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Division algebras with dimension 2^t, t$\in \mathbb{N}$

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Abstract

In this paper we find a field such that the algebras obtained by the Cayley-Dickson process are division algebras of dimension $2^t, \forall t \in \mathbb{N}$.

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From Frobenius Theorem and from the remark given by Bott and Milnor in 1958, we know that for $n \in \{1, 2, 4\}$ we find the real division algebras over the real field $\mathbb{R}$. These are: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$(the real quaternion algebra), $\mathbb{O}$(the real octonions algebra). They are unitary and alternative algebras. In 1978, Okubo gave an example of a division non alternative and non unitary real algebra with dimension 8, namely the real *pseudo-octonions* algebra.(See[7]). Here we find a field such that the algebras obtained by the Cayley-Dickson process are division algebras of dimension $2^t, \forall t \in \mathbb{N}$. First of all we describe shortly the Cayley-Dickson process.

Definition 1. Let U be an arbitrary algebra. The vector spaces morphism $\phi : U \to U$ is called an **involution** of the algebra U if $\phi(\phi(x)) = x$ and $\phi(xy) = \phi(y)\phi(x), \forall x, y \in U$.

Let U be a arbitrary finite dimensional algebra with unity, $1 \neq 0$, with an involution $\phi : U \to U, \phi(a) = \overline{a}$, where $a + \overline{a}$ and $a\overline{a}$ belong in $K \cdot 1$, for all a in U. Let $\alpha \in K$, be a non zero fixed element . Over the vector space $U \oplus U$, we define the multiplication:

$$(a_1, a_2)(b_1, b_2) = \left(a_1 b_1 - \alpha \overline{b_2} a_2, a_2 \overline{b_1} + b_2 a_1\right). \tag{1}$$

Key Words: Division algebra; Cayley-Dickson process.

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In this way we obtain an algebra structure over $U \oplus U$. We denote the obtained algebra by (U, α) and it is called **the derivate algebra** obtained from the algebra U by **Cayley-Dickson process.** It is proved easily that the algebra U is isomorphic with a subalgebra of the algebra (U, α), and $\dim (U, \alpha) = 2 \dim U$. We denote $v = (0, 1) \in U$ and we obtain that $v^2 = -\alpha \cdot 1$, then $(U, \alpha) = U \oplus Uv$. In the next, we denote the elements of the form $\alpha \cdot 1$ by α and each of these elements is in U.

Let $x = a_1 + a_2 v \in (U, \alpha)$. Denoting $\overline{x} = \bar{a}_1 - a_2 v$, we remark that $x + \overline{x} = a_1 + \overline{a_1} \in K \cdot 1$, $x\overline{x} = a_1\overline{a_1} + \alpha a_2 \overline{a_2} \in K \cdot 1$. The map:

$$\psi : (U, \alpha) \to (U, \alpha) \quad \psi(x) = \bar{x}\,,$$

is an involution of the algebra (U, α), which extends the involution ϕ. If $x, y \in (U, \alpha)$, we have $\overline{xy} = \overline{y}\,\overline{x}$.

For $x \in U$ we denote $t(x) = x + \overline{x} \in K$, $n(x) = x\overline{x} \in K$, and we call them **the trace**, respectively **the norm** of the element x from U. If $z \in (U, \alpha)$, so that $z = x + yv$, then $z + \overline{z} = t(z) \cdot 1$ and $z\overline{z} = \overline{z}z = n(z) \cdot 1$, where $t(z) = t(x)$ and $n(z) = n(x) + \alpha n(y)$. From this, we have that $(z + \overline{z}) z = z^2 + \overline{z}z = z^2 + n(z) \cdot 1$, therefore

$$z^2 - t(z) z + n(z) = 0, \forall z \in (U, \alpha)\,,$$

so that each algebra obtained by the Cayley-Dickson process is a **quadratic algebra**. We remark that all algebras obtained by this process are flexible and power-associative algebras.

If in the Cayley-Dickson process we take $U = K$, $char K \neq 2$ and the involution $\psi(x) = x$, we obtain at the step t an algebra of dimension 2^t.

At the step 0, we obtaine the field K. This algebra has dimension 1.

At the step 1, we obtaine $\mathbb{K}(\alpha) = (K, \alpha), \alpha \neq 0$. This algebra has dimension 2. If the polynomial $X^2 + \alpha$ is irreducible over K , then $\mathbb{K}(\alpha)$ is a field, otherwise $\mathbb{K}(\alpha) = K \oplus K$ and it is a non division algebra.

