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Stability of Discrete Shocks for Difference Approximations to Systems of Conservation Laws

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Abstract

In this paper we analyse the stability of weak discrete stationary shocks. The difference approximation is conservative, dissipative and k-th order accurate ($k = 1$ or $k = 3$). In Sections 1-3, following the previous results of D. Michelson we adapt the multistep methods Runge-Kutta and Adams-Bashforth to solve systems of conservation laws. In section 4 we analyse the stability of the numerical method and provide maximal values for the Courant-Friederichs-Levy number. The numerical examples presented in Section 5 confirm the robustness of the algorithm.

Subject Classification: 65M12, 65M06, 35L65, 35L67.

1 Introduction

Let us consider a system of conservation laws

$$u_t + f(u)_x = 0, \tag{1.1}$$

where $f : R^n \longrightarrow R^n$ is a smooth vector function and the unknown function $u = u(x,t) \in R^n$ depends on $-\infty < x < \infty$ and $t > 0$. We are interested in

Key Words: Discrete shocks; System of conservation laws; Asymptotic stability; High order dissipation.

studying the solution of (1.1) when it develops stationary shocks. By planar stationary shock one means the solutions

$$u = u_L, x < x_0; \quad u = u_R, x > x_0, \tag{1.2}$$

where u_L, u_R satisfy the Rankine-Hugoniot condition

$$f(u_L) = f(u_R). \tag{1.3}$$

Suppose that (1.1) is approximated by a dissipative system

$$u_t + f(u)_x = -(i\partial_x)^{\frac{k+1}{2}}[A(u)(i\partial_x)^{\frac{k+1}{2}}]u, \tag{1.4}$$

where k is an odd number. Stationary solutions u_{st} of (1.4) which attain the above limits u_L and u_R as $x \to -\infty$ and $x \to \infty$ correspondingly are called *stationary viscous shocks*. The travelling shocks by a change of variables $x - c \cdot t \to x$ are reduced to the stationary ones. Furthermore, if the difference $|u_R - u_L|$ is small, the shock is called *weak*. We assume that u_L, u_R lie in a small neighbourhood of a point u_0 such that the differential $df[u_0]$ of f at u_0 has *distinct real* eigenvalues with a zero eigenvalue $\lambda_1(u_0) = 0$. Without loss of generality we may assume that

$$df[u_0] = diag(\lambda_1(u_0), \lambda_2(u_0), ..., \lambda_n(u_0)). \tag{1.5}$$

The eigenvalue $\lambda_1(u)$ should be also *genuinely nonlinear*, which means that its e_1 directional derivative is not null, i.e.

$$d\lambda_1[u_0] \cdot e_1 \neq 0, \tag{1.6}$$

where $e_1 = (1, 0, ..., 0)^T \in R^n$. We assume that the pair u_L , u_R is *entropy satisfying*, i.e.

$$\lambda_1(u_L) > 0 > \lambda_1(u_R) \tag{1.7}$$

and the right hand side of (1.4) is dissipative in a proper sense. Under the above assumptions in [2] the authors proved the existence of viscous shocks for cases $k = 1$ and $k = 3$.

The following natural question appears - are these shocks asymptotically stable ? Namely, given an initial condition $u(x, 0)$ which is a small perturbation of the stationary viscous shock $u_{st}(x)$, will $u(x, t)$ tend to $u_{st}(x)$ as $t \to \infty$? For $k = 1$ this problem was studied by Goodman in [1]. Under the asssumption of zero mean perturbation, i.e.

