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LADDER FUNCTORS WITH AN APPLICATION TO REPRESENTATION-FINITE ARTINIAN RINGS

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Introduction

Ladders were introduced by Igusa and Todorov for the investigation of representation-finite artinian algebras and algebras over an algebraically closed field [7]. They prove a radical layers theorem [7] which exhibits the graded structure of Auslander-Reiten sequences. In a second article [8] they obtain a characterization of the Auslander-Reiten quivers of representation-finite artinian algebras. Their construction of ladders starts with an irreducible morphism $f_0\colon A_0 \to B_0$ in a module category $\mathcal{A}$. So f_0 factors through a right almost split map $u\colon \vartheta B_0 \to B_0$. Assume that $f_0 = ug$ with a split monomorphism g. Then g can be written as $g = \binom{1}{0}$ with respect to a decomposition $\vartheta B_0 = A_0 \oplus B_1$. This gives a pullback

$$\begin{array}{ccc} A_1 & \longrightarrow & A_0 \\ \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle f_0} \\ B_1 & \longrightarrow & B_0 \end{array}$$

which completes the first step of a ladder. Under favorite circumstances, the ladder can be extended. In the given situation, Igusa and Todorov [7] solved

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the extension problem by a careful examination of the bimodules of irreducible maps between objects.

Recently, Iyama [9] improved the construction as follows. Let $\mathcal{A}$ be a category with left and right almost split sequences (see §1). He calls a morphism f_0 in $\mathcal{A}$ *special* if for each morphism $r\colon A_0 \to B_0$ in $\operatorname{Rad}^2\mathcal{A}$, $f_0 + r$ is isomorphic to f_0 as a two-termed complex. Then it follows in a quite elementary way that each step f_n of the ladder, after splitting off trivial complexes $X \to 0$, admits a continuation f_{n+1} which is again special. Such ladders have been far-reaching enough to get a solution of Igusa and Todorov's problem in dimension one. Namely, they yield a characterization of the Auslander-Reiten quivers of representation-finite orders over a complete discrete valuation domain [10].

In [19] we modify the theory of ladders in such a way that a functorial approach becomes possible. Apart from being functorial, this method has a two-fold advantage. Firstly, it applies to arbitrary morphisms $f_0 \in \operatorname{Rad}\mathcal{A}$, and secondly, it provides a kind of ladders with the property that the commutative squares between two steps are pullbacks and pushouts. Therefore, our ladders establish a bridge between almost split sequences and arbitrary short exact sequences.

In the present article, the method will be applied to the artinian situation. This gives a quick proof of Igusa and Todorov's characterization of the Auslander-Reiten quivers belonging to representation-finite artinian algebras. More generally, every cotilting module ${}_\Lambda U$ over a left artinian ring Λ defines a full subcategory $\mathbf{lat}(U)$ of Λ-**mod**, consisting of the Λ-modules $M \in \Lambda$-**mod** finitely cogenerated by U. For example, the category of representations of a poset in the sense of Nazarova, Roĭter [11], and Gabriel [6], and (generalized) vector space categories [20], are of that type. For a ring R, let R-**proj** denote the category of finitely generated projective left R-modules. We prove that the categories $\mathbf{lat}(U)$ with finitely many indecomposable objects can be characterized by two properties: They are equivalent to R-**proj** for an artinian ring R; and they have left and right almost split sequences for all of their objects.

1 τ-Rings and strict τ-categories

An additive category $\mathcal{A}$ is said to be a *Krull-Schmidt* category, if every object is a finite direct sum of objects with local endomorphism rings. Then the ideal $\operatorname{Rad}\mathcal{A}$ generated by the non-invertible morphisms between indecomposable objects in $\mathcal{A}$ is called the *radical* of $\mathcal{A}$. A morphism $f\colon A \to B$ in $\mathcal{A}$ is said to be *right* (*left*) *almost split* [4] if $f \in \operatorname{Rad}\mathcal{A}$, and every morphism $C \to B$ in (resp. $A \to C$) in $\operatorname{Rad}\mathcal{A}$ factors through f. The class of indecomposable objects will be denoted by $\operatorname{Ind}\mathcal{A}$, and $\operatorname{ind}\mathcal{A}$ will be a fixed representative system of the isomorphism classes in $\operatorname{Ind}\mathcal{A}$. If $\operatorname{ind}\mathcal{A}$ is finite for a Krull-Schmidt category $\mathcal{A}$, then $R := \operatorname{End}(\bigoplus \operatorname{ind}\mathcal{A})^{\mathrm{op}}$ is a semiperfect ring with $\mathcal{A} \approx R$-**proj**, the category of finitely generated projective left R-modules. Note that the functor $P \mapsto P^* := \operatorname{Hom}_R(P, R)$ provides a natural duality

$$(R\text{-}\mathbf{proj})^{\mathrm{op}} \approx R^{\mathrm{op}}\text{-}\mathbf{proj}. \tag{1}$$

We define a *τ-ring* as a semiperfect ring R such that, as a left or right R-module, $\operatorname{Rad} R$ satisfies the following conditions:

$$\left.\begin{array}{l}\operatorname{Rad} R \text{ is finitely presented}\\ \operatorname{pd}(\operatorname{Rad} R) \leqslant 1\\ \operatorname{Ext}_R(\operatorname{Rad} R, R) \text{ is semisimple.}\end{array}\right\} \tag{2}$$

This means that every simple R-module S has a minimal projective resolution

$$0 \to P_2 \xrightarrow{v} P_1 \xrightarrow{u} P_0 \twoheadrightarrow S \tag{3}$$

in $\mathcal{A} := R$-**proj** (resp. $\mathcal{A} := R^{\mathrm{op}}$-**proj**) such that $u, v \in \mathcal{A}$ have the following properties:

$$\left.\begin{array}{l}v = \ker u\\ u \text{ is right almost split}\\ v \text{ is left almost split.}\end{array}\right\} \tag{4}$$

A complex $P_2 \xrightarrow{v} P_1 \xrightarrow{u} P_0$ in a Krull-Schmidt category $\mathcal{A}$ that satisfies (4) is said to be a *right almost split sequence* for P_0. In a dual way, *left almost split sequences* are defined. So the definition of a τ-ring just states that R-**proj** has left and right almost split sequences for each of its objects. Krull-Schmidt categories with this property are known as *strict τ-categories* [9]. Since a right almost split sequence for an object A is unique up to isomorphism, it will be denoted by

$$\tau A \xrightarrow{v_A} \vartheta A \xrightarrow{u_A} A. \tag{5}$$

Similarly, a left almost split sequence for A is denoted by

$$A \xrightarrow{u^A} \vartheta^- A \xrightarrow{v^A} \tau^- A. \tag{6}$$

More generally, for a morphism $f\colon A \to B$ in a Krull-Schmidt category $\mathcal{A}$, we call $k\colon K \to A$ a *weak kernel* if $fk = 0$ and every morphism $k'\colon K' \to A$

with $fk' = 0$ factors through k. If, in addition, each $g\colon C \to K$ with $kg = 0$ lies in $\operatorname{Rad}\mathcal{A}$, then k is unique up to isomorphism (see [16], Proposition 7), and we write $\operatorname{wker} f := k$. If a sequence (5) satisfies (4) except that $\ker u$ is replaced by $\operatorname{wker} u$, we speak of a *right τ-sequence* for A. In a dual way, *weak cokernels*, $\operatorname{wcok} f$, and *left τ-sequences* (6) are defined. A Krull-Schmidt category with left and right τ-sequences for each of its objects is said to be a *τ-category* [9].

