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Boundedness conditions of Hausdorff h-measure in metric spaces

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Abstract

The fractal dimensions are very important characteristics of the fractal sets. A problem which arises in the study of the fractal sets is the determination of their dimensions. The Hausdorff dimension of this type of sets is difficult to be determined, even if the Box dimensions can be computed. In this article we present some boundedness conditions on the Hausdorff h-measure of a set, using their Box dimensions.

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1 Background

The calculus of the dimensions is fundamental in the study of fractals. The Hausdorff measures and the h-measures, the box dimensions, the packing dimensions are widely used and in many articles the relations between them are given ([5] - [8]).

In the papers [1] - [4] we gave some boundedness conditions for a class of fractal sets, in $\mathbf{R}^n$. This type of conditions is important in order to prove theorems concerning the module and the capacities and the relations between them ([10]).

In this paper we work in metric spaces and we give some boundedness conditions of the Hausdorff h - measures.

Definition 1. *Let* (X, d) *be a metric space.*

If $r_0 > 0$ *is a given number, then, a continuous function* $h(r)$, *defined on* $[0, r_0)$, *nondecreasing and such that* $\lim\limits_{r\to 0} h(r) = 0$ *is called* a *measure function.*

Key Words: Hausdorff dimension; Box dimension; Equivalence relation.

If $0 < \delta < \infty$, E is a subset of (X, d) and h is a measure function, then, the Hausdorff h-measure of E is defined by:

$$H_h(E) = \lim_{\delta \to 0} \inf \left\{ \sum_i h(|U_i|) : E \subseteq \bigcup_i U_i : 0 < |U_i| < \delta \right\}.$$

where $|\ |$ *denotes the diameter of the set* U_i.

Particularly, when $h(r) = r^s, 0 < s < \infty$, then the s-dimensional Hausdorff measure of E, denoted by $H^s(E)$, is obtained.

The Hausdorff dimension of a nonempty set $E \subset X$ is the number defined by

$$\dim_H E = \inf \{s : H^s(E) = 0\} = \sup \{s : H^s(E) = \infty\}.$$

Remark. There are definitions where the covering of the set E is made with balls. The relation between the new measure, denoted by H'_h and H_h is: $H_h(E) \le H'_h(E)$. Thus,

$$H'_h(E) < \infty \Rightarrow H_h(E) < \infty,$$

$$H'_h(E) = 0 \Rightarrow H_h(E) = 0,$$

and

$$H_h(E) > 0 \Rightarrow H'_h(E) > 0.$$

Definition 2. *Let β be a positive number and E be a nonempty and bounded subset of the metric space (X, d). Let $N_\beta(E)$ be the smallest number of sets of diameter at most β that cover E. Then the upper and lower Box dimension of E are defined by:*

$$\overline{\dim_B} E = \overline{\lim_{\beta \to 0}} \frac{\log N_\beta(E)}{-\log \beta}; \quad \underline{\dim_B} E = \underline{\lim_{\beta \to 0}} \frac{\log N_\beta(E)}{-\log \beta}.$$

If these limits are equal, the common value is called the Box dimension of E and is denoted by $\dim_B E$.

Definition 3. *Let $\varphi_1, \varphi_2 > 0$ be functions defined in a neighborhood of $0 \in \mathbf{R}^n$. We say that φ_1 and φ_2 are equivalent and we denote by: $\varphi_1 \sim \varphi_2$, for $x \to 0$, if there exist $r > 0$, $Q > 0$, satisfying:*

$$\frac{1}{Q} \varphi_1(x) \le \varphi_2(x) \le Q \varphi_1(x), (\forall) x \in \mathbf{R}^n, |x| < r,$$

where for $x \in \mathbf{R}^n$, $x = (x_1, ..., x_n)$, $|x| = \sum_{i=1}^n x_i^2$.

An analogous definition can be given for $x \to \infty$. In this case, $\varphi_1 \sim \varphi_2$ means that the previous inequalities are valid in all the space.

Remark. In what follows, if U is a set in a metric space, particularly in $\mathbf{R}^n$, $|U|$ means the diameter of U and if $x \in \mathbf{R}^n$, $|x|$ has the significance given in the definition 3.