$$\int_{-\infty}^{\infty} (u(x, 0) - u_{st}(x))dx = 0,$$

he proved that the shock is asymptotically stable. Liu in [3] removed the above restriction, but assumed that u_L and u_R are connected by a non-degenerate sum of n shocks. Then he showed that $u(x,t)$ tends asymptotically to a sum of travelling viscous shocks.
Let us consider the stability problem for the shock solution of the scalar equation

$$u_t + \left(\frac{u^2}{2}\right)_x = -u_{xxxx} \ . \tag{1.8}$$

The stationary shock u_{st} satifies the equation

$$u_{xxx} = \frac{1}{2}(1-u^2), \quad u(\mp\infty) = \pm 1, \tag{1.9}$$

and is not monotone. As a result, the energy method in the L_2 space is not applicable to (1.8). We study the stability of u_{st} on a finite interval $|x| \le l$, where

$$\delta_0^{-1}\varepsilon^{-1} \le l \le \delta_0\varepsilon^{-k} \tag{1.10}$$

or

$$\delta_0^{-1}\varepsilon^{-1} \le l \le \delta_0\varepsilon^{-k-1} \tag{1.11}$$

Here ε^{-1} is proportional to the width of the boundary layer

$$\varepsilon^{-1} \sim (\lambda_1(u_L)^{-\frac{1}{k}} \sim |u_R - u_L|^{-\frac{1}{k}} \tag{1.12}$$

and δ_0 is a small constant. The boundary conditions should also include

$$P\left(f(u) + (i\partial x)^{\frac{k+1}{2}}[A(u)(i\partial x)^{\frac{k+1}{2}}u]\right) = 0, \qquad x = \pm l, \tag{1.13}$$

where $P : R^n \to R^n$ is a projector on the first n_1 components of u. This implies that the integral

$$\int_{-l}^{l} Pu dx$$

is conserved in time. The restriction (1.10) applies if $n_1 = 1$ or $n_1 = n$, thus for $k = 1$, we shall consider only these cases.

Remark 1.1 *Note that the stationary solution u_{st} tends exponentially fast to u_L, u_R with respect to the variable εx, thus the boundaries $x = \pm l$ are practically at infinity.*

Remark 1.2 *The real motivation for the shock stability problem comes from the aerodynamic computations. Hence instead of the continuous model* (1.4) *we will consider a discrete approximation of* (1.1). *The continuous problem could be solved in a similar way. The discrete settings will follow the paper in [2] where the author proved the existence of weak discrete shocks for dissipative approximations with* $k = 1$ *and* $k = 3$. *The most popular numerical schemes for shock wave computations produce monotone shock profiles, thus they exclude high order approximations. The results indicate that the high order schemes, although producing one overshoot of about* 20% *in the shock layer, converge exponentially fast to the stationary solution.*

2 The Difference Approximation

The system of conservation laws (1.1) is approximated by the difference scheme

$$G(\{E^j u(x,t)\}) = 0, \quad j = (j_1, j_2) \in J \subset Z^2, \quad j_2 \leq 0, \tag{2.1}$$

where J is a finite set, $G = G(\{u_j\})$ is a smooth vector (at least C^3) function of vector variables $u_j \in R^n, \quad j \in J$, and $E^j = E_x^{j_1} E_t^{j_2}$ is a shift operator

$$E^j u(x,t) = u(x + j_1 h, t + j_2 h). \tag{2.2}$$

We consider $u(x,t)$ in (1.4) as a grid function defined on a uniform grid with a mesh size h

$$D_h = I_h \times (R_+)_h \tag{2.3}$$

in the half-strip $[-l, l] \times [0, \infty)$. The scheme G should be *conservative*, i.e.

$$G(\{E^j u(x,t)\}) = (E_x - I)G_1(\{E^j u(x,t)\}) + (E_t - I)G_2(\{E^j u(x,t)\}) \tag{2.4}$$

with the multi-index j in G_1,and G_2 varying over corresponding subsets of J. We assume that G is consistent with (1.1)

$$G_1(\{u\}) = f(u), \quad G_2(\{u\}) = u, \tag{2.5}$$

where $\{u\}$ stands for the set $\{u_j\}$ of vectors $u_j = u$. Since the domain is bounded, there are boundary conditions of the form