At the step 2, we obtaine $\mathbb{H}(\alpha, \beta) = (\mathbb{K}(\alpha), \beta), \beta \neq 0$, the generalized quaternion algebra. This algebra has dimension 4. This is an associative algebra, but it is not a commutative algebra.

At the step 3, we obtaine $\mathbb{O}(\alpha, \beta, \gamma) = (\mathbb{H}(\alpha, \beta), \gamma), \gamma \neq 0$, the generalized octonion algebra or the Cayley-Dickson algebra. This algebra has dimension 8. This algebra is alternative(i.e. $x^2 y = x(xy)$ and $yx^2 = (yx)x$) but it is a non associative and non commutative algebra.

At the step 3, we obtain the generalized sedenion algebra, which is a non-alternative algebra, but it is a flexible and power-associative algebra. This algebra has dimension 16.

Definition 2. Let U be an arbitrary algebra over the field K. U is called **a division algebra** if and only if $U \neq 0$ and the equations:

$$ax = b,\ ya = b, \forall a, b \in U, a \neq 0,$$

have unique solutions in U.

From the beginning, we remark that if we started from a finite field, K, we don't obtained a division algebra, for any step of the Cayley-Dickson process. Indeed, we find the quaternion algebra. This algebra is always an associative and non commutative algebra. If this algebra is a division algebra it became a finite field. By Wedderburn's theorem, this field is a commutative field, false. Then we put the problem if we get the division algebra for all steps of the Cayley-Dickson process. If the field K is infinite, the answer is positive and we show that in the next.

Remark 3. If an algebra U is a finite dimensional algebra, then U is a division algebra if and only if from the relation $xy = 0$ it results that x or y must be zero.

Let $X_1, X_2, ..., X_t$ be t algebraically independent elements over K. For $i \in \{1, 2, ..., t\}$ we build the algebra U_i over the field $F = K(X_1, X_2, ..., X_t)$ putting $\alpha_j = X_j$ with $j = \overline{1, t}$. Let $U_0 = F$ and the involution $\psi(x) = x$. We prove by induction on i that U_i is a division algebra for i=$\overline{1, t}$.

Case 1.

For $j = 1$ we have $U_1 = U_0 \oplus U_0$ with $\alpha_1 = X_1$, $v_1 = (0, 1) \in U_1$, $x = a + bv_1$, $y = c + dv_1$, $\overline{x} = a - bv_1$ and the multiplication

$$xy = ac - \alpha_1 db + (bc + da) v_1.$$

If x, y are nonzero elements in U_1 such that $xy = 0$ we have

$$ac - X_1 db = 0 \tag{2}$$

and

$$bc + da = 0. \tag{3}$$

Since a, b, c, d are in F, from the relations (2) and (3), we have that a, b, c, d are non identically zero elements. Indeed, since $x \neq 0$ and $y \neq 0$ we have the possibilities:

i) $a = c = 0$ and $b, d \neq 0 \Rightarrow b = d = 0$, false.
ii) $a = d = 0$ and $b, c \neq 0 \Rightarrow b = c = 0$, false.
iii) $b = c = 0$ and $a, d \neq 0 \Rightarrow a = d = 0$, false.
iv) $b = d = 0$ and $a, c \neq 0 \Rightarrow a = c = 0$, false.

These have the form:

$$a = f(X_1, ..., X_t), \;\; b = g(X_1, ..., X_t),$$

$$c = h(X_1, ..., X_t), \;\; d = r(X_1, ..., X_t), f, g, h, r \in F$$

Since we can multiply the relations (2) and (3) with the great common divisor of denominators of f, g, h, r, we may suppose that f, g, h, r are in $K[X_1, ..., X_t]$. Replacing the elements a, b, c, d in the relation (2), we have: $x_1 \mid f$ or $x_1 \mid h$.

That means $X_1 \mid ac$. It results $X_1 \mid a$ or $X_1 \mid c$ or $X_1 \mid a$ and $X_1 \mid c$.