Proposition 1 ([9], 2.3). *Let R be a τ-ring, and let S be a simple R-module with* $\operatorname{pd} S = 2$. *Then* $\operatorname{Ext}^i_R(S, R) = 0$ *for $i < 2$, and* $\operatorname{Ext}^2_R(S, R)$ is simple.

Proof. For a minimal projective resolution (3) of S, consider the projective resolution
$$0 \to P^* \xrightarrow{i^*} P_1^* \xrightarrow{v^*} P_2^* \twoheadrightarrow \operatorname{Ext}^2_R(S, R)$$
of the semisimple R-module $\operatorname{Ext}^2_R(S, R)$. Then $u^* = i^* p^*$ for some $p\colon P \to P_0$, and $u = pi$. This gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P_2 \arrow[r, "v"] \arrow[d, equal] & P_1 \arrow[r, "i"] \arrow[d, equal] & P \arrow[r, two heads] \arrow[d, "p"] & C \arrow[d, dashed, "e"] \\
0 \arrow[r] & P_2 \arrow[r, "v"] & P_1 \arrow[r, "u"] & P_0 \arrow[r, two heads, "c"] & S
\end{tikzcd}$$

with $C := \operatorname{Ext}^2_R(\operatorname{Ext}^2_R(S, R), R)$, where the horizontal sequences are projective resolutions. Our assumption $v \neq 0$ implies that $cp \neq 0$. Hence e is epic, and so S is a direct summand of the semisimple R-module C. Since $\operatorname{Ext}_R(C, R) = 0$, we infer that $\operatorname{Ext}_R(S, R) = 0$. Moreover, $cp \neq 0$ implies that p is a split epimorphism. Hence u^* is monic, and $\operatorname{Ext}_R(S, R) = 0$ shows that $u^* = \ker v^*$. Thus e is an isomorphism. Since the complex (3) is indecomposable, this completes the proof. $\square$

Proposition 1 shows that any right almost split sequence $P_2 \to P_1 \to P_0$ with P_0 indecomposable and $P_2 \neq 0$ is left almost split with P_2 indecomposable.

2 Ladder functors

An additive category $\mathcal{A}$ is said to be *preabelian* if every morphism in $\mathcal{A}$ has a kernel and a cokernel. Kernels (cokernels) in $\mathcal{A}$ will be depicted by $\rightarrowtail$

(resp. $\twoheadrightarrow$). Monic and epic morphisms will be called *regular*. A sequence of morphisms

$$A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$$

in $\mathcal{A}$ with $a = \ker b$ and $b = \operatorname{cok} a$ is said to be *short exact*. Since every commutative square

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ C & \xrightarrow{d} & D \end{array} \tag{7}$$

in $\mathcal{A}$ corresponds to a complex

$$A \xrightarrow{\binom{a}{-b}} B \oplus C \xrightarrow{(c\ d)} D, \tag{8}$$

we call (7) a *left* (*right*) *almost split square* resp. a *left* (*right*) *τ-square* if the corresponding property holds for (8). We call (7) *exact* if (8) is a short exact sequence. An object $Q \in \mathcal{A}$ is said to be *projective* (*injective*) if the functor $\operatorname{Hom}_{\mathcal{A}}(Q, -)$ (resp. $\operatorname{Hom}_{\mathcal{A}}(-, Q)$) preserves short exact sequences. The full subcategories of projective (injective) objects will be denoted by $\mathbf{Proj}(\mathcal{A})$ (resp. $\mathbf{Inj}(\mathcal{A})$). We say that $\mathcal{A}$ has *strictly enough projectives* (*injectives*) [14] if for each object $A \in \mathcal{A}$ there is a cokernel $P \twoheadrightarrow A$ with $P \in \mathbf{Proj}(\mathcal{A})$ (resp. a kernel $A \rightarrowtail I$ with $I \in \mathbf{Inj}(\mathcal{A})$).

Let $\mathcal{A}$ be a Krull-Schmidt category. The morphisms in $\mathcal{A}$ form an additive category $\operatorname{Mor}(\mathcal{A})$ with morphisms φ: $b \to c$ given by commutative squares (7). Let $[\mathcal{A}]$ be the ideal of morphisms φ: $b \to c$ in $\mathcal{A}$ which are homotopic to zero, i. e. for which there exists a morphism h: $C \to B$ in $\mathcal{A}$ with $a = hb$ and $d = ch$. It is easy to see that $[\mathcal{A}]$ consists of the morphisms which factor through an object 1_E: $E \to E$ in $\operatorname{Mor}(\mathcal{A})$. Every object of $\operatorname{Mor}(\mathcal{A})$ is isomorphic to $e \oplus 1_E$ for some $e \in \operatorname{Rad}\mathcal{A}$. Therefore, the homotopy category $\operatorname{Mor}(\mathcal{A})/[\mathcal{A}]$ is equivalent to a full subcategory $\mathsf{M}(\mathcal{A})$, consisting of the objects $e \in \operatorname{Mor}(\mathcal{A})/[\mathcal{A}]$ with $e \in \operatorname{Rad}\mathcal{A}$. There are two natural full embeddings $(\ )^+$: $\mathcal{A} \hookrightarrow \mathsf{M}(\mathcal{A})$ and $(\ )^-$: $\mathcal{A} \hookrightarrow \mathsf{M}(\mathcal{A})$ which map an object $A \in \mathcal{A}$ to A^+: $0 \to A$ and A^-: $A \to 0$, respectively. So we have two full subcategories $\mathcal{A}^+$ and $\mathcal{A}^-$ of $\mathsf{M}(\mathcal{A})$ which are equivalent to $\mathcal{A}$:

$$\mathcal{A}^+ \hookrightarrow \mathsf{M}(\mathcal{A}) \hookleftarrow \mathcal{A}^-. \tag{9}$$

By $\operatorname{Rad}^+\mathsf{M}(\mathcal{A})$ (resp. $\operatorname{Rad}^-\mathsf{M}(\mathcal{A})$) we denote the ideal of morphisms $b \to c$ in $\mathsf{M}(\mathcal{A})$ given by a commutative square (7) with $d \in \operatorname{Rad}\mathcal{A}$ (resp. $a \in \operatorname{Rad}\mathcal{A}$).