$$S_L(\{E^j u(-l,t)\}, u_L) = 0, \quad S_R(\{E^j u(l,t)\}, u_R) = 0, \tag{2.6}$$

where S_L and S_R depend smoothly on its arguments and j belongs to a finite set. The range of $j = (j_1, j_2)$ in S_L is such that $j_1 \geq 0, \ j_2 \leq 0$ while in S_R we

have $j_1 \le 0$ $j_2 \ge 0$, and the boundary conditions should be *consistent* with a constant solution ,

$$S_L(\{u\}), u) = S_R(\{u\}, u) = 0. \tag{2.7}$$

There are also the initial conditions

$$u(x,t) = u_{in}(x,t), \qquad x = j_1 \in [-l, l], \qquad t = j_2 \in [0, \Delta j_2 - 1], \tag{2.8}$$

where $-\Delta j_2$ is the minimal value of the index j_2 in the functions G, S_L and S_R. Note that we can always add to G, S_L, S_R, dummy variables u_j so that they have a common minimal j_2. Then we consider a family of initial boundary value problems which depend on the parameter $\varepsilon \sim (\lambda_1(u_L))^{\frac{1}{k}}$ with u_L, u_R satisfying (1.3) and (1.7) so that $u_L(\varepsilon = 0) = u_R(\varepsilon = 0) = u_0$ and l as in (1.10) or (1.11). More precisely

$$u_L = u_0 + \mu e_1, \qquad u_R = u_0 - \mu e_1 + O(\mu^2), \qquad \mu = \frac{b\varepsilon^k}{(d\lambda_1[u_0] \cdot e_1)}, \tag{2.9}$$

where b is the (positive) dissipation coefficient and e_1 is the unit vector as in (1.6). Equation (2.1) is defined for (x,t) in a subdomain $\overset{\circ}{D}_h$ such that $(x + j_1, t + j_2) \in D_h$ (see (2.2)) lies in D_h for all $j \in J$. We will assume that $h = 1$ so that $2l$ is the number of mesh points in I_h. If we denote by $x_{-l}, x_l - 1$ the left and right end points of $\overset{\circ}{D}_h$ and sum the equation in (2.1) for $x \in [x_{-l}, x_l - 1]$, we obtain the global conservation law

$$(E_t - I) \sum G_2(\{E^j u(x,t)\}) = G_1(\{E^j u(x_{-l}, t)\}) - G_1(\{E^j u(x_l, t)\}). \tag{2.10}$$

It would be natural to assume that

$$G_1(\{E^j u(x_{\pm l}, t)\}) - f(u_{R,L}) = 0, \tag{2.11}$$

i.e. (2.11) is a part of the boundary conditions from (2.6). More generally, we will assume that

$$P(G_1(\{E^j u(x_{\pm l}, t)\}) - f(u_{R,L})) = 0, \tag{2.12}$$

where $P = P(\mu) : R^n \to R^n$ is a projector which depends smoothly on μ such that $P(0)$ is the standard projection on the first n_1 components of u. As a result the solution of IBVP (2.1), (2.7), (2.9) satisfies

$$P \sum G_2(\{E^j(u(x,t)\}) = P \sum G_2(\{E^j(u_{in}\}). \tag{2.13}$$

If we linearize the above IBVP at the constant state $u = u_0$ and $\mu = 0$ with arbitrary l, then one should assume that the linear problem

$$dG[u_0]u = F, \quad dS_L[u_0] = g_L, \quad dS_R[u_0] = g_R,$$

is solvable in time. The resulting constant coefficient difference operators will be denoted by $dG[u_0](E_x, E_t)$, $dS[u_0](E_x, E_t)$ or simply $dG[u_0]$, $dS[u_0]$. The Laplace-Fourier symbol of $dG[u_0]$ is defined as $dG[u_0](e^{i\xi}, e^s)$.
According to [4] we introduce the following assumptions
Dissipativity: The symbol $dG[u](e^{i\xi}, e^s)$ is non-singular for all pairs (ξ, s) with $Re(s) \geq 0$ but $s = 0$ and $\xi = 0 \mod 2\pi$, where u is any vector in a neighbourhood of u_0.
Accuracy: The difference operator $dG[u_0]$ restricted to the x variable is exactly the k-th order accurate approximation of $df[u_0]\frac{\partial}{\partial x}$ in the direction $e_1 = (1, 0, \ldots, 0)^T$.
Under these assumptions in [4] the author proved that the initial boundary value problem is asymptotically stable.