In the second part of the paper we shall use the following results:

Lemma 1. ([6]) *If E is a set in $\mathbf{R}^2$, then*

$$\dim_H E \leq \underline{\dim}_B E \leq \overline{\dim}_B E.$$

Remark. The previous lemma remains true in a nonempty compact metric space.

2 Results

Theorem 1. *Let (X, d) be a nonempty compact metric space, with $\dim_H X = s$. Let h be a measure function such that there is $m > 0$, with $\frac{h(t)}{t^s} > m$. Suppose that there exist λ_0, $\alpha > 0$ such that for any set $E \subset X$, with $|E| < \lambda_0$, there is a mapping $\varphi : E \to X$ such that*:

$$\alpha d(x, y) \leq |E| d(\varphi(x), \varphi(y)), (\forall) x, y \in E.$$

Then $H_h(X) > 0$.

Proof. First, it will be proved that $H^s(X) \geq \alpha^s$.

Suppose that $0 \leq H^s(X) < \alpha^s$. Then, given $0 < \delta < \min\{\lambda_0, \frac{\alpha}{2}\}$, there are the sets $U_1, ..., U_k$, with $|U_i| < \delta$, for $i = 1, 2, ..., k$ and $X \subset \bigcup_{i=1}^k U_i$ such that

$$\sum_{i=1}^k |U_i|^s < \alpha^s$$

and so

$$\sum_{i=1}^k |U_i|^t < \alpha^t,$$

for some $t < s$.

By the hypotheses of the theorem there are the mappings $\varphi_i : U_i \to X$ such that

$$d(x, y) \leq \alpha^{-1} |U_i| d(\varphi(x), \varphi(y)), (\forall) x, y \in U_i \Rightarrow$$

$$|\varphi_i^{-1}(U_q)| = \sup d(\varphi_i^{-1}(x), \varphi_i^{-1}(y)) < \alpha^{-1} |U_i||U_q| < \frac{1}{2}\delta \Rightarrow$$

$$|\varphi_i^{-1}(U_q)|^t < \alpha^{-t}|U_i|^t|U_q|^t \Rightarrow$$

$$\sum_{i=1}^{k}\sum_{q=1}^{k}|\varphi_i^{-1}(U_q)|^t < \alpha^{-t}(\sum_{i=1}^{k}|U_i|^t)(\sum_{q=1}^{k}|U_q|^t) < \alpha^t.$$

But $X \subset \bigcup_{i,q=1}^{k}|\varphi_i^{-1}(U_q)|^t$. Therefore X has a covering by sets of diameter less than $\frac{1}{2}\delta$, with the same bound on the t-th power of the diameters. Repeating the argument, we see that there are coverings V_i of X, with diameters at most $2^{-n}\delta$, such that $\sum|V_i|^t < \alpha^t$. It follows that $H^t(X) < \alpha^t$ and $\dim_H X = t < s$, which is a contradiction.

So, $H^s(X) \geq \alpha^s > 0$.

If $\{U_i\}_{i\in\mathbf{N}} \subset X$ with $|U_i| < \delta$ such that $X \subset \bigcup_{i=1}^{\infty}$, then:

$$\sum_{i=1}^{\infty}h(|U_i|) = \sum_{i=1}^{\infty}\left\{\frac{h(|U_i|)}{|U_i|^s}\cdot|U_i|^s\right\} > m\sum_{i=1}^{\infty}|U_i|^s \Rightarrow$$

$$H_h(X) \geq \alpha H^s(X) \geq \ m\cdot\alpha^s > 0.$$

Proposition 1. *Let (X,d) be a nonempty compact metric space, with $\dim_H X = s$. Let h be a measure function such that there is $M>0$, $\frac{h(t)}{t^s} < M$. Then $H_h(X) \leq M\cdot H^s(X)$.*

Proof. Let $\delta > 0$ and $\{U_i\}_{i\in\mathbf{N}^*}$ be a covering of X with sets with $|U_i| < \delta$, $(\forall)i \in \mathbf{N}^*$.

$$\sum_{i=1}^{\infty}h(|U_i|) = \sum_{i=1}^{\infty}\left\{\frac{h(|U_i|)}{|U_i|^s}\cdot|U_i|^s\right\} < M\sum_{i=1}^{\infty}|U_i|^s \Rightarrow$$

$$H_h(X) \leq \ M\cdot\ H^s(X).$$

Remarks. 1. In the theorem 1 it was also proved that $H^s(X) > 0$.

