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A new simple proof for an inequality of Cebyshev type

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Abstract

We give here a simple proof of a well-known integral version of Cebyshev inequality. Using the same method, we give a lower bound in the case of increasing functions and then in the case of convex functions. We also establish a result at limit which shows that the constant 1/12 is sharp, in the sense that it cannot be replaced by a smaller one.

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It is mentioned in [2, pp. 297] the following inequality of Cebyshev type:

Theorem 1. *Let $f, g : [a, b] \to \mathbf{R}$ be derivable functions, with bounded derivatives on $[a, b] \subseteq \mathbb{R}$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)\, dx - \frac{1}{b-a} \int_a^b f(x)\, dx \cdot \frac{1}{b-a} \int_a^b g(x)\, dx \right| \leq$$

$$\leq \frac{(b-a)^2}{12} \cdot ||f'|| \cdot ||g'|| \, ,$$

where $||f'|| = \sup\limits_{x \in [a,b]} |f'(x)|\, , \ ||g'|| = \sup\limits_{x \in [a,b]} |g'(x)|\, .$

The constant $1/12$ is the best possible one in the sense that it cannot be replaced by a smaller one.

Key Words: Cebyshev inequality; Riemann integrals.

In order to prove the inequality, we denote

$$\phi(t) = (t-a)\int_a^t f(x)g(x)\ dx - \int_a^t f(x)\ dx \cdot \int_a^t g(x)\ dx \quad , \quad t \in [a,b]\,.$$

As we will see, the function ϕ plays a key role in what follows. We have

$$\phi'(t) = \int_a^t f(x)g(x)\ dx + (t-a)f(t)g(t) - f(t)\int_a^t g(x)\ dx - g(t)\int_a^t f(x)\ dx$$

which can be written as

$$\phi'(t) = \int_a^t \left[f(t) - f(x)\right]\left[g(t) - g(x)\right]\ dx.$$

With Lagrange theorem, we have

$$|f(t) - f(x)| \le ||f'||\,(t-x) \quad , \quad |g(t) - g(x)| \le ||g'||\,(t-x),$$

so

$$|\phi'(t)| = \left|\int_a^t \left[f(t) - f(x)\right]\left[g(t) - g(x)\right]\ dx\right| \le$$

$$\le \int_a^t |f(t) - f(x)| \cdot |g(t) - g(x)|\ dx \le$$

$$\le ||f'|| \cdot ||g'|| \cdot \int_a^t (t-x)^2\ dx = \frac{(t-a)^3}{3} \cdot ||f'|| \cdot ||g'||\,.$$

In consequence,

$$|\phi'(t)| \le \frac{(t-a)^3}{3} \cdot ||f'|| \cdot ||g'|| \quad , \quad \forall\ t \in [a,b]\,.$$

Now,

$$|\phi(b)| = |\phi(b) - \phi(a)| = \left|\int_a^b \phi'(t)\ dt\right| \le$$

$$\le \int_a^b |\phi'(t)|\ dt \le ||f'|| \cdot ||g'|| \cdot \int_a^b \frac{(t-a)^3}{3}\ dt = \frac{(b-a)^4}{12} \cdot ||f'|| \cdot ||g'||\,.$$

We obtain

$$|\phi(b)| \le \frac{(b-a)^4}{12} \cdot ||f'|| \cdot ||g'||$$

or

$$\left|(b-a)\int_a^b f(x)g(x)\ dx - \int_a^b f(x)\ dx \cdot \int_a^b g(x)\ dx\right| \le$$

$$\leq \frac{(b-a)^4}{12} \cdot ||f'|| \cdot ||g'||$$

and the required inequality follows by dividing with $(b-a)^2$.□

It is well-known that a basic form of Cebyshev inequality asserts that

$$\frac{1}{b-a}\int_a^b f(x)g(x)\ dx - \frac{1}{b-a}\int_a^b f(x)\ dx \cdot \frac{1}{b-a}\int_a^b g(x)\ dx \geq 0,$$

if f and g are monotone, with the same type of monotony. Moreover, we establish here the following stronger inequality:

Theorem 2 *Let $f, g : [a, b] \to \mathbf{R}$ be monotonically increasing. Assume further that f and g are derivable such that there exist*

$$\alpha = \inf_{x\in[a,b]} f'(x) \quad , \quad \beta = \inf_{x\in[a,b]} g'(x)$$

where α, β are nonnegative real numbers. Then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\ dx - \frac{1}{b-a}\int_a^b f(x)\ dx \cdot \frac{1}{b-a}\int_a^b g(x)\ dx \geq \frac{(b-a)^2}{12} \cdot \alpha\beta.$$

