d-1)! + \ldots$, d =dim A be the Hilbert polynomial of A (see [BH,4.1]). It follows that $\mathrm{P}_R(z) = (z+1)\mathrm{P}_A(z) = e(A)(z+1) z^{d-1}/(d-1)! + \ldots = de(A) z^d / d! + \ldots$. Since dim R =dim $A + 1$, we are done.

We will now compute a Gröbner basis of I_J.

Theorem 1.3 *Let $<$ be the reverse lexicographic order induced by $X_1 > \ldots > X_n > Y_1 > \ldots > Y_n$. If L is a Gröbner basis of J with respect to the term order $<$, then $L' \cup H$ is a Gröbner basis of I_J with respect to $<$.*

Proof. Let L be a Gröbner basis of J with respect to the reverse lexicographic order induced by $<$ on B. Then $L' \cup H$ is a Gröbner basis of I_J with respect to $<$ if the obvious inclusion $< in(L' \cup H) > \subset in(I_J)$ is an equality. For this aim it is enough to see that $\mathrm{H}(S/in(I_J), i)$ =$\mathrm{H}(S/ < in(L' \cup H) >, i)$ for all $i \in \mathbf{N}$. But $\mathrm{H}(S/in(I_J), i)$ =$\mathrm{H}(S/I_J, i) = \mathrm{H}(R, i) = (i+1)\mathrm{H}(A, i)$ by Macaulay Theorem [BH,4.2.4] and Lemma 1.2. Choose a monomial basis T of A. We need the following elementary lemma:

Lemma1.4 *T' is a monomial basis of $S/ < in(L' \cup H) >$ over K.*

Back to our proof note that $\mathrm{H}(S/ < in(L' \cup H) >, i) = |T'_i|$, where T'_i denotes the monomials of T' of degree i. If $u \in T_i$ then it gives exactly $(i+1)$-monomials $\{u^{(k)} | 0 \le k \le i\}$ in T'_i. Thus $|T'_i| = (i+1)|T_i| = (i+1)\mathrm{H}(A, i)$, which is enough.

We need the following lemma in the proof of Lemma 1.4.

Lemma 1.5 *Let $\mathcal{M}$ be the set of monomials of B. Then*

i) $\mathcal{M}'$ is a K-basis in $S/ < in(H) >$.

ii) If the linear K-space generated by $T \subset \mathcal{M}$ is an ideal in B then the linear K-space generated by T' in $S/ < in(H) >$ is an ideal too.

iii) Let $T \subset N \subset \mathcal{M}$. If N is contained in the ideal generated by T in B then N' is contained in the ideal generated by T' in S.

iv) Let $T, N \subset \mathcal{M}$. If $T \cap N = \emptyset$ then $T' \cap N' = \emptyset$.

Proof. i) Note that $in(H) = \{X_i Y_j | i > j\}$. By construction in $\mathcal{M}$ appear all monomials of type $X_1^{k_1} \cdots X_e^{k_e} Y_e^{s_e} \cdots Y_n^{s_n}$, these are exactly the monomials which are not divided by a monomial of type $X_i Y_j$ with $i > j$. But these are the monomials which are not in $< in(H) >$.

ii) An element of T' has the form $u^{(k)}$ for an $u \in T$, $0 \le k \le \deg u$ and it is enough to show that $X_i u^{(k)}$, $Y_j u^{(k)}$ belong to $T' + < in(H) >$. But if $X_i u^{(k)} \notin < in(H) >$ then as in i) it is contained in $\mathcal{M}'$ and moreover $X_i u^{(k)} = (X_i u)^{(k)} \in T'$ since $X_i u \in T$ by hypothesis. Similarly, if $Y_j u^{(k)} \notin < in(H) >$ then $Y_j u^{(k)} = (X_j u)^{(k+1)} \in T'$.

iii) Let $u^{(k)} \in N'$ for some $u \in N$, $0 \le k \le \deg u$. By hypothesis $u = vw$ for a $v \in T$ and a $w \in \mathcal{M}$. Then $u^{(k)} = v^{(s)} w^{(k-s)}$ for some $0 \le s \le k$ and so $u^{(k)}$ belongs to the ideal generated by T' in S.

iv) Let $\psi : S \to B$ be the retraction of $B \subset S$ given by $Y \to X$. Then $\psi(T') = T$ for $T \subset \mathcal{M}$. If $T' \cap N' \neq \emptyset$ then $\psi(T' \cap N') \subset \psi(T') \cap \psi(N') = T \cap N$ and so $T \cap N \neq \emptyset$.