2. The Theorem 1 and the Proposition 1 give boundedness conditions for the Hausdorff h-measure of a compact metric space X, if $h(t) \sim t^s$.

Indeed, if $h(t) \sim t^s$, there is $Q > 0$, satisfying:

$$\frac{1}{Q}\cdot t^s \leq h(t) \leq Q\cdot t^s, (\forall)t > 0.$$

In the hypotheses of the mentioned theorems, for $m = \frac{1}{Q}$ and $M = Q$,

$$0 < \frac{1}{Q}\cdot\alpha^s \leq \frac{1}{Q}\cdot H^s(X) \leq H_h(X) \leq Q\cdot\ H^s(X).$$

Theorem 2. *Let (X, d) be a nonempty compact metric space, with* $\dim_H X = s < \infty$*. Suppose that there exist $a, r_0 > 0$ such that for any ball B in X of radius $r < r_0$ there is a mapping $\psi : E \to B$ such that:*

$$ard(x, y) \leq d(\psi(x), \psi(y)), (\forall)x, y \in X.$$

Let h be a measure function such that there is $M > 0$, with $\frac{h(t)}{t^s} < M$. Then $H_h(X) < Ms$.

Proof. Following the proof of the theorem 4 [6], it results that

$$\overline{\dim}_B X = \underline{\dim}_B X = s$$

and $H^s(X) < \infty$. Using the relation (5), it results that $H_h(X) < Ms$.

Examples.

1. *Self-similar sets.* For $i = 1, ..., k$, let $\psi_i : \mathbf{R}^n \to \mathbf{R}^n$ be contracting similarity transformations, i.e.

$$d(\psi_i(x), \psi_i(y)) = c_i d(x, y),$$

where $0 < c_i < 1$ and d is the Euclidean metric. Then, there is a unique nonempty compact set $F \subset \mathbf{R}^n$ that is self-similar ([8]), i.e.

$$F = \bigcup_{i=1}^{k} \psi_i(F).$$

If $s = \dim_H(F)$ and h is a measure function as in the Theorem 2, then $H_h(F) < \infty$.

2. *Dynamical repeller.* If f is a $C^{1+\eta}$ conformal mapping on a Riemann manifold with mixing repeller J ([5]), $s = dim_H J$ and h is a measure function such that there is $M > 0$, with $\frac{h(t)}{t^s} < M$, then $H_h(J) < \infty$.

In ([5]) it was proved that in the previous hypotheses, $0 < H^s(J) < \infty$. Using the Theorem 2, it results $H_h(J) < \infty$.

Theorem 3. *Let (X, d) be a nonempty metric space, $E \subset X$, $E \neq \emptyset$, compact and h be a measure function such that $H'_h(E) < \infty$. Let $\boldsymbol{F}$ be the family of the closed sets in the topology induced by the metric. Suppose that there is $\varphi : \mathbf{F} \to \mathbf{R}_+$ such that φ is subadditive and φ satisfies the conditions:*

a. $\varphi(F) \geq 0, (\forall)F \subset \mathbf{F}$.

b. If $F \supset E$, then $\varphi(F) \geq b > 0$, where b is a constant.

c. There is a constant, $k \neq 0$, such that $\varphi(F) \leq kh(|F|)$.

Then, $H_h'(E) \geq b/k$.

Proof. Let $\delta > 0$. If $\{U_i\}$ is a sequence of open discs that cover E, with $|U_i| < \delta$, it will be proved that $\Sigma_{U_i} h(U_i) \geq \frac{b}{k}$.