Lemma 1. *Let $\mathcal{A}$ be a Krull-Schmidt category. A morphism φ: $b \to c$ in $\mathsf{M}(\mathcal{A})$ given by (7) is invertible if and only if (8) is a split short exact sequence.*

Proof. Assume first that (8) is a split short exact sequence. Then there are morphisms $\binom{e}{g}$: $D \to B \oplus C$ and $(f\ \ -h)$: $B \oplus C \to A$ with

$$(c\ d)\tbinom{e}{g} = 1, \quad (f\ \ -h)\tbinom{a}{-b} = 1, \quad \tbinom{a}{-b}(f\ \ -h) + \tbinom{e}{g}(c\ d) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right). \tag{10}$$

This gives six equations in $\mathcal{A}$. Five of these equations, except $ah = ed$, imply that

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "c"] & A \arrow[d, "b"] \\ D \arrow[r, "g"] & C \end{tikzcd} \tag{11}$$

is an inverse of φ. Conversely, let (11) be an inverse of φ. Then there are morphisms e: $D \to B$ and h': $C \to A$ with

$$\begin{aligned} 1 - af &= ec \qquad & 1 - dg &= ce \\ 1 - fa &= h'b \qquad & 1 - gd &= bh'. \end{aligned} \tag{12}$$

Since $b, c \in \operatorname{Rad} \mathcal{A}$, this implies that a and d are isomorphisms. Hence (8) is a split short exact sequence. $\square$

Remark. Without use of the Krull-Schmidt property, the proof can be completed as follows. Equations (12) remain valid if we replace h' by $h := h' - f(ah' - ed)$. In fact,

$f(ah' - ed)b = fah'b - fedb = fa(1 - fa) - feca = f(1 - af - ec)a = 0$ and
$bf(ah' - ed) = bfah' - bfed = b(1 - h'b)h' - gced = bh'gd - g(1 - dg)d = 0.$

Now (10) follows, since $ah - ed = ah' - ed - af(ah' - ed) = (1 - af)(ah' - ed) = ec(ah' - ed) = ecah' - eced = edbh' - e(1 - dg)d = ed(1 - gd) - e(1 - dg)d = 0.$ $\square$

Let $\mathcal{A}$ be a strict τ-category, and let a: $A_1 \to A_0$ be an object in $\mathsf{M}(\mathcal{A})$. Any decomposition $A_0 = C \oplus P$ defines a morphism π_C: $a \to \overline{a}$, given by a commutative square

$$\begin{tikzcd} A_1 \arrow[r, "1"] \arrow[d, "a"] & A_1 \arrow[d, "\overline{a}"] \\ C \oplus P \arrow[r, "(1\ 0)"] & C. \end{tikzcd}$$

In [19] we define a morphism

$$\lambda_{C,a} : L_C a \to a \tag{13}$$

in $\mathsf{M}(\mathcal{A})$ with the following universal property:

$$(U)\begin{cases}\pi_C\lambda_{C,a} \in \mathrm{Rad}^+\mathsf{M}(\mathcal{A}), \text{ and for every } \varphi\colon\ x \to a \text{ with } \pi_C\varphi \in \mathrm{Rad}^+\mathsf{M}(\mathcal{A}) \\ \text{there is a unique factorization } \varphi = \lambda_{C,a}\varphi'.\end{cases}$$

Let us repeat the construction of (13). For any decomposition $A_1 = B \oplus U$, we can write a as a matrix $a = \left(\begin{smallmatrix} b & r \\ s & q \end{smallmatrix}\right)\colon\ B \oplus U \to C \oplus P$. We choose U as a maximal direct summand of A_1 such that $r \in \mathrm{Rad}^2\mathcal{A}$. Then we have a right almost split square

$$\begin{tikzcd} C' \arrow[r, "f'"] \arrow[d, "b'"] & B \arrow[d, "b"] \\ B' \arrow[r, "f"] & C. \end{tikzcd} \tag{14}$$

Thus $r = (f\ \ b)\left(\begin{smallmatrix} t \\ t' \end{smallmatrix}\right)$ with $t, t' \in \mathrm{Rad}\,\mathcal{A}$. We modify $B \oplus U$ by $\begin{pmatrix} 1 & -t' \\ 0 & 1 \end{pmatrix} \in \mathrm{Aut}(B \oplus U)$, replacing the matrix of a by $\begin{pmatrix} b & r \\ s & q \end{pmatrix}\begin{pmatrix} 1 & -t' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & ft \\ s & p \end{pmatrix}$ with $p := q - st'$. Then (13) is given by the commutative square

$$\begin{tikzcd}[column sep=huge] C' \oplus U \arrow[r, "\begin{pmatrix} f' & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} b' & t \\ sf' & p \end{pmatrix}"'] & B \oplus U \arrow[d, "\begin{pmatrix} b & ft \\ s & p \end{pmatrix}"] \\ B' \oplus P \arrow[r, "\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}"] & C \oplus P. \end{tikzcd} \tag{15}$$

Notice the symmetric structure of (15). We apply (13) in two particular cases. First, we choose P as the largest direct summand of A_0 with $\tau P = 0$. Then we simply write $\lambda_a\colon\ La \to a$ instead of (13). Together with its dual, we obtain a pair of additive functors $L, L^-\colon\ \mathsf{M}(\mathcal{A}) \to \mathsf{M}(\mathcal{A})$ with natural transformations

$$L \xrightarrow{\lambda} 1 \xrightarrow{\lambda^-} L^-. \tag{16}$$

In fact, let $\underline{\mathrm{Rad}}\,\mathcal{A}$ (resp. $\overline{\mathrm{Rad}}\,\mathcal{A}$) be the ideal of morphisms $r + s \in \mathcal{A}$ such that $r \in \mathrm{Rad}\,\mathcal{A}$, and s factors through an object $Q \in \mathcal{A}$ with $\tau Q = 0$ (resp. $\tau^- Q = 0$). By $\underline{\mathrm{Rad}}^+\mathsf{M}(\mathcal{A})$ (resp. $\overline{\mathrm{Rad}}^-\mathsf{M}(\mathcal{A})$) we denote the ideal of morphisms φ: $b \to c$ in $\mathsf{M}(\mathcal{A})$ given by (7) such that $d \in \underline{\mathrm{Rad}}\,\mathcal{A}$ (resp. $a \in \overline{\mathrm{Rad}}\,\mathcal{A}$). Then the universal property (U) specializes to

$$(\mathrm{U}_\lambda)\begin{cases}\lambda_a \in \underline{\mathrm{Rad}}^+\mathsf{M}(\mathcal{A}) \text{ for each object } a \in \mathsf{M}(\mathcal{A}), \text{ and every morphism } x \to a \\ \text{in } \underline{\mathrm{Rad}}^+\mathsf{M}(\mathcal{A}) \text{ factors through } \lambda_a \text{ in a unique manner.}\end{cases}$$

Therefore, a morphism $\varphi\colon a \to b$ in $\mathsf{M}(\mathcal{A})$ determines a commutative square

$$\begin{tikzcd} La \arrow[r, "\lambda_a"] \arrow[d, "L\varphi"] & a \arrow[d, "\varphi"] \\ Lb \arrow[r, "\lambda_b"] & b \end{tikzcd} \tag{17}$$

with a unique morphism $L\varphi$. This shows that L is a functor with a natural transformation $\lambda\colon L \to 1$.