3 Numerical schemes

The following approximation was sugested in [5]: the derivative $f(u)_x$, in (1.1) is approximated by

$$F(\{E^\alpha u\}) = \frac{1}{\Delta x}(D_x^{(k+1)} f(u) + (-1)^{k_1} K(E_x^{\frac{1}{2}} - E_x^{-\frac{1}{2}})^{(k+1)} u), \tag{3.1}$$

where $k_1 = \frac{k+1}{2}$ and $(\Delta x)^{-1} D_x^{(k+1)}$ is $k+1$ order central difference approximation of the operator ∂_x and $K = O^*(1)$ is a positive constant. For example

$$D_x^{(k+1)} = \sum_{j=1}^{k_1} c_j (E_x^j - E_x^{-j}), \quad c_j = \frac{(-1)^{j-1}(k_1!)^2}{j(k_1 - j)!(k_1 + j)!}, \tag{3.2}$$

which has a maximal accuracy for the $k+2$ point lattice. The equation (1.1) is then replaced by

$$u_t + F(\{E^\alpha u\}) = 0, \tag{3.3}$$

for which we can apply a classical O.D.E. solver. Let us consider now the equation (3.3) in the space of grid functions $u(x), x \in I_h$. We will approximate the equation (3.3) by an m-th O.D.E. solver with $m \geq k$. For $k = 1$ the approximation for the derivative becomes

$$F(\{E^\alpha u\}) = \frac{1}{\Delta x}\left[\frac{1}{2}(E_x^1 - E_x^{-1}) f(u) - K_1(E_x^1 - 2I + E_x^{-1})u\right],$$

whereas for $k = 3$

$$F(\{E^{\alpha}u\}) = \frac{1}{\Delta x}\left[\frac{1}{12}(E_x^{-2} - 8E_x^{-1} + 8E_x^{1} - E_x^{2})f(u)\right.$$

$$\left.+K_1(E_x^{-2} - 4E_x^{-1} + 6I - 4E_x^{1} + E_x^{2})u\right].$$

Let us consider now instead of (1.1) the equation (3.3). We then can apply as solver the m-th order Adams-Bashforths multi-step method

$$(I - E_t^{-1}) + \Delta t \sum_{i=1}^{m} d_i E_t^{-i} F(u) = 0, \qquad m \geq k, \tag{3.4}$$

or an explicit r-stage Runge-Kutta method of the form

$$k_q = \Delta t F\left(u + \sum_{j=1}^{q-1} \alpha_{qj} k_j\right), \quad E_t u = u + \sum_{q=1}^{r} \alpha_{r+1,q} k_q, \quad \alpha_{q,q-1} \neq 0. \tag{3.5}$$

Both methods must be conservative, dissipative and k-th order accurate in space and with k an odd number. Moreover, both schemes clearly satisfy the accuracy and dissipativity assumptions stated in the previous section. According to the boundary conditions in the case of the Runge-Kutta method because of the global conservation laws we have to use the following ones

$$PF_1\left(u + \sum_{j=1}^{q-1} \alpha_{qj} k_j\right)\Big|_{x=x_{\pm l}} - Pf(u_{L,R}) = 0, \quad 2 \leq q \leq r+1. \tag{3.6}$$