If $X_1 \mid a$ and $X_1 \mid c$ let i_1, i_2 be the greatest powers such that $a = X_1^{i_1} f_1$, $c = X_1^{i_1} h_2$, $f_1, h_1 \in K[X_1, ..., X_t]$. We replace in the relation (2), and we have:

$$X_1^{i_1+i_2} f_1 h_1 - X_1 db = 0.$$

Since $i_1 + i_2 \geq 2$, we obtain that $X_1 \mid db$. We have the possibilities:

i) $X_1 \mid d$ and $X_1 \nmid b$. It results that there is a great power i_3 such that $d = X_1^{i_3} r_1$, $r_1 \in K[X_1, ..., X_t]$. We replace in the relations (2) and (3) and we obtain:

$$X_1^{i_1+i_2} f_1 h_1 - X_1^{i_3+1} r_1 b = 0, \tag{2.1}$$

$$b X_1^{i_2} h_1 + X_1^{i_1+i_3} f_1 r_1 = 0. \tag{3.1}$$

a) If $i_2 < i_1 + i_3$ then $X_1 \mid bh_1$. Since $X_1 \nmid b$, we obtain $X_1 \mid h_1$. It results h_1 identically zero, so that $c = 0$, false.

b) $i_2 = i_1 + i_3$. From the relation (2.1), we have:

$$X_1^{2i_1+i_3} f_1 h_1 - X_1^{i_3+1} r_1 b = 0.$$

It results that $X_1 \mid r_1 b$. Since $X_1 \nmid b$, we have r_1 identically zero, then $d = 0$, false.

c) $i_2 > i_1 + i_3$. It result $X_1 \mid f_1 r_1$, then f_1 or r_1 are identically zero, false.

ii) $X_1 \nmid d$ and $X_1 \mid b$. Let i_4 be the greatest power such that $b = X_1^{i_4} g_1$, $g_1 \in K[X_1, ..., X_t]$. We replace in the relations (2) and (3) and we obtain:

$$X_1^{i_1+i_2} f_1 h_1 - X_1^{i_4+1} d g_1 = 0, \tag{2.2}$$

$$X_1^{i_2+i_4} g_1 h_1 + X_1^{i_1} d f_1 = 0. \tag{3.2}$$

a) If $i_1 < i_4 + i_2$ then $X_1 \mid df_1$, therefore f_1 is identically zero, false.

b) $i_1 = i_4 + i_2$. Then we have $X_1^{2i_2+i_4} f_1 h_1 - X_1^{i_4+1} d g_1 = 0$, therefore $X_1 \mid dg_1$, so that g_1 is identically zero, false.

c) $i_1 > i_4 + i_2$, then $X_1 \mid g_1h_1$, false.

iii) $X_1 \mid d$ and $X_1 \mid b$. We replace in the relations (2) and (3) and we have:

$$X_1^{i_1+i_2} f_1h_1 - X_1^{i_3+i_4+1} r_1g_1 = 0, \tag{2.3}$$

$$X_1^{i_2+i_4} g_1h_1 + X_1^{i_1+i_3} r_1f_1 = 0. \tag{3.3}$$

a) $i_1 + i_3 < i_4 + i_2$. Then $X_1 \mid r_1f_1$, false.
b) $i_1 + i_3 > i_4 + i_2$. Then $X_1 \mid g_1h_1$, false.
c) $i_1 + i_3 = i_4 + i_2$. We have:
c_1) $i_1 + i_2 > i_3 + i_4 + 1$, then $X_1 \mid r_1g_1$, false.
c_2) $i_1 + i_2 < i_3 + i_4 + 1$, then $X_1 \mid f_1h_1$, false.
c_3) $i_1 + i_2 = i_3 + i_4 + 1$. Since $i_1 + i_3 = i_4 + i_2$, we add this last two relations and we have $i_1 + 2i_2 + i_4 = 2i_3 + i_1 + i_4 + 1$, false.

Then we have $X_1 \mid a$ or $X_1 \mid c$. We suppose that $X_1 \mid a$ or $X_1 \nmid c$. From the relation (3), we have $X_1 \mid b$, therefore $X_1 \mid g$. Let s_1, s_2 be the greatest numbers such that $X_1^{s_1} \mid f, X_2^{s_2} \mid g$ and $X_1^{s_1+1} \nmid f, X_1^{s_2+1} \nmid g$. Then $f = X_1^{s_1} f_1$ and $g = X_1^{s_2} g_1$ with $f_1, g_1 \in K[X_1, ..., X_t]$. Replacing in the relation (2), we obtain:

$$X_1^{s_1} f_1c - X_1^{s_2+1} dg_1 = 0. \tag{2.4}$$

We have the cases:

1) $s_1 > s_2 + 1$. We have $X_1 \mid dg_1$ then $X_1 \mid r, r = X_1r_1, r_{1 \in K[X_1,...,X_t]}$. We replace in the relation (3) and we obtain:

$$X_1^{s_2} g_1c + X_1^{s_1+1} r_1f_1 = 0. \tag{3.4}$$

Then $X_1 \mid g_1c$ therefore $X_1 \mid g_1$, so that g_1 is identically zero, that means $b = 0$, false.