By the symmetry of (15), the universal property of λ admits a certain converse. Namely, every morphism $\varphi\colon La \to d$ in $\overline{\mathrm{Rad}}^-\mathsf{M}(\mathcal{A})$ factors uniquely through λ_a ([19], Proposition 4). In particular, every morphism $\psi\colon La \to b$ satisfies $\lambda_b^-\psi \in \overline{\mathrm{Rad}}^-\mathsf{M}(\mathcal{A})$. Therefore, ψ induces a commutative square

$$\begin{tikzcd} La \arrow[r, "\lambda_a"] \arrow[d, "\psi"] & a \arrow[d, "\psi'"] \\ b \arrow[r, "\lambda_b^-"] & L^-b \end{tikzcd} \tag{18}$$

with a unique ψ', and by symmetry, the correspondence $\psi \mapsto \psi'$ is bijective. Consequently, (18) together with (17) and its dual shows that L is left adjoint to L^-. We call L, L' the *ladder functors* of $\mathsf{M}(\mathcal{A})$.

Another special case of (13) arises when we set $P = 0$. Then we obtain a pair of functors $\widehat{L}$, $\widehat{L}^-\colon \mathsf{M}(\mathcal{A}) \to \mathsf{M}(\mathcal{A})$ with natural transformations

$$\widehat{L} \xrightarrow{\widehat{\lambda}} 1 \xrightarrow{\widehat{\lambda}^-} \widehat{L}^- \tag{19}$$

such that $\widehat{\lambda}_a := \lambda_{A_0,a}$ for any object $a\colon A_1 \to A_0$. Here the universal property (U) specializes to

$$(\mathrm{U}_{\widehat{\lambda}})\begin{cases}\widehat{\lambda}_a \in \mathrm{Rad}^+\mathsf{M}(\mathcal{A}), \text{ and every morphism } x \to a \text{ in } \mathrm{Rad}^+\mathsf{M}(\mathcal{A}) \text{ factors} \\ \text{uniquely through } \widehat{\lambda}_a.\end{cases}$$

The usefulness of L, L^- has been shown in [19]. An application of $\widehat{L}, \widehat{L}^-$ will be given in the next section.

Let $\varphi\colon b \to c$ be a morphism (7) in $\mathsf{M}(\mathcal{A})$. We call φ a *pullback* (*pushout*) *morphism* if (7) is a pullback (pushout). If (7) is an exact square, we call φ

an *exact* morphism. Note that these concepts are invariant under homotopy. In fact, a homotopy $h\colon C \to B$ in (7) amounts to an isomorphic change of the complex (8):

$$\begin{tikzcd} A \arrow[r, "\binom{a}{-b}"] \arrow[d, equal] & B \oplus C \arrow[r, "(c\ d)"] \arrow[d, "\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right)"] & D \arrow[d, equal] \\ A \arrow[r, "\binom{a-hb}{-b}"] & B \oplus C \arrow[r, "(c\ \ d\text{-}ch)"] & D. \end{tikzcd} \tag{20}$$

By [19], Propositions 3 and 4, and Corollary 3 of Proposition 5, we have

Proposition 2. *Let $\mathcal{A}$ be a strict τ-category. Then λ_a is exact, and $\widehat{\lambda}_a$ is a pullback morphism for any object $a \in \mathsf{M}(\mathcal{A})$. Moreover, L preserves exact morphisms.*

For a full subcategory $\mathcal{C}$ of an additive category $\mathcal{A}$, a morphism $f\colon A \to B$ in $\mathcal{A}$ is said to be $\mathcal{C}$-*epic* ($\mathcal{C}$-*monic*) if every morphism $C \to B$ (resp. $A \to C$) with $C \in \mathcal{C}$ factors through f. In [19], Proposition 2, we characterize pullback morphisms in $\mathsf{M}(\mathcal{A})$ as $\mathcal{A}^-$-epic monomorphisms. By $[\mathcal{C}]$ we denote the ideal of $\mathcal{A}$ generated by the morphisms 1_C with $C \in \mathcal{C}$.

Let $\mathcal{A}$ be a strict τ-category. We define $\mathbf{Proj}_\tau(\mathcal{A})$ (resp. $\mathbf{Inj}_\tau(\mathcal{A})$) as the full subcategory of objects $Q \in \mathcal{A}$ with $\tau Q = 0$ (resp. $\tau^- Q = 0$). By Proposition 1 we have the inclusions

$$\mathbf{Proj}(\mathcal{A}) \subset \mathbf{Proj}_\tau(\mathcal{A}); \quad \mathbf{Inj}(\mathcal{A}) \subset \mathbf{Inj}_\tau(\mathcal{A}). \tag{21}$$

By the universal properties (U_λ) and $(\mathrm{U}_{\widehat{\lambda}})$ there are unique natural transformations κ, κ^- which make the following triangles commutative:

$$\begin{tikzcd} L \arrow[r, "\lambda"] & 1 \arrow[r, "\lambda^-"] \arrow[dr, "\widehat{\lambda}^-"'] & L^- \arrow[d, "\kappa^-"] \\ \widehat{L} \arrow[u, "\kappa"] \arrow[ur, "\widehat{\lambda}"'] & & \widehat{L}^-. \end{tikzcd} \tag{22}$$

More generally, there are natural transformations $\lambda^n\colon L^n \to 1$ and $\widehat{\lambda}^n\colon \widehat{L}^n \to 1$ for each $n \in \mathbb{N}$ with components

$$\lambda_a^n := \lambda_a \lambda_{La} \cdots \lambda_{L^{n-1}a}\,; \quad \widehat{\lambda}_a^n := \widehat{\lambda}_a \widehat{\lambda}_{\widehat{L}a} \cdots \widehat{\lambda}_{\widehat{L}^{n-1}a}. \tag{23}$$

As in (22) we find a unique natural transformation $\kappa^n\colon \widehat{L}^n \to L^n$ with $\lambda^n \kappa^n = \widehat{\lambda}^n$ for any given n.