In (3.5) we may consider the fourth-stage Runge Kutta method given by

$$x_{n+1} = x_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$

where

$$\begin{cases} k_1 = \Delta t \cdot F(x) \\ k_2 = \Delta t \cdot F\left(x + \frac{1}{2} \cdot k_1\right) \\ k_3 = \Delta t \cdot F\left(x + \frac{1}{2} \cdot k_3\right) \\ k_4 = \Delta t \cdot F(x + k_3) \end{cases}$$

whereas in (3.4) a possible choice is given by the Adams-Bashforth explicit method

$$x_{n+1} = x_n + \frac{\Delta t}{24}\left(55F(x) - 59F(x - \Delta t) + 37F(x - 2\Delta t) - 9F(x - 3\Delta t)\right). \tag{3.7}$$

Runge-Kutta	$(-2.78, 0)$
Adams - Bashforth	$(-1.25, 0)$

Table 1: Intervals of stability.

where the first values for this method can be predicted with values like in Runge-Kutta method. In Table 1 we list the interval of absolute stability for the fourth-order four stage Runge-Kutta method and the fourth order Adams-Bashforth method.

Remark 3.1 *If we now make the comparison on the basis of the same number of function evaluation over a given interval we can apply the Adams-Bashforth method with half steplength of the Runge-Kutta method. From the point of view of the asymptotic stability characteristics, Runge-Kutta method are much better than the Adams-Bashforth method, and method like Runge-Kutta are easily programmed for a computation with a fixed steplength. In contrast to multi-step methods, the Runge Kutta method being one-step method, requires only the value at the last time point of the approximate solution and allows one to carry out calculations under initial conditions which are natural for the equation which we want to solve. Because the method does not make use of information concerning the solution of the previous nodes of the grid , it is in general less economical than Adams-Bashforth methods.*

The Runge-Kutta method employed here is not exactly of the type (2.1) with the boundary conditions (2.12). If we eliminate the intermediate steps k_q the resulting scheme should be rewritten as:

$$(E_t - I)u + (E_x - I)G_1(\{E^j u(x,t)\}, x) = 0 \tag{3.8}$$

One can consider the equation (3.8) for boundary points (i.e., not for inner points) as artificial boundary conditions where the flux G_1 changes its form as a discrete function of x.

4 Stability analysis of the methods

Let us consider the expression of $F(\{E^\alpha u\})$ from the equation (3.1). It is clear that we have $F(u) = (E_x - I)F_1(u)$ and hence the scheme is in the conservation form (2.4). Moreover, if we denote by $d\hat{F}[u_0]$ the Fourier symbol of the differential of F at u_0, The general form of the eigenvalues λ of $\Delta t d\hat{F}[u_0]$

will be

$$\lambda = \frac{\Delta t}{\Delta x}\left(\lambda_i(u_0)\sqrt{-1}\sum_{j=1}^{k_1} 2c_j \sin(j\xi) + K_1\left(2\sin\frac{\xi}{2}\right)^{k+1}\right). \tag{4.1}$$

In the case of m-th order Adams-Bashforth explicit method

$$(I - E_t^{-1})u + \Delta t\sum_{i=1}^{m} d_i E_t^{-i} F(u) = 0, \tag{4.2}$$

and for the stability of the Cauchy problem one needs that the roots z of the characteristic equation

$$1 - z^{-1} + \sum d_i z^{-i}\lambda = 0, \tag{4.3}$$

should satisfy $|z| \leq 1$, or in other words $-\lambda$ should belong to the domain Ω of absolute stability of the multi-step method. For small values of ξ, if $m > k+1$, $-\lambda$ from the formula (4.1) always belong to Ω. This will restrict the choice of $\lambda_i(u_0)\frac{\Delta t}{\Delta x}$ and $K_1 \cdot \frac{\Delta t}{\Delta x}$.

