An. Şt. Univ. Ovidius Constanţa Vol. 14(1), 2006, 67–70

A Note on the Isomorphism of Modular Group Algebras of p-Mixed Abelian Groups with Divisible p-Components

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Abstract

We give a new conceptual proof of the following classical fact due to Karpilovsky (Contemp. Math., 1982) but over finite fields: Let G be a p-mixed abelian group with divisible p-component. If F is a finite field of $char(F) = p$, then $FH \cong FG$ as F-algebras for another group H forces that $H \cong G$.

Subject Classification: Primary 20C07, 16S34, 16U60; Secondary 20K10, 20K21, 20K25.

I. Introduction

Traditionally, suppose that FG is the group algebra of an abelian group G over a field F of positive characteristic, for instance p. For such an algebra FG, the letter $V(FG)$ is reserved for the group of normalized units in FG. As usual, throughout this brief article, $G[p]$ denotes the socle of the p-primary component G_p of G and $f_\tau(G)$ denotes the Ulm-Kaplansky p-invariants of G for any ordinal number τ; it is simply observed that $f_\tau(G) = f_\tau(G_p)$. All other unexplained exclusively notions and notations are standard and follow essentially the cited in the references papers.

As a point of departure in this theme concerning the *Isomorphism Problem* for p-mixed groups, we recall some of the major achievements. The principal known results in that aspect given in a chronological order are these: In [12] and [13] it was argued that if G is a p-mixed abelian group whose p-torsion part G_p is the direct product of a divisible group and a bounded group, then the isomorphism class of G is determined by FG; notice that this assertion was refined in [3] to p-mixed splitting groups with simply presented components. In [15] it was proved that if G is an abelian p-mixed group of torsion-free rank

Key Words: Group algebra; p-mixed abelian group; Group isomorphism.
Received: December, 2005

one for which G_p is countable, then FG recovered the isomorphic class of G. Later on, in [11], Hill and Ullery generalized the assertion for p-mixed abelian groups of torsion-free rank one whose p-component possesses length $< \omega_1 + \omega_0$, where ω_1 is the first uncountable ordinal and ω_0 is the first infinite ordinal, respectively.

In [1] we have extended these attainments to totally projective p-component G_p of length strictly less than ω_1^2. This restriction on $length(G_p)$ was omitted in [9] but when F coincides with the field of p-elements. In ([7], [8]) we strengthened all of the foregoing quoted results for torsion-free rank one to p-mixed Warfield groups since the abelian groups of torsion-free rank one with simply presented primary components are known to be Warfield.

However, in the current situation, the examined by us group G may have torsion-free rank greater than one, whence it is not necessarily a Warfield group. Nevertheless, the p-component of torsion of our group has to be of necessity divisible. But we do not use the method presented in [12] and, thereby, we need a different approach.

Before stating and proving the main statement, we must establish some preliminaries.

II. Preliminary Technicalities

We list here three affirmations necessary for the verification of the main result. First of all, we start with a group-theoretic claim on the cancellation property in abelian groups with specific Ulm-Kaplansky functions.

Group Theorem ([10]). *Let $B \times G \cong C \times H$, where $B \cong C$ is a coproduct of countable abelian p-groups, and G and H are arbitrary abelian groups. For every Ulm-Kaplansky p-invariant of B, assume that it is either finite or else the corresponding Ulm-Kaplansky p-invariants of G and H are zero. Then, $G \cong H$.*

The next well-known classical theorem concerns the separating of G as a direct component of $V(FG)$ (for more details, see also [11], [15]; [2], [3] and [5]).

Direct Factor Theorem. *Let G be a p-mixed abelian group so that G_p is divisible and F a perfect field of $char(F) = p \neq 0$. Then $V(FG)/G$ is a coproduct of countable p-groups and G is a direct factor of $V(FG)$.*

We end the preliminary technicalities with

Theorem of Invariants ([6]). *Let G be a nontrivial abelian group and let F be a finite field of $char(F) = p > 0$. Then, for each ordinal $\tau \geq 0$, the Ulm-Kaplansky p-invariants of $V(FG)/G$ are computed only in terms of F and G. In particular, if $G_p^{p^\tau} \neq 1$, $G^{p^\tau} \neq G^{p^{\tau+1}}$ and $|G^{p^\tau}| \geq \aleph_0$ then $f_\tau(V(FG)/G) = |G^{p^\tau}|$ while if $|G^{p^\tau}| < \aleph_0$ then $f_\tau(V(FG)/G) < \aleph_0$.*

Moreover, $\forall \tau \geq 0$, $f_\tau(V(FG)/G)$ can be determined from FG.

The next equivalence is our crucial tool.

Proposition. For the non-trivial mixed abelian group G the following two conditions are equivalent:

1) G_p is divisible;

2) $\forall \tau \geq 0 : |G^{p^\tau}| \geq \aleph_0 \Rightarrow f_\tau(G) = 0 \iff G^{p^\tau}[p] = G^{p^{\tau+1}}[p]$.

Proof. 1) $\Rightarrow$ 2) is straightforward.