Proposition 3. *Let $\mathcal{A}$ be a strict τ-category. For each object $a \in \mathsf{M}(\mathcal{A})$, and $n \in \mathbb{N}$, the morphism $\kappa_a^n\colon \widehat{L}^n a \to L^n a$ is $\mathcal{A}^+$-epic modulo $[\mathbf{Proj}_\tau(\mathcal{A})^+]$.*

Proof. Let A be an object in $\mathcal{A}$. Then every morphism $\varphi\colon A^+ \to L^n a$ in $\mathsf{M}(\mathcal{A})$ satisfies $\lambda_a^n \varphi = \rho + \sigma$ with $\rho \in (\mathrm{Rad}^+ \mathsf{M}(\mathcal{A}))^n$ and $\sigma \in [\mathbf{Proj}_\tau(\mathcal{A})^+]$. Hence $(\mathrm{U}_{\widehat{\lambda}})$ gives $\rho = \widehat{\lambda}_a^n \rho'$ for some $\rho'\colon A^+ \to \widehat{L}^n a$. Since λ_a^n is $\mathbf{Proj}_\tau(\mathcal{A})^+$-epic by (U_λ), we get $\sigma = \lambda_a^n \sigma'$ for some $\sigma' \in [\mathbf{Proj}_\tau(\mathcal{A})^+]$. Therefore, $\lambda_a^n(\varphi - \kappa_a^n \rho' - \sigma') = 0$, and thus $\varphi = \kappa_a^n \rho' + \sigma'$. □

3 Artinian τ-rings

Let R be a τ-ring with $\mathcal{A} := R$-**proj**. We define $\mathrm{Fix}\, L$ (resp. $\mathrm{Fix}\, L^-$, $\mathrm{Fix}\, \widehat{L}$, $\mathrm{Fix}\, \widehat{L}^-$) as the full subcategory of objects $a \in \mathsf{M}(\mathcal{A})$ for which λ_a (resp. λ_a^-, $\widehat{\lambda}_a$, $\widehat{\lambda}_a^-$) is an isomorphism. (Note that a morphism $\varphi\colon b \to c$ in $\mathsf{M}(\mathcal{A})$ given by (7) is invertible if and only if a and d are invertible in $\mathcal{A}$.) By the definitions, a: $A_1 \to A_0$ belongs to $\mathrm{Fix}\, L$ (resp. $\mathrm{Fix}\, \widehat{L}$) if and only if $\tau A_0 = 0$ (resp. $A_0 = 0$).

The category $\mathsf{M}(\mathcal{A})$ is closely related to the categories R-**mod** and **mod**-R of finitely presented left resp. right R-modules. There are two additive functors

$$R\text{-}\mathbf{mod} \xleftarrow{\mathrm{Cok}} \mathsf{M}(\mathcal{A}) \xrightarrow{\mathrm{Cok}^-} (\mathbf{mod}\text{-}R)^{\mathrm{op}} \tag{24}$$

given by the cokernel of a: $A_1 \to A_0$ in R-**mod** and $\mathrm{Cok}^- a := \mathrm{Cok}(a^*)$.

Proposition 4. *For a τ-ring R with $\mathcal{A} := R$-**proj**, the functors (24) induce equivalences*

$$R\text{-}\mathbf{mod} \approx \mathsf{M}(\mathcal{A})/[\mathcal{A}^-]; \quad (\mathbf{mod}\text{-}R)^{\mathrm{op}} \approx \mathsf{M}(\mathcal{A})/[\mathcal{A}^+]. \tag{25}$$

In particular, an object $a \in \mathsf{M}(\mathcal{A})$ satisfies $\mathrm{Cok}\, a = 0$ if and only if $a \in \mathcal{A}^-$.

Proof. Since the functors (24) are full and dense, we only have to show that a morphism $\varphi\colon b \to c$ given by (7) belongs to $[\mathcal{A}^-]$ if and only if there exists a morphism $h\colon C \to B$ in $\mathcal{A}$ with $d = ch$. If such an h exists, φ admits a factorization

$$\begin{array}{ccccc} A & \xrightarrow{\ 1\ } & A & \xrightarrow{a-hb} & B \\ \Big\downarrow b & & \Big\downarrow & & \Big\downarrow c \\ C & \longrightarrow & 0 & \longrightarrow & D. \end{array}$$

The converse is trivial. □

Since $\widehat{\lambda}_a$: $\widehat{L}a \to a$ is a pullback morphism for every object $a \in \mathsf{M}(\mathcal{A})$, and the embedding R-**proj** $\hookrightarrow R$-**mod** preserves pullbacks, there is a natural embedding $\mathrm{Cok}(\widehat{L}a) \hookrightarrow \mathrm{Cok}\, a$. More precisely, we have (cf. [9], Theorem 4.1)

Proposition 5. *Let R be a τ-ring. For any object $a \in \mathsf{M}(R\text{-}\mathbf{proj})$,*

$$\mathrm{Cok}(\widehat{L}a) = \mathrm{Rad}(\mathrm{Cok}\, a). \tag{26}$$

Proof. Put $\mathcal{A} := R$-**proj**, and assume that $\widehat{\lambda}_a$ is given by a commutative square

$$\begin{array}{ccc} B_1 & \xrightarrow{f_1} & A_1 \\ \Big\downarrow{\scriptstyle \widehat{L}a} & & \Big\downarrow{\scriptstyle a} \\ B_0 & \xrightarrow{f_0} & A_0. \end{array}$$

Then $f_0 \in \mathrm{Rad}\,\mathcal{A}$ implies that $\mathrm{Cok}(\widehat{L}a) \subset \mathrm{Rad}(\mathrm{Cok}\, a)$. Conversely, let p: $P \twoheadrightarrow \mathrm{Rad}\, A_0$ be a projective cover in R-**mod**. Consider the natural epimorphisms c: $A_0 \twoheadrightarrow \mathrm{Cok}\, a$ and d: $B_0 \twoheadrightarrow \mathrm{Cok}(\widehat{L}a)$, and the inclusion i: $\mathrm{Cok}(\widehat{L}a) \hookrightarrow \mathrm{Cok}\, a$. Then p induces a morphism φ: $P^+ \to a$ in $\mathrm{Rad}^+\mathsf{M}(\mathcal{A})$. By $(\mathrm{U}_{\widehat{\lambda}})$ there is a morphism φ': $P^+ \to \widehat{L}a$ with $\varphi = \widehat{\lambda}_a\varphi'$. This gives morphisms g: $P \to B_0$ and h: $P \to A_1$ with $p - f_0 g = ah$. Hence $\mathrm{Rad}(\mathrm{Cok}\, a) = cp(P) = cf_0g(P) = idg(P) \subset \mathrm{Cok}(\widehat{L}a)$. □

