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# EXAMPLES OF STRING-THEORETIC EULER NUMBERS

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**Abstract**

This note gives a short motivation to study stringy invariants and presents results on string theoretic Euler numbers of the 3-dimensional simple singularities, obtained by D. I. Dais and the author in [8].

## 0. Introduction

Deformation an resolution of singularities is one of the central topics in algebraic geometry. The local question to study isolated singularities is closely related to the the investigation of properties of their resolutions, which is of global nature. The local part can be well formulated in terms of commutative algebra, for the other geometric techniques related to cohomology show their strength.

Let $X$ be a Calabi-Yau variety, i.e. a $d$-dimensional, compact complex variety such that $\omega_X \cong \mathcal{O}_X$, $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, d-1$, which has at most canonical singularities. If $f : \tilde{X} \to X$ is a crepant, projective resolution, then an expected mirror partner $\tilde{Y} \to Y$ of $f$ should have the property that Hodge numbers satisfy the condition

$$h^{p,q}(\tilde{X}) = h^{d-p,q}(\tilde{Y}), \quad 0 \leq p, q \leq d.$$

It is reasonable to ask whether $h^{p,q}(\tilde{X})$ are independent on the resolution $\tilde{X} \to X$. The affirmative answer results from the work of Batyrev and Kontsevich (see [2]). Unfortunately, there exist examples of Calabi-Yau varieties which do not admit a crepant resolution.
Working in the singular category, it is possible to define string-theoretic Hodge invariants which arise from appropriate correction terms in the neighbourhood of singularities.

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## 1. The definitions

Let $X$ be a complex variety, not necessarily compact nor smooth. The cohomology groups $H_c^k(X,\mathbb{C})$ with compact support have a natural mixed Hodge structure which gives rise to the $E$-polynomial

$$E(X;u,v) := \sum_{0\le p,q\le d} e^{p,q}(X)u^p v^q, \quad e^{p,q}(X) := \sum_{0\le k\le 2d} h^{p,q}(H_c^k(X,\mathbb{C}))$$

of $X$. This polynomial encodes invariants like $e(X) = E(X;1,1)$, the topological Euler characteristic of $X$. Now suppose $X$ has at most log-terminal singularities and $f : \tilde{X} \to X$ is a resolution of singularities having exceptional divisors $D_i$ ($i \in I = \{1,\dots,r\}$) which are smooth, irreducible with only normal crossings. Then the string-theoretic $E$-function of $X$ is defined by

$$E_{str}(X;u,v) := \sum_{J\subseteq I} E(D_j^o;u,v) \prod_{j\in J} \frac{uv-1}{(uv)^{a_j+1}-1},$$

where $a_j$ are the discrepancy coefficients,

$$K_{\tilde{X}} - f^*K_X = \sum_{i\in I} a_i D_i, \quad a_i > -1,$$

and $D_J := \cap_{j\in J} D_j$ for $J \neq \emptyset$, $D_\emptyset := \tilde{X}$ and $D_J^o := D_J - \cup_{i\notin J} D_i$. The rational number

$$e_{str}(X) := \lim_{u,v\to 1} E_{str}(X;u,v) = \sum_{J\subseteq I} e(D_j^o) \prod_{j\in J} \frac{1}{a_j+1}$$

is said to be the string-theoretic Euler number of $X$, and the natural number

$$\mathrm{ind}_{str}(X) := \min\{q \mid e_{str}(X) = \frac{p}{q},\ p,q \in \mathbb{Z},\ q>0\}$$

is defined to be the stringtheoretic Index of $X$. In fact, by Batyrev's results, $E_{str}$ is independent on the choice of the resolution, which gives a good sense to the definition of $e_{str}(X)$ and $\mathrm{ind}_{str}(X)$.

Motivated by the examples, in 1997 Batyrev asked the question whether (among all relevant $X$) the number $\mathrm{ind}_{str}(X)$ is bounded by a constant, depending only on the dimension of $X$.

## 2. Stringy invariants of 3-dimensional ADE's

Let $(X, x)$ be a simple, 3-dimensional hypersurface singularity, i.e. one of the singularities of type ADE ([1],[14]). Their canonical resolutions $f : \tilde{X} \to X$ are known since a long time (cf. [16], [12]). Using detailed information on those resolutions, all discrepancy coefficients can be obtained. The $E$-polynomials are

$$E(\tilde{X} - \{x\}; u, v) = E(D^o_\emptyset; u, v) = (uv)^d (E(L; u^{-1}, v^{-1});$$

they only on the Link $L$ of the singularity $(X, x)$. For the given case, this is nothing but

$$E\left(X - \{x\}; u, v\right) = (uv - 1)\ \left[1 + \left(1 + h^{1,1}\left(H^2(L, \mathbb{C})\right)\right)\ uv + (uv)^2\right].$$

From the resolution data, remaining terms of $E_{str}$ are obtained by calculations on the surfaces $D_i$. This gives rise to a complete list of the algebraic functions $E_{str}(X; u, v)$. Especially, we obtain that among $A_n$-singularities, as well as for $D_n$, the number $\mathrm{ind}_{str}$ has arbitrarily large values, thus answering Batyrev's question.

Here is an example how the results look like. Let $X = A_n$ be defined by the equation $x_1^{n+1} + x_2^2 + x_3^2 + x_4^2$ in 4-space, $n \geq 1$ and $x$ the origin. Then

(i) $n$ even implies

$$E_{str}\left(X; u, v\right) = w^3 + w - 1 + \sum_{i=2}^{\frac{n}{2}} \frac{(w-1)\left(w^2-1\right)}{w^{i+1}-1} + \frac{(w-1)w^2}{w^{n+3}-1}$$
$$+ (w-1)\left(w^2-1\right)\left[\sum_{i=1}^{\frac{n}{2}-1} \frac{1}{(w^{i+1}-1)(w^{i+2}-1)} + \frac{1}{(w^{\frac{n}{2}+1}-1)(w^{n+3}-1)}\right],$$

(ii) $n$ odd, $n > 1$ implies

$$E_{str}\left(X; u, v\right) = (w-1)\ (w+1)^2 + w$$
$$+ \left(w^2-1\right)\left[\sum_{i=2}^{\frac{n-1}{2}} \frac{(w-1)}{w^{i+1}-1} + \frac{w}{w^{\frac{n+3}{2}}-1} + \sum_{i=1}^{\frac{n-1}{2}} \frac{(w-1)}{(w^{i+1}-1)(w^{i+2}-1)}\right].$$

The stringy Euler numbers are obtained as

(i) $e_{str}(X) = 2 - \dfrac{3}{n+3}$, and

(ii) $e_{str}(X) = 2$, respectively.

Therefore, the following values for string theoretic indices of $A_n$ are obtained,

$$\mathrm{ind}_{str}(X) = \begin{cases} 1, & \text{if } n \equiv 1 \,(\mathrm{mod}\ 2) \\ n+3, & \text{if } n \equiv 2 \text{ or } 4 \,(\mathrm{mod}\ 6) \\ \frac{n}{3}+1, & \text{if } n \equiv 0 \,(\mathrm{mod}\ 6). \end{cases}$$

### 3. Application

Globally, from the previous results string-theoretic Euler numbers can be obtained for complex 3-folds with known Hodge numbers, if they have only ADE-singularities. Among them, there are symmetric hypersurfaces like Segre's cubic ($e_{str} = 14$), Burkhart's quartic [6], where $e_{str} = 34$, van Straten's quintic [20], where $e_{str} = 60$, or Knörrer's quadric's [15] with $e_{str} = 6$ and $e_{str} = 8\frac{41}{864}$, respectively.

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