2) $s_1 < s_2 + 1$. From the relation (2.4), we have $X_1 \mid f_1c$, it results that $X_1 \mid f_1$, then f_1 is identically zero therefore $a = 0$, false.

3) $s_1 = s_2 + 1$. From the relation (3.4), we obtain $X_1 \mid g_1c$ then g_1 is identically zero, therefore $b = 0$, false.

We obtain that U_1 is a division algebra.

Case 2.

For $j = 2$ we have $U_2 = U_1 \oplus U_1$ with $\alpha_2 = X_2$, $v_2 = (0,1) \in U_2$, $x = a + bv_2$, $y = c + dv_2$, $\bar{x} = \bar{a} - bv_2$ and the multiplication

$$xy = ac - \alpha_2 \bar{d}b + (b\bar{c} + da)\, v_2.$$

If x, y are nonzero elements in U_2 such that $xy = 0$ we have

$$ac - X_2\bar{d}b = 0 \tag{4}$$

and

$$b\bar{c} + da = 0. \tag{5}$$

Like in Case 1, is obviously that a, b, c, d are nonzero elements. Let $\{u_1, u_2, u_3, u_4\}$ be a basis in U_2 over F. Then, we can write:

$$a = \sum_{j=1}^{4} f_j(X_1, ..., X_t)u_j, \;\; b = \sum_{j=1}^{4} g_j(X_1, ..., X_t)u_j,$$

$$c = \sum_{j=1}^{4} h_j(X_1, ..., X_t)u_j, \;\; d = \sum_{j=1}^{4} r_j(X_1, ..., X_t)u_j,$$

where the elements $f_j(X_1, ..., X_t)$, $g_j(X_1, ..., X_t)$, $h_j(X_1, ..., X_t)$, $r_j(X_1, ..., X_t)$ belong in F. We remark that $f_j(X_1, ..., X_t)$, $g_j(X_1, ..., X_t)$, $h_j(X_1, ..., X_t)$, $r_j(X_1, ..., X_t)$ can be chosen in $K[X_1, ..., X_t]$. We replace the elements a, b, c, d in the relation (4) and we obtain $X_2 \mid ac$.

Since U_1 is a division algebra, like in the **Case 1**, we have $X_2 \mid a$ or $X_2 \mid c$. We suppose that $X_2 \mid a$ and $X_2 \nmid c$. It results that $X_2 \mid f_j$, $\forall j = \overline{1,4}$, therefore $f_j = X_2 f_j'$. Since $X_2 \mid a$ and $X_2 \nmid c$, from the relation (5), we have $X_2 \mid b$, then $X_2 \mid g_j$, $\forall j = \overline{1,4}$, and $g_j = X_2 g_j'$. Let s_1 be the greatest power of X_2 such that $X_2^{s_1} \mid f_j$, $\forall j = \overline{1,4}$, and s_2 be the greatest power of X_2 such that $X_2^{s_1} \mid g_j, \forall j = \overline{1,4}$. Therefore we have $X_2^{s} \mid f_j$, $\forall j = \overline{1,4}$, and there is an index $t_1 \in \{1,2,3,4\}$ such that, $X_2^{s_1+1} \nmid f_{t_1}$. In the same way, we have $X_2^{s} \mid g_j$, $\forall j = \overline{1,4}$, and there is an index $t_2 \in \{1,2,3,4\}$ such that, $X_2^{s_2+1} \nmid f_{t_2}$. We replace in the relation (4) and we have the cases:

1) $s_1 > s_2 + 1$. We simplify with $X_2^{s_2+1}$, and we have that $X_2 \mid r_j, \forall j = \overline{1,4}$. Replacing in the relation (5), we simplify and, since $X_2 \nmid c$ $\exists$ t_3 such that $X_2 \nmid h_{t_3}$. We have $X_2^{s_2+1} \mid g_j, \forall j = \overline{1,4}$, then g_j are identically zero for all $j \in \{1,2,3,4\}$, false.

