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Isomorphism of noncommutative group algebras of torsion-free groups over a field

P. V. Danchev

Abstract

The isomorphism problem for group algebras over a field with arbitrary characteristic of some special classes of torsion-free non-abelian groups is explored. Specifically, the following are proved: Suppose F is a field and G is a torsion-free group with centre $C(G)$ such that $FG \cong FH$ as F-algebras for any group H. Then it is shown that H is torsion-free (provided that FG is without zero divisors), and if G is soluble we have even more that $C(G) \cong C(H)$. The latter extends classical results due to Higman(1940)-May(1969) when G is torsion-free abelian.

Moreover, if G is an R-group or a D-group, then the above F-isomorphism yields that so is H.

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I. Introduction and preliminaries. Throughout this article, let we assume that F is an arbitrary field (of arbitrary characteristic) with multiplicative group $U(F)$, and G is a multiplicatively written group, possibly non-abelian, with centre $C(G)$. Denote by FG the group algebra of G over F with centre $Z(FG)$, with unit group $U(FG)$ and corresponding normed unit group $V(FG)$, and with a relative augmentation two-sided ideal $I(FG; A)$ with respect to the normal subgroup $A \triangleleft G$. The further notions, notations and terminology are as in ([4], [6], [7]) and ([1], [14]).

A classical long-standing conjecture, due to Kaplansky, also named *Zero Divisors Problem*, is that FG is without zero divisors provided G is torsion-free (for instance, see [14]).

Conjecture (KAPLANSKY). *If G is torsion-free, then FG has no zero divisors ($\Longleftrightarrow$ has no nilpotents).*

Key Words: Group rings; Units; Centres, Torsion-free groups; Soluble groups; D-groups; R-groups.

The best principal results at this moment, concerning large major sorts of groups for which the above question holds in the affirmative, are obtained for the so-called soluble (= solvable in other terms) torsion-free groups (see, for example, [3]; [8] and [13]) - in particular for all torsion-free abelian groups (e.g. [9]) - as well as for the so-called *up*-groups (cf. [14]). These two group classes are independent. However, it is well-known that all free groups and all torsion-free (locally) nilpotent groups are right-ordered hence *up*-groups.

Another old-standing conjecture, due to A.I.Malcev, is that termed as *Trivial Units Problem*, which asserts the following.

Conjecture (MALCEV). *If G is torsion-free, then FG possesses only trivial units, i.e.*

$$U(FG) = G \times U(F).$$

The best known principal result in that aspect is due to Mihovski-Dimitrova ([11],[12]), argued for *up*-groups. It is still unknown whether or not this does hold true for soluble torsion-free groups. Important special case is for torsion-free abelian groups which is completely settled in [5] and [9], respectively.

Nevertheless, in the general situation, Mihovski has established in [10] a remarkable weaker variant for the trivial central units. Actually, his result is stronger than that formulated below and it gives a comprehensive satisfactory description of $C(U(FG)) = U(Z(FG))$ too.

Theorem (MIHOVSKI). *Let G be torsion-free so that FG is without zero divisors. Then*

$$C(U(FG)) = C(G) \times U(F).$$

We shall use in the sequel this decomposition freely, without further comments.

Our goals at this stage are to derive that $C(G)$ and $F(G/C(G))$ are structural invariants of FG, provided G is soluble torsion-free. This generalizes a well-documented in ([5],[1],[9]) theorem of Higman-May when G is torsion-free abelian. In particular, as a consequence, we apply here these affirmations to metabelian (soluble of class two = nilpotent of class two) torsion-free groups. Finally, we show that some group characteristic parameters for G, such as being a D-group or an R-group, can be invariantly recovered from FG.

These main facts will be proceed by proving in the following paragraph. They were announced in ([2], Section 2).

II. Central results. We start here with a well-known *Isomorphism Problem* (see, for instance, [14]).

Conjecture (ISOMORPHISM). *If G is torsion-free, then $FG \cong FH$ are F-isomorphic for any group H if and only if $G \cong H$.*

The apparent connection between the last two conjectures may be demonstrated via the following simple, but a key technical tool (see e.g. [14], p.675, Theorem 3.1). We give below a new confirmation of this fact.