Let us consider first the case when $k = 1$. Then the eigenvalue λ from (4.1) takes the form :

$$\lambda = \frac{\Delta t}{\Delta x}\left(\lambda_i(u_0)\sqrt{-1}\cdot\sin(\xi) + 4\cdot K_1 \sin^2\left(\frac{\xi}{2}\right)\right).$$

We consider the maximum of the values $\lambda_i(u_0), i = 1, \ldots, n$ over all domain of computation and first for simplicity the value of $K_1 = 0$. In order to compute the domain Ω of absolute stability in the case of 4-th order Runge-Kutta explicit method we need to have:

$$\left|1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!}\right| \leq 1. \tag{4.4}$$

In this case we can get the maximum value for λ should not exceed $\sqrt{8}$. Furthermore, if we fix the value of $\lambda_i(u_0)\frac{\Delta t}{\Delta x}$ at the above maximal value we have to compute the maximal value of $K_1 \neq 0$ such that the multi-step method remain stable. In Table 2 we show some of the values $\lambda_i(u_0)\frac{\Delta t}{\Delta x}$ and $K_1\frac{\Delta t}{\Delta x}$ In Figure 1 we plot the graph of the left hand side expression in (4.4). It is clear that the value of the expression is less than one and the method remains stable. For $k = 3$ the relation (4.1) becomes

$$\lambda = \frac{\Delta t}{\Delta x}\left(i\cdot\lambda_i(u_0)\cdot 2\cdot\left(\frac{8}{12}sin(\xi) - \frac{1}{12}sin(2\xi)\right) + 16\cdot K_1\cdot sin^4\left(\frac{\xi}{2}\right)\right).$$

$\lambda_i(u_0)\frac{\Delta t}{\Delta x}$	$K_1\frac{\Delta t}{\Delta x}$
1.5	0.2
1	0.4
0.5	1

Table 2: Maximal values for stability for $k = 1$.

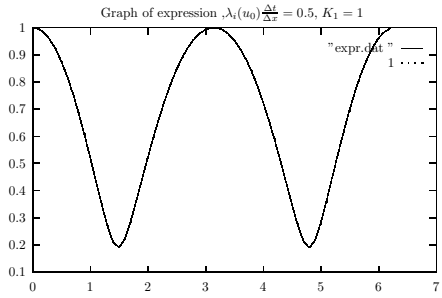

Figure 1: Graph of the expression (4.3) in case when $k = 1$.

We also ask if the following inequality holds

$$\left|1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!}\right| \leq 1, \tag{4.5}$$

for all $\xi \in [0, 2\pi]$ and some $K_1 \neq 0$. Some values for $\lambda_i(u_0)\frac{\Delta t}{\Delta x}$ and $K_1\frac{\Delta t}{\Delta x}$ are presented in Table 3. Furthermore, in Figure 2 we show the graph of left hand side of (4.5).

In all the numerical experiments presented in the next section of the paper

$\lambda_i(u_0)\frac{\Delta t}{\Delta x}$	$K_1\frac{\Delta t}{\Delta x}$
1.5	0.1
1	0.17
0.5	0.3

Table 3: Maximal values for stability for $k = 3$.

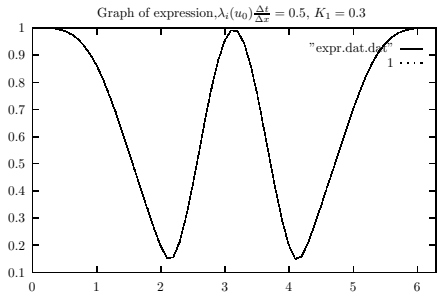

Figure 2: Graph of expression (4.3) when $k = 3$.
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we make this analysis of the eigenvalue λ to provide a small enough value for the time step Δt which should ensure the convergence of the multistep method.

5 Numerical examples

5.1 Example 1

Many problems in mechanics, in particular in gas dynamics lead us to the study of nonlinear hyperbolic conservation laws. One of the simplest conservation law (nonlinear hyperbolic partial differential equation) is the Burgers' equation (see e.g. [6])

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \qquad x \in R, \qquad t > 0.$$

We shall approximate the solution of the Burgers equation for a strong shock

$$u(0,t) = u_0(x) = \begin{cases} u_L & , x \geq 0 \\ u_R & , x < 0 \end{cases},$$

where $u_L = 1, \quad u_R = -1$. When we consider $K_1 = 1$ we eliminate this oscillations with a 4-th order term which smears the profiles of the numerical approximation like in Figure 3.