2) $s_1 < s_2 + 1$. We simplify by $X_2^{s_1}$, and we have $X_2 \mid f_j, \forall j = \overline{1,4}$, since $X_2 \nmid c$, false.

3) $s_1 = s_2 + 1$. Replacing in the relation (5), we simplify with $X_2^{s_2}$, and we obtain that $X_2 \mid g_j, \forall j = \overline{1,4}$, since $X_2 \nmid c$, false.

Therefore U_2 is a division algebra.

Case i.

By the induction step, we suppose that U_{i-1} is a division algebra. We have $U_i = U_{i-1} \oplus U_{i-1}$, with $\alpha_i = X_i$, $v_i = (0,1) \in U_i$, $x = a + bv_i$, $y = c + dv_i$, $\bar{x} = \bar{a} - bv_i$ and the multiplication

$$xy = ac - \alpha_i \bar{d} b + (b\bar{c} + da)\, v_i.$$

If x, y are nonzero elements in U_i such that $xy = 0$, we have that

$$ac - X_i \bar{d} b = 0 \tag{6}$$

and

$$b\bar{c} + da = 0. \tag{7}$$

But a, b, c, d are nonzero elements, since $x \neq 0$ and $y \neq 0$.

Let $\{u_1, u_2, ..., u_q\}$ be a basis in U_{i-1} over $F, q = 2^{i-1}$. Then we can write:

$$a = \sum_{j=1}^{q} f_j(X_1, ..., X_t) u_j, \ \ b = \sum_{j=1}^{q} g_j(X_1, ..., X_t) u_j,$$

$$c = \sum_{j=1}^{q} h_j(X_1, ..., X_t) u_j, \ \ d = \sum_{j=1}^{q} r_j(X_1, ..., X_t) u_j,$$

where the elements $f_j(X_1, ..., X_t), g_j(X_1, ..., X_t), h_j(X_1, ..., X_t), \ r_j(X_1, ..., X_t)$ belong to F. We remark that $f_j(X_1, ..., X_t)$, $g_j(X_1, ..., X_t)$, $h_j(X_1, ..., X_t)$, $r_j(X_1, ..., X_t) \in K[X_1, ..., X_t]$. Replacing the elements a, b, c, d in the relation (6), we obtain $X_i \mid ac$

Like in the **Case 1.**, since U_{i-1} is a division algebra, we have $X_i \mid a$ or $X_i \mid c$. We suppose that $X_i \mid a$ or $X_i \nmid c$. It results that $X_i \mid f_j, \forall j = \overline{1, q}$, and $f_j = X_i f_j'$. Since $X_i \mid a$ or $X_i \nmid c$, by the relation (7), we have $X_i \mid g_j, \forall j = \overline{1, q}$, and $g_j = X_i g_j'$. Let s_1 be the greatest power of X_i such that $X_i^{s_1} \mid f_j$, $\forall j = \overline{1, q}$, and s_2 be the greatest power of X_i such that $X_i^{s_2} \mid g_j, \forall j = \overline{1, q}$. Then we have $X_i^s \mid f_j$, $\forall j = \overline{1, q}$, and there is an index t_1 such that $X_i^{s_1+1} \nmid f_{t_1}$. Analogously, $X_i^s \mid g_j$, $\forall j = \overline{1, q}$, and there is an index t_2 such that $X_i^{s_2+1} \nmid f_{t_2}$. Replacing in the relation (6) we have the cases:

1) $s_1 > s_2 + 1$. We simplify by $X_i^{s_2+1}$, and it results that $X_i \mid r_j, \forall j = \overline{1, q}$. Replacing in the relation (7), we simplify and, since $X_i \nmid c$, $\exists \ t_3$ such that $X_i \nmid h_{t_3}$. We have $X_i^{s_2+1} \mid g_j, \forall j = \overline{1, q}$, then g_j are identically zero for all $j \in 1, ..., t$, false.

2) $s_1 < s_2 + 1$. We simplify with $X_i^{s_1}$, and we have $X_i \mid f_j, \forall j = \overline{1, q}$, since $X_i \nmid c$, false.

3) $s_1 = s_2 + 1$. Replacing in the relation (7), and simplify with $X_i^{s_2}$, then we have that $X_i \mid g_j, \forall j = \overline{1, q}$, since $X_i \nmid c$, false.

Therefore the algebra U_i is a division algebra and, by Cayley-Dickson process, $\dim U_i = 2^i, 1 \leq i \leq n$.

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