Proof of Lemma 1.4 Let $D \subset \mathcal{M}$ be the set of monomials from $in(J)$ and $C = in(L)$. By hypothesis we have $T \cup D = \mathcal{M}$ and $T \cap D = \emptyset$ and using Lemma 1.5 i),iv) we get $T' \cup D' = \mathcal{M}$ is a K-basis in $S/ < in(H) >$ and $T' \cap D' = \emptyset$. Thus T' is a K-basis in $S/ < D', in(H) >$ because the linear K-space generated by D' in $S/ < in(H) >$ is an ideal by Lemma 1.5 ii). But $in(L' \cup H) >=< D', in(H) >$ by Lemma 1.5 iii), which is enough.

Corollary 1.6 *If J has a quadratic Gröbner basis, then so does I_J.*

We would like to remark that if L is a reduced Gröbner basis, then $L' \cup H$ need not be reduced as shows the following:

Example 1.7 Let $A = K[X_1, X_2, X_3]/(X_1X_2 - X_3^2)$. Then $L = \{X_1X_2 - X_3^2\}$ is a reduced Gröbner basis of J, but $L' \cup H$ is not reduced, since $X_1Y_2 = in(X_1Y_2 - X_3Y_3)$ appears in $X_1Y_2 - X_2Y_1$.

2. Depth of Rees algebras

As above, let $B = K[X]$, $A = B/J = K[x]$, $x = (x_1, \ldots, x_n)$, $S = K[X, Y]$, $R = S/I_J = K[x, y] \subset A[t]$, where $y = xt$.

Lemma 2.1 (after [GS, 2.7]) *Suppose $x_1, \ldots, x_r$, $r \geq 1$ is a regular sequence on A and let $f_i := x_i - y_{i-1}$, $1 \leq i \leq r$, $y_0 = 0$. Then the sequences $\{f_1, \ldots, f_r\}$, $\{f_1, \ldots, f_{r-1}, y_r\}$ are regular on R. In particular depth $R \geq$depth A.*

Proof. Apply induction on r. Clearly $x_1 = f_1$ is regular on $R \subset A[t]$ and by symmetry y_1 is too. Suppose $r > 1$. Let

$0 \to (x_r) \to R/(y_r) \to R/(x_r, y_r) \to 0$

be the canonical exact sequence. We have $x_rR/(y_r) \cong (X_r, Y_r, I_J)/(Y_r, I_J) \cong (X_r)/(X_r) \cap (Y_r, I_J) \cong S/((Y_r, I_J) : X_R)(-1)$. Note that $((Y_r, I_J) : X_r) \supset (Y_1, \ldots, Y_n)$ because $X_rY_j - X_jY_r \in I_J$. Thus $((Y_r, I_J) : X_r) = (Y_1, \ldots Y_n, (J : X_r)) = (Y, J)$, x_r being regular on B. Hence $x_rR/(y_r) \cong S/(Y, J)(-1) \cong A(-1)$ which yields the following exact sequence:

(*) $0 \to A(-1) \to R/(y_r) \to R/(x_r, y_r) \to 0$.

By induction hypothesis, we have $\{f_1, \ldots, f_{r-1}\}$ regular on $R/(x_r, y_r)$. Since $\{f_1, \ldots, f_{r-1}\}$ acts on A as $\{x_1, \ldots, x_{r-1}\}$ it is also regular on A and so on $R/(y_r)$ by (*). Since x_r is regular on A it is also regular on R as well as y_r (see case $r = 1$). Thus $\{f_1, \ldots, f_{r-1}, y_r\}$ is regular on R.

Suppose that $\{f_1, \ldots, f_r\}$ is not regular on R. Then there exists a prime ideal $P \subset R$ associated to $(f_1, \ldots, f_{r-1})$ and containing f_r. Since $\{f_1, \ldots, f_{r-1}, y_r\}$ is regular it follows $y_r \notin P$. We claim that $P \supset (x_1, \ldots, x_n)$. Otherwise, let $x_j \notin P$ for a $1 \leq j \leq n$. By induction on $1 \leq i \leq r$ we see that $j > i$ and $(x_1, \ldots x_i, y_1, \ldots, y_i) \subset PR_P$. Indeed, if $i = 1$ then $x_1 = f_1 \in P$ and so $j > 1$ and $x_jy_1 = x_1y_j \in PR_P$. Thus $y_1 \in PR_P$. Suppose $1 < i \leq r$. By induction hypothesis on i we have $j > i - 1$ and $(x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1}) \subset PR_P$. Since $f_e \in P$, $1 \leq e \leq r$ it follows $x_i \in PR_P$. Thus $j > i$ and $y_i \in PR_P$ because $x_jy_i = x_iy_j \in PR_P$. This completes our induction on i. It follows $y_r \in PR_P$ which is a contradiction.