Remark 5.1 *By applying multi-step methods like Runge-Kutta or Adams Bash-forth we can see that they smear the profile of the numerical solution. Finally, in this case the methods converge to a stationary solution which attains the limits u_L and u_R when $x \to -\infty$ and $x \to +\infty$, respectively. These numerical solutions are called* stationary viscous shocks.

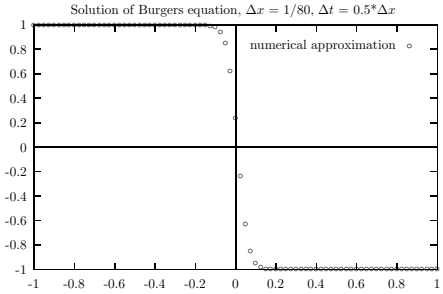

Figure 3: Stationary solution at time $T = 1.1$ using Runge-Kutta 4-th order for $K_1 = 0$.

5.2 Example 2

This example comes from a classical problem in gas dynamics. In this case we apply the Runge-Kutta scheme to the one dimensional system,

$$u_t + f(u)_x = 0,$$

where

$$u = (\rho, \rho u, E)^T,$$

and

$$f(u) = (\rho u, P + \rho u^2, u(E + P))^T,$$

augmented with $P = (\gamma - 1)(E - \frac{1}{2}\rho u^2)$. Here ρ, u, P and E are respectively, the density, velocity, pressure and total specific energy. The initial data is of the Riemann form

$$u(x, 0) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x > 0 \end{cases}$$

The first Riemann problem was proposed by Sod [6]. If we denote $m = \rho u$ then we have

$$u = (\rho, m, E)^T,$$

and the function

$$f(u) = \left(m, \quad P + \frac{m^2}{\rho}, \quad \frac{m}{\rho}(E + P)\right)^T.$$

The initial data is taken as

$$(\rho_L, u_L, P_L) = (1, 0, 1), \quad (\rho_R, u_R, P_R) = (0.125, 0, 0.1).$$

Numerical approximations for density in both cases $k = 1$ and $k = 3$ are shown in Figure 4. In Figure 4 the numerical approximation for the density is presented for both cases $k = 1$ and $k = 3$. This approximation is obtained using the Runge-Kutta method. We can observe that the solution is composed from three types of shocks , a rarefaction wave, a contact discontinuity and a simple shock. Furthermore, in Figures 5 and 6 we show the numerical solutions for the velocity and pressure profiles at the same final time $T = 2.0$.

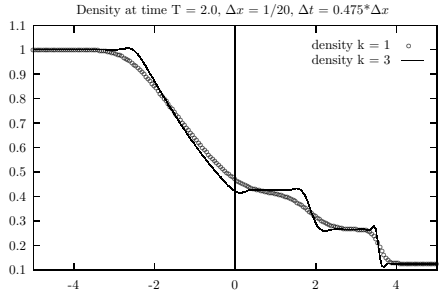

Figure 4: Plot of the density in cases $k = 1$ and $k = 3$..

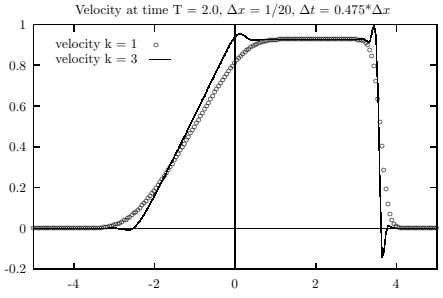

Figure 5: Plot of the velocity in cases $k = 1$ and $k = 3$.