Lemma. *Given $FG \cong FH$ as F-algebras and $V(FG) = G$. Then $G \cong H$.*

Proof. Suppose $\phi : FH \to FG$ is a F-isomorphism. It is well-known that it can be chosen to preserve the augmentation, so the restriction $\phi : V(FH) \to V(FG) = G$ is a group isomorphism. Clearly $\phi(H) \subseteq G = \phi(V(FH))$ since $H \subseteq V(FH)$. What suffices to detect is that $\phi(H) = G$. In fact, take $g \in G$ whence there is $\sum_{h\in H} \alpha_h h \in V(FH)$ with $\phi(\sum_{h\in H} \alpha_h h) = g$. But $\phi(\sum_{h\in H} \alpha_h h) = \sum_{h\in H} \phi(\alpha_h h) = \sum_{h\in H} \alpha_h \phi(h)$ with all group coefficients $\phi(h) \in \phi(H) \subseteq G$. Thereby, $g = \phi(h)$ for some $h \in H$ belonging to the sum, and we are done.

Thus, by what we have stated above, the isomorphism problem holds for *up*-groups (for more details see [11, 12]).

It seems to the author that the above posed general isomorphism question is very difficult, so he feels that of some interest and importance is the following mild modification which is also left-open yet.

Problem. *Let G be torsion-free. Then whether $FH \cong FG$ as F-algebras for some group H implies $C(H) \cong C(G)$.*

We claim that this problem has an affirmative answer under the additional hypothesis that so does the Kaplansky problem. This will be showed below. First of all, we need some preliminary technicalities.

Proposition 1. *If (G is torsion-free such that) FG is without zero divisors and $FG \cong FH$ as F-algebras for a group H, then H is torsion-free.*

PROOF. It is no harm in assuming that $FG = FH$, where $H \subseteq V(FG)$ is a normalized group basis for FG. If we tolerate $h^m = 1$ for some $m = order(h) \in \mathbb{N}$ and $1 \neq h \in H$, it is plainly checked that $1 - h^m = 0 \iff (1-h)(1 + h^{m-1} + h^{m-2} + ... + h) = 0$ where evidently $1 + h^{m-1} + h^{m-2} + ... + h \neq 0$. Thus FG contains a non-trivial zero divisor, which is a contradiction. Henceforth, H must be torsion-free and everything is proved.

Remark. *In non-commutative rings the formula $x^n - y^n = (x-y).(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ does not hold in general when $x \neq 1$ and $y \neq 1$, for each natural number n. When either $x = 1$ or $y = 1$, it is true.*

Corollary 1. *If FG has not zero divisors, then G is torsion-free.*

Remark. *When G is abelian, the converse is valid as well (see [9], p. 141, Lemma 1). We also indicate that our foregoing proof of Proposition 1 is different from that of May and that it may be directly obtained from Proposition 3 listed below.*

We now come to our central assertion.

Theorem 1 (Isomorphism). *Supposing (that G is torsion-free so) that FG possesses no zero divisors, the F-isomorphism $FG \cong FH$ for any group H implies $C(G) \cong C(H)$.*

Proof. We observe that Proposition 1 ensures that H is torsion-free (it is not necessary but however this is an immediate consequence). Without loss of generality, we may presume that $FG = FH$. Thus, since both FG and FH are with no zero divisors, the Mihovski's attainment guarantees that $C(U(FG)) = C(G) \times U(F) = C(H) \times U(F) = C(U(FH))$, and consequently $C(G) \cong C(H)$, as wanted. The proof is completed after all.

Corollary 2. *Let G be soluble torsion-free. Then $FG \cong FH$ as F-algebras for some group H implies that $C(G) \cong C(H)$.*

Proof. Combining the above quoted facts, all central units in FG are trivial because FG does not have zero divisors ([3], [8] and [13]). Hence the claim.

Corollary 3 (HIGMAN [5]-MAY [9, p.142]). *Let G be a torsion-free abelian group and $FG \cong FH$ as F-algebras for a group H. Then $G \cong H$.*

Proof. Since each abelian group G is soluble and $C(G) = G$ as well as $C(H) = H$ being obviously abelian, Theorem 1 is applicable to finish the proof.

Remark. *Certainly, $FG \cong FH$ gives $Z(FG) \cong Z(FH)$, but $Z(FG) \supset F(C(G))$ and similarly $Z(FH) \supset F(C(H))$. Therefore, the result of Higman-May is not directly applicable. However, $Z(FG) \cong Z(FH)$ will imply that $F(C(G)) \cong F(C(H))$ as it was shown above.*

Next, we shall summarize below certain group-theoretic facts much needed for our good presentation, namely:

(1) [7, p.89] - A group G is metabelian if and only if $G/C(G)$ is abelian.

(2) [7, p.411-413] - For every nilpotent torsion-free group G, every two elements $a, b \in G$ and every two positive integers k and l, the equality $a^k b^l = b^l a^k$ implies $ab = ba$.