We are now concerned with the left implication. Since G is a true mixed group, it contains a torsion-free element. Hence, for any $0 \leq n < \omega$ we observe that $|G^{p^n}| \geq \aleph_0$. Hypothesis means that $G^{p^n}[p] = G^{p^{n+1}}[p]$. Consequently, G_p is divisible. In fact, given $g \in G_p$ of order p^m for some $m \in \mathbf{N}$ whence $g^{p^{m-1}} \in G^{p^{m-1}}[p] = G^{p^m}[p]$. Thus $g^{p^{m-2}} \in G^{p^{m-1}}[p]$ etc. after a finite number of steps, $g \in G^p[p] \subseteq G_p^p$, as expected. This completes the proof. $\diamondsuit$

Although the given limitations are rather restrictive on the Ulm-Kaplansky functions, they are realistic and, there are many examples of abelian groups G having divisible p-component G_p for various primes p that are not Warfield groups.

II. Main Result and Proof

All the groundwork has now been laid and thus we have at our disposal all the information necessary for confirmation of the

Main Theorem (Karpilovsky, 1982). *Suppose F is a finite field of $char(F) = p$ and G is a p-mixed abelian group such that G_p is divisible. Then $FH \cong FG$ are F-isomorphic for an arbitrary group H if and only if $H \cong G$.*

Proof. Via the isomorphism $FG \cong FH$ we write $FG = FH$, whence $V(FG) = V(FH)$. According to [14] (or to [4] as well), we conclude that H is p-mixed, that $f_\tau(G) = f_\tau(H)$ and that $|G^{p^\tau}| = |H^{p^\tau}|$. Thus, utilizing the Proposition alluded to above, H_p must be also divisible. Applying now the Direct Factor Theorem, we write

$$(V(FG)/G) \times G = (V(FH)/H) \times H,$$

where both $V(FG)/G$ and $V(FH)/H$ are coproducts of countable p-groups. Note that the maximal divisible subgroups both of $V(FG)/G$ and $V(FH)/H$ can be recaptured from $FG = FH$ (see, e.g., [4]). Therefore, by the Theorem Invariants, we deduce that $V(FG)/G \cong V(FH)/H$. What we need to show is that the remaining requirements of the Group Theorem are satisfied. In fact, if for some ordinal number $\tau \geq 0$ we have $G_p^{p^\tau} = 1$ or $G^{p^\tau} = G^{p^{\tau+1}}$, one can easily infer that $f_\tau(V(FG)/G) = f_\tau(G) = 0$. If now $G_p^{p^\tau} \neq 1$ and $G^{p^\tau} \neq G^{p^{\tau+1}}$, it follows in virtue of Theorem of Invariants that $f_\tau(V(FG)/G) = |G^{p^\tau}|$ whenever $|G^{p^\tau}| \geq \aleph_0$. Hence, in conjunction with our additional restrictions on G from the text, it must be that $f_\tau(G) = 0$. Otherwise, if $|G^{p^\tau}| < \aleph_0$, it follows at once that $f_\tau(G) < \aleph_0$, and employing once again the already used Theorem Invariants it is obvious that $f_\tau(V(FG)/G) < \aleph_0$. That is why the wanted restrictions in the Group Theorem are really realized. Finally, $G \cong H$, thus finishing the proof. ♢

References

[1] P. Danchev, *Isomorphism of modular group algebras of totally projective abelian groups*, Commun. Algebra **28** (2000) (5), 2521–2531.

[2] P. Danchev, *Modular group algebras of coproducts of countable abelian groups*, Hokkaido Math. J. **29** (2000) (2), 255–262.

[3] P. Danchev, *Invariants for group algebras of splitting abelian groups with simply presented components*, Compt. rend. Acad. bulg. Sci. **55** (2002) (2), 5–8.

[4] P. Danchev, *Invariants for group algebras of abelian groups*, Rend. Circ. Mat. Palermo **51** (2002) (3), 391–402.

[5] P. Danchev, *Commutative group algebras of direct sums of countable abelian groups*, Kyungpook Math. J. **44** (2004) (1), 21–29.

[6] P. Danchev, *Ulm-Kaplansky invariants for $S(RG)/G_p$*, Bull. Inst. Math. Acad. Sinica **32** (2004) (2), 133–144.

[7] P. Danchev, *A note on the isomorphism of modular group algebras of global Warfield abelian groups*, Bol. Soc. Mat. Mexicana **10** (2004) (1), 49–51.

[8] P. Danchev, *Isomorphic commutative group algebras of p-mixed Warfield groups*, Acta Math. Sinica **21** (2005) (4), 913–916.

[9] P. Danchev, *The solution to W. May's problem for isomorphism of commutative group algebras of mixed groups over finite fields*, to appear.

[10] R. Goebel and W. May, *Cancellation of direct sums of countable abelian p-groups*, Proc. Amer. Math. Soc. **131** (2003) (9), 2705–2710.

[11] P. Hill and W. Ullery, *On commutative group algebras of mixed groups*, Commun. Algebra **25** (1997) (12), 4029–4038.

[12] G. Karpilovsky, *On commutative group algebras*, Contemp. Math. **9** (1982), 289–294.

[13] G. Karpilovsky, *Unit Groups of Group Rings*, North-Holland, Amsterdam, 1989.

[14] W. May, *Commutative group algebras*, Trans. Amer. Math. Soc. **136** (1969) (1), 139–149.

[15] W. Ullery, *On group algebras of p-mixed abelian groups*, Comm. Algebra **20** (1992) (3), 655–664.

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