Corollary. *A τ-ring R with $\mathcal{A} := R$-**proj** is artinian if and only if there is an $n \in \mathbb{N}$ with $\widehat{L}^n\mathcal{A}^+ \subset \mathcal{A}^-$. For such an n, every object $a \in \mathsf{M}(\mathcal{A})$ satisfies $\widehat{L}^n a \in \mathcal{A}^-$ and $L^n a \in \mathrm{Fix}\, L$.*

Proof. Note that R is artinian if and only if $\mathrm{Rad}^n R = 0$ for some $n \in \mathbb{N}$. So the first statement follows by Propositions 4 and 5. Furthermore, $\widehat{L}^n a \in \mathcal{A}^-$ holds for each object $a \in \mathcal{A}$. By Proposition 3, κ_a^n is $\mathcal{A}^+$-epic modulo $[\mathbf{Proj}_\tau(\mathcal{A})^+]$. Therefore, $\widehat{L}^n a \in \mathcal{A}^-$ implies that

$$\mathrm{Hom}_{\mathsf{M}(\mathcal{A})}(\mathcal{A}^+, L^n a) \subset [\mathbf{Proj}_\tau(\mathcal{A})^+],$$

whence $L^n a \in \mathrm{Fix}\, L$. □

Proposition 6. *For an artinian τ-ring R, the category R-**proj** is preabelian and has strictly enough projectives and injectives.*

Proof. A morphism $f \in \mathcal{A} := R$-**proj** can be regarded as an object $f \in \mathrm{Mor}(\mathcal{A})$. So f is isomorphic to some $1_C \oplus a$ with $a \in \mathrm{Rad}\,\mathcal{A}$. Therefore, a (co-)kernel of a gives a (co-)kernel of f. By the above Corollary, there is an $n \in \mathbb{N}$ with $\widehat{L}^n\mathcal{A}^+ \subset \mathcal{A}^-$. In particular, $\widehat{L}^n a = K^-$ for some object $K \in \mathcal{A}$. Since $\widehat{\lambda}^n_a\colon \widehat{L}^n a \to a$ is a pullback morphism by Proposition 2, this gives a kernel of $a \in \mathcal{A}$. Now let A be an object in $\mathcal{A}$. By the above Corollary, $L^n A^+ \in \mathrm{Fix}\,L$. Since $\lambda^n_{A^+}\colon\ L^n A^+ \to A^+$ is exact by Proposition 2, we get a short exact sequence $B \overset{i}{\rightarrowtail} P \twoheadrightarrow A$ with $i = L^n A^+$. To show that P is projective, consider a short exact sequence $X \overset{x}{\rightarrowtail} Y \overset{y}{\twoheadrightarrow} Z$ in $\mathcal{A}$ and a morphism $f\colon\ P \to Z$. We may assume without loss of generality that $x \in \mathrm{Rad}\,\mathcal{A}$. Then y determines an exact morphism $\varphi\colon\ x \to Z^+$, and we have to show that $f^+\colon\ P^+ \to Z^+$ factors through φ. By (U_λ) we have $f^+ = \lambda^n_{Z^+}\psi$ for some $\psi\colon\ P^+ \to L^n Z^+$. So it remains to be shown that ψ factors through $L^n\varphi$. Proposition 2 implies that $L^n\varphi$ is exact. By [16], Corollary of Proposition 8, every cokernel $D \twoheadrightarrow Q$ with $\tau Q = 0$ splits. Since $L^n Z^+ \in \mathrm{Fix}\,L$, Lemma 1 shows that $L^n\varphi$ is an isomorphism. Hence P is projective. The rest follows by duality. □

Remark. A preabelian category with strictly enough projectives and injectives is also called a *strict PI-category* [14]. Such categories form an important class of almost abelian categories (see [14], §5).

As a consequence, we get the following extension of Igusa and Todorov's theorem ([8], Theorem 3.4).

Corollary. *For a ring R with $\mathcal{A} := R$-**proj**, the following are equivalent:*

(a) *R is an artinian τ-ring such that u_P is not epic for each $P \in \mathrm{Ind}\,\mathcal{A}$ with $\tau P = 0$.*

(b) *There exists an artinian ring Λ with Λ-**mod** $\approx \mathcal{A}$.*

Proof. (a) ⇒ (b): Define $Q := \bigoplus(\mathbf{Proj}(\mathcal{A}) \cap \mathrm{ind}\,\mathcal{A})$ and $\Lambda := \mathrm{End}_{\mathcal{A}}(Q)^{\mathrm{op}}$. Then Λ is artinian, and $\mathbf{Proj}(\mathcal{A}) \approx \Lambda$-**proj**. So it suffices to prove that $\mathcal{A}$ is abelian, i. e. that every regular morphism $r\colon\ A \to B$ in $\mathcal{A}$ is invertible (see [14], Proposition 12). In $\mathrm{Mor}\,\mathcal{A}$ we have a decomposition $r \cong 1_C \oplus a$ with $a \in \mathrm{Rad}\,\mathcal{A}$. By the Corollary of Proposition 5, there is an $n \in \mathbb{N}$ with $L^n a \in \mathrm{Fix}\,L$. Now (a) implies that $L^n a$ is not epic, unless $L^n a \in \mathcal{A}^-$. Since a is epic, we get $L^n a \in \mathcal{A}^-$. As a is monic, the exactness of λ^n_a gives $L^n a = 0$, whence $a = 0$.

(b) ⇒ (a): By Auslander's general existence theorem ([3], Theorem 3.9), there is an almost split sequence $\mathbb{E}\colon\ A \rightarrowtail B \twoheadrightarrow C$ in the category Λ-**Mod** of all Λ-modules for each non-projective $C \in \mathrm{Ind}(\Lambda\text{-}\mathbf{mod})$. Since A is finitely generated by [13], Corollary (4.4), $\mathbb{E}$ is an almost split sequence in Λ-**mod**.