That is why, using these two assertions, we get

Claim 1. *If G is nilpotent torsion-free, then $G/C(G)$ is torsion-free.*

Proof. Indeed, take $x = gC(G)$ for $g \in G$. If there is $s \in \mathbb{N}$ such that $x^s = C(G)$ we elementarily have $g^s \in C(G)$ and thus $g^s a = ag^s$ for every $a \in G$. Furthermore (2) insures that $ga = ag$ and so $g \in C(G)$. Therefore, x is the identity element and the proof is complete.

After this, we concentrate on

Proposition 2. *Assume that G is torsion-free such that FG has no zero divisors (in particular G is soluble torsion-free). Then $FG \cong FH$ as F-algebras for some group H assures that*

(a) $F(G/C(G)) \cong F(H/C(H))$;

(b) $G/C(G) \cong H/C(H)$ *and H is metabelian, provided G is metabelian.*

Proof. We may without harm of generality assume that $FG = FH$. Thus, as we have verified above, $C(G)\times U(F) = C(H)\times U(F)$. Furthermore, $F(C(G)\times U(F)) = F(C(G)) = F(C(H)) = F(C(H)\times U(F))$, and hence $I(FG; C(G)) = FG.I(F(C(G)); C(G)) = FH.I(F(C(H)); C(H)) = I(FH; C(H))$. Finally, we infer at once that $F(G/C(G)) \cong FG/I(FG; C(G)) = FH/I(FH; C(H)) \cong F(H/C(H))$, as desired.

Now, we are ready to attack the second point. Since G is metabelian, (1) and (a) give that $H/C(H)$ is abelian, i.e. H is metabelian. On the other hand, the torsion-freeness of $G/C(G)$ follows in virtue of the group Claim 1. Finally, by what we have already shown above in (a) along with Corollary 3, $G/C(G) \cong H/C(H)$. This ends the proof.

Remark. *Letting $FG \cong FH$. Since the metabelian groups are soluble, in accordance with Corollary 2 we also have that $C(G) \cong C(H)$. But the metabelian torsion-free groups are known to be up-groups (see [14]), so $G \cong H$ as it has been extracted in ([11, 12]) and mentioned above as well. Thus the foregoing deduced two isomorphism relationships pertaining to the centres of metabelian groups, namely $C(G) \cong C(H)$ and $G/C(G) \cong H/C(H)$, are already simple consequences, although we have exploited different methods for their confirmation (compare also with Theorem (Isomorphism) in [2]).*

We indicate that if G is free and $FG \cong FH$ as F-algebras, then $G/G' \cong H/H'$ where G' is the commutant (= commutator subgroup) of G. This is so since $FG \cong FH$ forces that $F(G/G') \cong F(H/H')$ (e.g. [14]), and moreover G/G' is abelian torsion-free, hence Corollary 3 works.

We continue with some properties of the group basis which can be retrieved from the group algebra. First and foremost, we need some conventions. The following two definitions are well-known, but they are included here only for the sake of completeness, even for the readers having no a full information (e.g. cf. [7], pp.410 and 403).

Definition 1. The group G is said to be an *R-group* if for every $g \in G$ and natural n the equation $x^n = g$ has at most (= precisely) one solution in G, i.e., in other words, for each two elements $g, h \in G$ and positive integer n we have $g^n = h^n \iff g = h$.

Evidently, every R-group is torsion-free and all abelian groups without torsion are R-groups. More generally, Malcev and Chernikov have proved that every nilpotent torsion-free group is an R-group (see [7], p.413). Moreover, each free group is an R-group.

Definition 2. The group G is called a *D-group* (= a divisible group) if for every $g \in G$ and $n \geq 0$ the equation $x^n = g$ has at least one solution in G.

As we have concluded above in Proposition 2, if G is metabelian torsion-free and $FH \cong FG$ as F-algebras then H is metabelian, too. Even more, $H \cong G$. In Proposition 1 it has been proved an analogous assertion for a torsion-free group but provided extraordinary that FG has no zero divisors (i.e. that the Kaplansky Conjecture holds positively).

We shall establish below statements of this type in virtue of another technique such that to be a priori unknown that either G and H are isomorphic or that FG does not possess zero divisors. So, we are ready to process the following.

Proposition 3 (Structure). *Suppose there exists an F-isomorphism $FG \cong FH$ for groups G and H. If G is one of the following:*
(c) *torsion-free;*
(d) *free;*
(e) *finitely generated torsion-free;*
(f) *an R-group;*
(g) *a torsion-free D-group,*
then so does H.

Because of big technical difficulties, we shall give a proof of this attainment elsewhere (perhaps in a subsequent appropriate paper).

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13, General Kutuzov Street,
bl. 7, floor 2, apart. 4,
4003 Plovdiv
Bulgaria
e-mail: pvdanchev@yahoo.com