Proof. With the previous notations, we have $\phi \geq 0$ with

$$\phi'(t) = \int_a^t [f(t) - f(x)]\,[g(t) - g(x)]\ dx. \tag{1}$$

We use again Lagrange theorem to obtain

$$|f(t) - f(x)| \geq \alpha(t-x) \quad , \quad |g(t) - g(x)| \geq \beta(t-x)$$

and then

$$\phi'(t) \geq \alpha\beta \int_a^t (t-x)^2\ dx = \frac{(t-a)^3}{3} \cdot \alpha\beta.$$

By integrating with respect to t on $[a, b]$, we deduce

$$\phi(b) \geq \frac{(b-a)^4}{12} \cdot \alpha\beta.$$

Finally, the required inequality follows by dividing with $(b-a)^2$.□

Let us assume for now that f and g are twice derivable and there exist

$$\alpha_2 = \inf_{x\in[a,b]} f''(x) \quad , \quad \beta_2 = \inf_{x\in[a,b]} g''(x),$$

where α_2 and β_2 are nonnegative. According with the Taylor theorem, we have

$$f(t)-f(x)=f'(x)(t-x)+\frac{f''(\xi)}{2}(t-x)^2 \geq \frac{f''(\xi)}{2}(t-x)^2 \geq \frac{\alpha_2}{2}(t-x)^2$$

and

$$g(t)-g(x)=g'(x)(t-x)+\frac{g''(\xi)}{2}(t-x)^2 \geq \frac{g''(\eta)}{2}(t-x)^2 \geq \frac{\beta_2}{2}(t-x)^2.$$

If we substitute these estimations in (1), we derive

$$\phi'(t) \geq \frac{\alpha_2\beta_2}{4}\int_a^t (t-x)^4 \, dx = \frac{(t-a)^5}{20}\cdot\alpha_2\beta_2. \tag{2}$$

We can give the following similar inequality for twice derivable functions:

Theorem 3 *Let $f,g:[a,b]\to\mathbf{R}$ be twice derivable and monotonically increasing. Assume further that*

$$\alpha_2=\inf_{x\in[a,b]} f''(x) \quad , \quad \beta_2=\inf_{x\in[a,b]} g''(x)$$

where α_2,β_2 are nonnegative real numbers. Then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx \geq \frac{(b-a)^4}{120}\cdot\alpha_2\beta_2.$$

Proof. By integrating the inequality (2) with respect to t in $[a,b]$, we deduce

$$\phi(b) \geq \frac{\alpha_2\beta_2}{20}\int_a^b (t-a)^5 \, dt = \frac{(b-a)^6}{120}\cdot\alpha_2\beta_2$$

and the required inequality follows by dividing with $(b-a)^2$.□

In the sequel we use a new method to show that the constant $1/12$ is the best possible. Normally, this it proved if we can find a particular case when the equality arise. To give an example, let us remark that

$$f(x)=g(x)=x$$

in case $a=0$, $b=1$ provide:

$$\int_0^1 x^2 \, dx-\int_0^1 x \, dx\cdot\int_0^1 x \, dx=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}.$$

We also can prove the sharpeness of the constant 1/12 by giving the following

Theorem 3 *Let* $f, g : [a, b] \to \mathbf{R}$ *be derivable, with continuous derivatives at* a. *Then*

$$\lim_{t\to a} \frac{1}{(t-a)^2}\left[\frac{1}{t-a}\int_a^t f(x)g(x)\ dx - \frac{1}{t-a}\int_a^t f(x)\ dx \cdot \frac{1}{t-a}\int_a^t g(x)\ dx\right] =$$

$$= \frac{1}{12}\cdot f'(a)g'(a).$$

Proof. The required limit can be written as

$$\lim_{t\to a}\frac{\phi(t)}{(t-a)^4}.$$

In order to use l'Hospital rule, let us compute

$$\lim_{t\to a}\frac{\phi'(t)}{4(t-a)^3} = \frac{1}{4}\lim_{t\to a}\frac{\int_a^t [f(t)-f(x)]\,[g(t)-g(x)]\ dx}{(t-a)^3} =$$

$$= \frac{1}{4}\lim_{t\to a}\frac{f'(\xi_{t,x})g'(\eta_{t,x})\int_a^t (t-x)^2\ dx}{(t-a)^3} = \frac{1}{4}\cdot f'(a)g'(a)\lim_{t\to a}\frac{\frac{1}{3}(t-a)^3}{(t-a)^3} =$$

$$= \frac{1}{12}\cdot f'(a)g'(a).$$

□

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