By [21], Theorem 4, Λ-**mod** has a finitely generated injective cogenerator. Therefore, the dual argument implies that R is a τ-ring. By Harada and Sai's lemma ([12], 2.2), $\operatorname{Rad} R$ is nilpotent. Hence R is artinian. Since $\mathcal{A} \approx \Lambda$-**mod**, this proves (a). $\square$

More generally, we get a characterization of arbitrary artinian τ-rings. Let Λ and Γ be left and right coherent rings, respectively (see [1], §19). By [17], Proposition 10, this means that Λ-**mod** and **mod**-Γ are abelian categories. A bimodule ${}_\Lambda U_\Gamma$ is said to be *cotilting* (cf. [5]) if ${}_\Lambda U$ and U_Γ are finitely presented with $\Lambda = \operatorname{End}(U_\Gamma)$ and $\Gamma = \operatorname{End}({}_\Lambda U)^{\mathrm{op}}$ such that for each $M \in \Lambda$-**mod** and $N \in$ **mod**-Γ,

$$\operatorname{Ext}_\Lambda(U, U) = \operatorname{Ext}_\Gamma(U, U) = \operatorname{Ext}^2_\Lambda(M, U) = \operatorname{Ext}^2_\Gamma(N, U) = 0. \qquad (27)$$

Since Γ is determined by ${}_\Lambda U$, the module ${}_\Lambda U$ is said to be a *cotilting module*. By $\mathbf{lat}(U)$ we denote the full subcategory of Λ-**mod** consisting of the modules $M \in \Lambda$-**mod** which are finitely cogenerated by ${}_\Lambda U$ (i. e. which admit an embedding $M \hookrightarrow U^n$ for some $n \in \mathbb{N}$). Then $\mathbf{lat}(U)$ is equivalent to the category of right Γ-modules $N \in$ **mod**-Γ which are finitely cogenerated by U_Γ (see Appendix).

Theorem 1. *For every artinian τ-ring R there exists a cotilting bimodule ${}_\Lambda U_\Gamma$ over artinian rings Λ, Γ such that R-**proj** $\approx \mathbf{lat}(U)$. Conversely, if ${}_\Lambda U$ is a cotilting module over a left artinian ring Λ with* $\operatorname{ind}(\mathbf{lat}(U))$ *finite, then Λ and $\Gamma := \operatorname{End}_\Lambda(U)^{\mathrm{op}}$ are artinian, and up to Morita equivalence, there is a unique artinian τ-ring R with R-**proj** $\approx \mathbf{lat}(U)$.*

Proof. Let R be an artinian τ-ring with $\mathcal{A} := R$-**proj**. We set $P := \bigoplus(\mathbf{Proj}(\mathcal{A}) \cap \operatorname{ind}\mathcal{A})$ and $I := \bigoplus(\mathbf{Inj}(\mathcal{A}) \cap \operatorname{ind}\mathcal{A})$. Then $\Lambda := \operatorname{End}_\mathcal{A}(P)^{\mathrm{op}}$ and $\Gamma := \operatorname{End}_\mathcal{A}(I)^{\mathrm{op}}$ are artinian. By Proposition 6 and the cotilting theorem ([14], Theorem 6; see Appendix), ${}_\Lambda U_\Gamma := \operatorname{Hom}_\mathcal{A}(P, I)$ is a cotilting bimodule with $\mathcal{A} \approx \mathbf{lat}(U)$.

Conversely, let ${}_\Lambda U_\Gamma$ be a cotilting bimodule with Λ left artinian such that $\operatorname{ind}\mathcal{A}$ is finite for $\mathcal{A} := \mathbf{lat}(U)$. We set $R := \operatorname{End}_\mathcal{A}(\bigoplus \operatorname{ind}\mathcal{A})^{\mathrm{op}}$. Then R-**proj** $\approx \mathcal{A}$. Consider $\mathcal{A}$ as a full subcategory of Λ-**mod**. Then $\mathbf{Proj}(\mathcal{A}) = \Lambda$-**proj** by [15], Lemma 4. Let $C \in \operatorname{Ind}\mathcal{A}$ be non-projective. Then there is a cokernel $c\colon C'' \twoheadrightarrow C'$ and a morphism $f\colon C \to C'$ in $\mathcal{A}$ such that f does not factor through c. By [14], Proposition 12, $\mathcal{A}$ is an almost abelian category (see Appendix). Therefore, the pullback of c and f yields a non-split short exact sequence $A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$ in $\mathcal{A}$. Consequently, there is an indecomposable direct summand D of A such that the projection $a'\colon A \twoheadrightarrow D$ does not factor through a. So the pushout of a and a' yields a non-split short exact sequence $D \overset{d}{\rightarrowtail} E \overset{e}{\twoheadrightarrow} C$ in $\mathcal{A}$. By the lemma of Harada and Sai (see [12], 2.2), $\operatorname{Rad} R$ is nilpotent. Hence there exists a morphism $g\colon D \to D'$ in $\operatorname{Ind}\mathcal{A}$ that does not factor through d such that for each non-invertible $h\colon D' \to D''$ in $\operatorname{Ind}\mathcal{A}$, the composition hg factors through d. So the pushout of d and g yields a left almost split sequence $D' \rightarrowtail E' \twoheadrightarrow C$. For $P \in \mathbf{Proj}(\mathcal{A})$, the right almost split sequence in $\mathcal{A}$ is given by $0 \to (\operatorname{Rad}\Lambda)P \to P$. If we regard $\mathcal{A}$ as a full subcategory of $(\textbf{mod-}\Gamma)^{\mathrm{op}}$, the preceding arguments can be dualized. Therefore, [18], Lemma 8, implies that $\mathcal{A}$ is a strict τ-category. Hence R is an artinian τ-ring. Since the rings Λ, Γ are of the form eRe for some idempotent $e \in R$, they are artinian as well. Finally, $R\textbf{-proj} \approx \mathcal{A}$ implies that R is unique up to Morita equivalence. □

Appendix: The general cotilting theorem

In this appendix we give a brief explanation and a short proof of the cotilting theorem ([14], Theorem 6). Let Λ (resp. Γ) be a left (resp. right) coherent ring. Then $\Lambda\textbf{-mod}$ and $\textbf{mod-}\Gamma$ are abelian categories (see [17], Proposition 10). Every bimodule ${}_\Lambda U_\Gamma$ with ${}_\Lambda U$ and U_Γ finitely presented gives rise to an adjoint pair of additive functors

$$\Lambda\textbf{-mod} \underset{F}{\overset{E}{\rightleftarrows}} (\textbf{mod-}\Gamma)^{\mathrm{op}} \tag{28}$$

with $E := \operatorname{Hom}_\Lambda(-, U)$ and $F := \operatorname{Hom}_\Gamma(-, U)$. Conversely, we have the following version of Watt's theorem.