Then $P \supset (x_1, \ldots, x_n, y_1, \ldots, y_{r-1})$ since $f_i \in P$. By induction hypothesis on r we have $\{f_2, \ldots, f_r\}$ regular on $R/(x_1, y_1)$. It follows depth $(R/(x_1, y_1))_P \geq r-1$ because $P \supset (f_2, \ldots, f_r, x_1, y_1)$. But $(R/(y_1))_P \cong (R/(x_1, y_1))_P$ because $y_r x_1 = x_r y_1 \in (y_1)$ and $y_r \notin P$. Thus depth $(R/(y_1))_P \geq r-1$ and so depth $(R_P) \geq r$ since y_1 is regular on R. This contradicts the choice of P as associated to $(f_1, \ldots, f_{r-1})$. Hence $\{f_1, \ldots, f_r\}$ is regular on R.

Remark 2.2 Note that $y_r(x)^r = (y)x_r(x)^{r-1} \equiv (y)y_{r-1}(x)^{r-1} \equiv \cdots \equiv (y)^r x_1 \equiv 0$ modulo $(f_1, \ldots, f_r)$. Thus if $\{f_1, \ldots, f_r, y_r\}$ would be regular on R then $(x)^r \subset (f_1, \ldots, f_r)$ and so $(x_1, \ldots, x_r)$ would be a m-primary ideal in A. Thus dim A =depth $A = r$. This is exactly the Cohen-Macaulay case investigated in [GS]. Here we are interested especially in the case when A is not Cohen-Macaulay.

Lemma 2.3 *Suppose depth $A = r$ and $x_1, \ldots x_s$ is regular for a s, $1 \leq s \leq r$ in A and depth $R/(x_1, \ldots, x_s, y_1, \ldots, y_s) \neq r - s$. Then depth $R = r + 1$.*

Proof. Apply induction on s. If $s = 1$ then we consider the exact sequence (*) from the proof of 2.1

$0 \to A(-1) \to R/(y_1) \to R/(x_1, y_1) \to 0.$

By Lemma 2.1 and our hypothesis we have depth $R/(x_1, y_1) \geq r$. As depth $A = r$ we obtain

depth $R/(y_1) \geq$ min {depth A, depth $R/(x_1, y_1)$} $\geq r$.

On the other hand $r =$ depth $A \geq$ min {depth $R/(y_1), 1+$ depth $R/(x_1, y_1)$} implies necessarily depth $R/(y_1) = r$ and so depth $R = r + 1$, y_1 being regular on R.

Suppose now $s > 1$. By induction hypothesis we get then depth $R/(x_s, y_s) = r > r - 1$. Applying again the case $s = 1$ it follows depth $R = r + 1$.

From Lemma 2.3 it follows

Proposition 2.4 *Suppose that depth $A \neq$depth R. Then the Rees algebra R' of $A[X']$, $X' = (X'_1, \ldots, X'_s)$ has depth $R' =$ depth $A + s + 1$.*

Lemma 2.5 *Let R' be the Rees algebra of $A' = A[X']$-the polynomial A-algebra in one variable X', r =depth A, $x_1, \ldots, x_r$ a regular sequence on A and $f_1, \ldots, f_r$ defined as in 2.1. Suppose that depth $R = r <$dim A. If depth $R' \neq$depth A' then depth R' =depth $A' + 1$.*

Proof. Note that $R'/(Y') \cong R[X']/(X'y)$. By Lemma 2.1 $\{f_1, \ldots, f_r, Y'\}$ forms a regular sequence in R' and so depth $R'/(Y') \geq r$. By hypothesis r =depth R and so depth $\bar{R} = 0$, where $\bar{R} := R/(f_1, \ldots, f_r)$. As dim $A > r$ we see that depth $(\bar{R}/H^0_{(x,y)}(\bar{R})) > 0$, where

$H^0_{(x,y)}(\bar{R}) = \{v \in \bar{R} | \mathrm{Ann}_{\bar{R}} v$ is (x,y)-primary$\}$.