Since E is a compact set,

$$(\exists) n \in \mathbf{N}^* : E = \bigcup_{i=1}^{n} U_i.$$

We can take closed discs, $U_i^{'}, U_i^{'} \supset U_i$, with the radius $\frac{\delta_i^{'}}{2}$ close enough to $\frac{|U_i|}{2}$, such that

$$h(|U_i^{'}|) < (1+\varepsilon) h(|U_i|),$$

where $\varepsilon > 0$ is small enough.

Then,

$$h(|U_i^{'}|) \geq \frac{1}{k} \varphi(|U_i^{'}|) \Rightarrow$$

$$\sum_{i=1}^{n} h(|U_i|) \geq \frac{1}{1+\varepsilon} \sum_{i=1}^{n} h(|U_i^{'}|) \geq \frac{1}{k(1+\varepsilon)} \sum_{i=1}^{n} \varphi(|U_i^{'}|) \geq$$

$$\geq \frac{1}{k(1+\varepsilon)} \varphi(\bigcup_{i=1}^{n} U_i^{'}) \geq \frac{b}{k(1+\varepsilon)}.$$

Thus, $H_h'(E) \geq \frac{b}{k}$.

Remark. The previous theorem remains true if $\mathbf{F}$ is replaced by the set $\mathbf{G}$ of the open sets.

The Theorem 3 is a generalization of the sufficiency of the Theorem 1 [9].

Theorem 4. *Let* (X, d) *be a nonempty metric space,* $E \subset X$, $E \neq \emptyset$, *compact and* h *be a measure function such that* $H_h'(E) < \infty$ and $h(t) \sim P(t) e^{T(t)}, t \geq 0$, *where* P *and* T *are the polynomials:*

$$P(t) = \sum_{j=1}^{p} a_j t^j, p \geq 1, a_1 \neq 0, T(t) = \sum_{j=0}^{m} b_j t^j,$$

with positive coefficients. Then $H_h'(E) > 0$.

The result remains true if $p \geq 2, a_1 = 0$ *and* $\delta > 0$.

Proof. Let us define the function:

$$\varphi : \mathbf{F} \to \mathbf{R}_+, \varphi(F) = |F|, (\forall) F \in \mathbf{F}.$$

It will be proved that the function φ satisfies the conditions of the Theorem 3.

Since $h(t) \sim P(t)e^{T(t)}, t \geq 0$, there is $Q > 0$ such that:

$$\frac{1}{Q} \cdot h(t) \leq P(t)e^{T(t)} \leq Q \cdot h(t), (\forall)t > 0.$$

We obtain easily the results:

a. $|F| \geq 0, (\forall)F \in \mathbf{F}$.

b. If $F \supset E$, then $\varphi(F) = |F| \geq |E|$.

So, b from the previous theorem is $|E| > 0$.

c.

$$\frac{\varphi(F)}{h(|F|)} = \frac{|F|}{h(|F|)} = \frac{|F|}{P(|F|)e^{T(|F|)}} \cdot \frac{P(|F|)e^{T(|F|)}}{h(|F|)} \leq$$

$$\leq Q \cdot \frac{|F|}{P(|F|)e^{T(|F|)}} < \frac{Q}{e^{b_0} \cdot a_2} = k.$$

Using the previous theorem we deduce that:

$$H_h'(E) \geq \frac{|E| \cdot e^{b_0} \cdot a_2}{Q} > 0.$$

Remark. Another function that could be used to prove the Theorem 4 is:

$$\psi : \mathbf{F} \to \mathbf{R}_+, \psi(F) = |F \bigcap E|, (\forall)F \in \mathbf{F}.$$

a. $|F| \geq 0, (\forall)F \in \mathbf{F}$.

b. If $F \supset E$, then $\psi(F) = \psi(E) = |E| > 0$.

c.

$$\frac{\psi(F)}{h(|F|)} = \frac{|F \bigcap E|}{h(|F|)} = \frac{|F \bigcap E|}{P(|F|)e^{T(|F|)}} \cdot \frac{P(|F|)e^{T(|F|)}}{h(|F|)} \leq$$

$$\leq Q \cdot \frac{|F|}{P(|F|)e^{T(|F|)}} < \frac{Q}{e^{b_0} \cdot a_2} = k.$$

So, ψ satisfies the hypotheses of the Theorem 4.

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