Lemma 2. *Every adjoint pair (28) is of the form* $E \cong \operatorname{Hom}_\Lambda(-, U)$ *and* $F \cong \operatorname{Hom}_\Gamma(-, U)$ *with a bimodule* ${}_\Lambda U_\Gamma$ *such that* ${}_\Lambda U$ *and* U_Γ *are finitely presented.*

Proof. Define $U_\Gamma := E({}_\Lambda\Lambda)$. Then the right operation of Λ on ${}_\Lambda\Lambda$ makes U into a (Λ, Γ)-bimodule. For $M \in \Lambda\textbf{-mod}$, consider a presentation $\Lambda^m \xrightarrow{a} \Lambda^n \twoheadrightarrow M$. Since E is a left adjoint, $EM = \operatorname{Cok}(Ea)$ in $(\textbf{mod-}\Gamma)^{\mathrm{op}}$. Thus $EM = \operatorname{Ker}\operatorname{Hom}_\Lambda(a, U)$ in $\textbf{mod-}\Gamma$, i. e. $E \cong \operatorname{Hom}_\Lambda(-, U)$. Hence $FN = \operatorname{Hom}_\Lambda(\Lambda, FN) \cong \operatorname{Hom}_\Gamma(N, E\Lambda) \cong \operatorname{Hom}_\Gamma(N, U)$ for all $N \in \textbf{mod-}\Gamma$. In particular, ${}_\Lambda U = \operatorname{Hom}_\Gamma(\Gamma, U) \cong F\Gamma$ is finitely presented. □

For a given bimodule ${}_\Lambda U_\Gamma$ we simply write $(\ )^*$ for both functors $\operatorname{Hom}_\Lambda(-, U)$ and $\operatorname{Hom}_\Gamma(-, U)$. Then the unit η and the counit ε of the adjunction are given by

$$\eta_M : M \to M^{**}; \quad \varepsilon_N : N \to N^{**} \tag{29}$$

for $M \in \Lambda$-**mod** and $N \in$ **mod**-Γ.

A pair

$$\mathcal{C} \underset{F}{\overset{E}{\rightleftarrows}} \mathcal{B} \tag{30}$$

of additive functors with $E \dashv F$ is said to be a *pre-equivalence* [15] if the unit is epic, and the counit is monic. Then (30) induces an equivalence $\operatorname{Im} F \approx \operatorname{Im} E$, and the category $\mathcal{A} := \operatorname{Im} F$ is *almost abelian*. This means that $\mathcal{A}$ is pre-abelian, and cokernels (resp. kernels) are stable under pullback (pushout) [14]. Furthermore, the full subcategory $\overline{\operatorname{Im} E}$ (resp. $\overline{\operatorname{Im} F}$) of subobjects (quotient objects) of objects in $\operatorname{Im} E$ (resp. $\operatorname{Im} F$) is abelian. If $\overline{\operatorname{Im} E} = \mathcal{B}$ and $\overline{\operatorname{Im} F} = \mathcal{C}$, we call (30) a *tilting*. In this case, up to isomorphism, the adjunction (30) is intrinsicly determined by the almost abelian category $\mathcal{A}$. In other words, tiltings and almost abelian categories are essentially the same thing (see [15], Theorem 1). In the particular case (28) we have the following characterization.

Theorem 2. *An adjoint pair (28) is a tilting if and only if the corresponding bimodule ${}_\Lambda U_\Gamma$ is cotilting. When these equivalent conditions hold,* $\mathbf{lat}(U)$ *is the corresponding almost abelian category.*

Proof. Let $\operatorname{Cog}_\Lambda U$ denote the class of finitely generated submodules of some $({}_\Lambda U)^n$. We show first that the conditions (27) can be replaced by

$$\operatorname{Ext}_\Lambda(M,U) = \operatorname{Ext}_\Gamma(N,U) = 0 \text{ for } M \in \operatorname{Cog}_\Lambda U \text{ and } N \in \operatorname{Cog}_\Gamma U. \tag{31}$$

Assume (31). For any $M \in \Lambda$-**mod**, there is a short exact sequence $M' \hookrightarrow \Lambda^n \twoheadrightarrow M$ with $M' \in \Lambda$-**mod**. Since an epimorphism $\Gamma^m \twoheadrightarrow U$ gives an embedding $\Lambda = \operatorname{Hom}_\Gamma(U,U) \hookrightarrow \operatorname{Hom}_\Gamma(\Gamma^m,U) = U^m$, we have $\Lambda^n \in \operatorname{Cog}_\Lambda U$. Hence $\operatorname{Ext}^2_\Lambda(M,U) = \operatorname{Ext}^1_\Lambda(M',U) = 0$. By duality, this proves (27). Conversely, let $M \hookrightarrow U^n \twoheadrightarrow C$ be a short exact sequence in Λ-**mod**. Then

$$\operatorname{Ext}^1_\Lambda(U^n,U) \to \operatorname{Ext}^1_\Lambda(M,U) \to \operatorname{Ext}^2_\Lambda(C,U)$$

is exact. Hence (27) implies (31).

Now let (28) be a tilting with corresponding bimodule ${}_\Lambda U_\Gamma$. Then ${}_\Lambda\Lambda \in \overline{\operatorname{Im} F}$ implies that there is an epimorphism $N^* \twoheadrightarrow {}_\Lambda\Lambda$. Since N^* is reflexive, i. e. η_{N^*} is invertible, we infer that Λ is reflexive. Hence $\operatorname{End}(U_\Gamma) = \Lambda$, and similarly, $\mathrm{End}({}_\Lambda U) = \Gamma$. Any embedding $M \hookrightarrow U^n$ in Λ-**mod** gives rise to a commutative diagram

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "\eta_M"] & U^n \arrow[d, "\eta_U"] \\ M^{**} \arrow[r] & (U^n)^{**}. \end{tikzcd}$$

Since η_U is an isomorphism, η_M is monic. On the other hand, an epimorphism $\Gamma^m \twoheadrightarrow N$ yields $N^* \hookrightarrow U^m$, i. e. $N^* \in \mathrm{Cog}_\Lambda U$. Therefore, $\mathrm{Cog}_\Lambda U$ consists of the reflexive modules in Λ-**mod**. So for a given $M \in \mathrm{Cog}_\Lambda U$, the modules in a short exact sequence $K \overset{i}{\rightarrowtail} \Lambda^k \overset{p}{\twoheadrightarrow} M$ are reflexive. Applying $(\ )^*$ gives $M^* \overset{p^*}{\rightarrowtail} U^k \overset{i^*}{\to} K^*$ with $p^* = \ker i^*$. As a submodule of K^*, $\mathrm{Cok}\, p^*$ is reflexive. Hence $(\mathrm{cok}\, p^*)^* \cong \ker p = i$, and thus $i^* = \mathrm{cok}\, p^*$. This proves that $\mathrm{Ext}_\Lambda(M, U) = 0$.

Conversely, let ${}_\Lambda U_\Gamma$ be cotilting, and $M \in \Lambda$-**mod**. Then a presentation $\Lambda^m \to \Lambda^n \overset{p}{\twoheadrightarrow} M$ leads to a short exact sequence $M^* \overset{p^*}{\rightarrowtail} U^n \twoheadrightarrow C$ and an embedding $C \hookrightarrow U^m$. By (31) it follows that $p^{**}\colon \Lambda^n \twoheadrightarrow M \overset{\eta_M}{\to} M^{**}$ is epic. Hence η_M is epic. Since Λ and Γ are reflexive, and every object in Λ-**mod** (resp. **mod**-Γ) is a factor module of a free module, (28) is a tilting. □

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