Choose a homogeneous element $u \in R$ which is regular on $\bar{R}/H^0_{(x,y)}(\bar{R})$. We may change A by $A \otimes_K K(Z)$ in order to suppose K infinite. By [BH,1.5.12] we may take u of degree 1.

We claim that $X' - u$ is regular on $\bar{R}[X']/(X'y)$. Indeed, if $q \in R$ satisfies $(X' - u)q \in (X'y)$ in $\bar{R}[X']$ then $X'q$, uq are zero in $\bar{R}[X']/(X'y)$. Then $q \in H^0_{(x,y)}(\bar{R})$, u being regular on $\bar{R}/H^0_{(x,y)}(\bar{R})$. Since $X'q$ is zero in $\bar{R}[X']/(X'y)$ we see that $q \in H^0_{(x,y)}(\bar{R}[X']/(X'y))$. But depth $R' \neq$depth A' by hypothesis. Then depth $R' > r + 1$ and so depth $\bar{R}[X']/(X'y) > 0$. It follows that $H^0_{(x,y)}(\bar{R}[X']/(X'y)) = 0$ and so q is zero in $\bar{R}[X']/(X'y)$, that is q is zero in $\bar{R}$ (apply to q the retraction $\bar{R}[X']/(X'y) \to \bar{R}$, $X' \to 0$ of the inclusion).

Now, let $q = \sum_{i=0}^{e} q_i X'^i \in R'[X']$ be such that $(X' - u)q = 0$ in $\bar{R}[X']/(X'y)$. The expression of q as a polynomial in $R[X']/(X'y)$ could be "unique" if we ask for $i > 0$ either $q_i \notin (y)$, or $q_i = 0$. From $(X' - u)q = \sum_{i=0}^{e+1}(q_{i-1} - q_i u)X'^i = 0$ it follows $q_i = 0$ for $1 \leq i \leq e$ by "unicity" of the expression of q. So we may suppose $q = q_0 \in R$, which was already settled. Note that $R'/(Y', X' - u) \cong R/(uy)$ and let $v \in R$ be inducing a nonzero element in $H^0_{(x,y)}(\bar{R})$. As above $v \notin y\bar{R}$ because otherwise $v \in H^0_{(x,y)}(\bar{R}[X']/(X'y)) = 0$. But then v induces a nonzero element in $H^0_{(x,y)}(\bar{R}/(uy))$, i.e. depth $\bar{R}/(uy) = 0$. Hence depth $R' = r + 2$, a regular sequence being $f_1 \ldots, f_r, Y', X' - u$.

Theorem 2.6 *Let R' be the Rees algebra of $A[X']$, $X' = (X'_1, \ldots, X'_s)$, $s \geq 1$. Then depth $A[X'] \leq$depth $R' \leq$depth $A[X'] + 1$.*

The proof follows from Lemma 2.1, Proposition 2.4 and Lemma 2.5 applied recursively.

Example 2.7 Let u, v be two algebraically independent elements over K and $A := K[u^4, u^3v, uv^3, v^4]$. Then dim $A = 2$ and $A \cong K[X_1, \ldots, X_4]/J$, where $J = (X_1X_4 - X_2X_3, X_3^3 - X_2X_4^2, X_2^2X_4 - X_1X_3^2, X_1^2X_3 - X_2^3)$. We see that $X_2^2(X_2, X_3, X_4) \subset (X_1) + J$ and so X_1 is maximal regular sequence in A, that is depth $A = 1$. By Proposition 1.1 we have $R = S/I_J$, where $S = K[X, Y]$, $I_J = J + J(Y) + T + H$, $J(Y)$ being obtained from J changing X by Y, H being as in 1.1, and $T = (X_1Y_4 - X_2Y_3, X_3^2Y_3 - X_2X_4Y_4, X_2^2Y_4 - X_1X_3Y_3, X_1^2Y_3 - X_2^2Y_2, X_3Y_3^2 - X_2Y_4^2, X_2Y_2Y_4 - X_1Y_3^2, X_1Y_1Y_3 - X_2Y_2^2)$. Then $x_2^2(x_2, x_3, x_4, y_1, y_2, y_3, y_4)$
$\subset (x_1)$ and so depth $R/(x_1) = 0$. Thus depth $A =$depth $R = 1$. It is not difficult to show that depth $R' = 2 =$depth A', but the Rees algebra R'' of $A'' := A[X', X'']$ has depth $= 1+$depth $A'' = 4$.

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