5.3 Example 3

The third example is for the second Riemann problem which was proposed by Lax (see [6]). In this example the initial data is given by

$$(\rho_L, u_L, P_L) = (0.445, 0.698, 3.528), \quad (\rho_R, u_R, P_R) = (0.5, 0, 0.571).$$

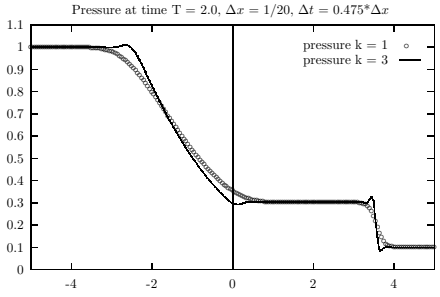

Figure 6: Plot of the pressure in cases $k = 1$ and $k = 3$.

We used again as solver the Runge-Kutta method. The numerical solutions of Riemann problem - density, velocity, pressure - are presented in Figures 7, 8, 9, respectively.

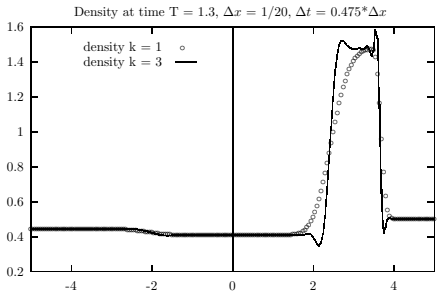

Figure 7: Plot of the density in cases $k = 1$ and $k = 3$.

5.4 Example 4

In this case we approximated the solution of two-dimensional Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^2}{2}\right)_y = 0,$$

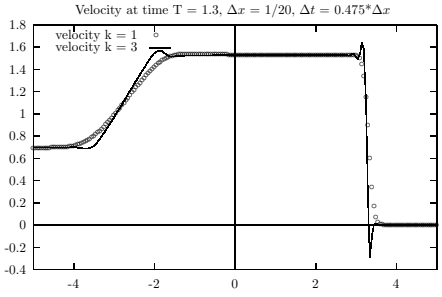

Figure 8: Plot of the velocity in cases $k = 1$ and $k = 3$.

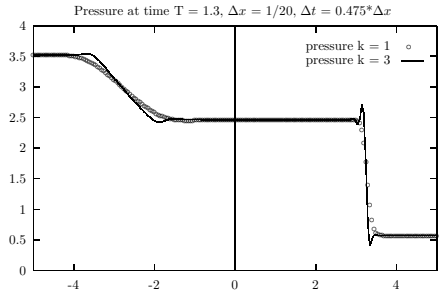

Figure 9: Plot of the pressure in cases $k = 1$ and $k = 3$.

subject to the initial conditions

$$u_0(x,y) = \begin{cases} 0.5 & -1 \le x < 0, -1 \le y < 0 \\ 0 & 0 \le x \le 1, -1 \le y < 0 \\ -1 & 0 \le x \le 1, 0 \le y \le 1 \\ -0.2 & -1 \le x < 0, 0 \le y \le 1. \end{cases}$$

To implement the boundary conditions at outflow boundary we used first order extrapolation. In Figure 10 we present the solution at the time $T = 1.$ for a mesh with 41×41 points.

6 Conclusions and further work

This paper investigates the stability of discrete shocks approximations for particular cases $k = 1$ and $k = 3$. Numerical examples are shown to emphasize

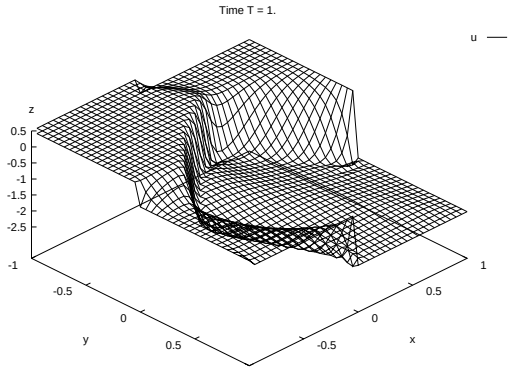

Figure 10: The 2D IBVP Burgers equation , T = 1, on a mesh 41×41

the robustness of the discrete approximations based on multi-step methods. Further work will be to apply the difference approximation to other two dimensional problems.

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