An. Şt. Univ. Ovidius Constanţa Vol. 14(1), 2006, 91–98

On non-tabular m-pre-complete classes of formulas in the propositional provability logic

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Abstract

In the present paper we construct an example of an m-pre-complete with respect to functional expressibility class of formulas in the propositional provability logic.

Subject Classification: 03F45, 03G25, 06E25.

There is a well known class of problems in mathematical logic, algebra, discrete mathematics and cybernetics dealing with the possibility of obtaining some functions (operations, formulas) from another ones by means of a fixed set of tools. The notion of expressibility of Boolean functions through other functions by means of superpositions goes back to the works of E. Post[14], [15]. He described all closed (with respect to superpositions) classes of 2-valued Boolean functions. The problem of completeness (with respect to expressibility) which requires to determine the necessary and sufficient conditions for all functions to be expressible via the given system of functions is also investigated. In 1956 ([6, p. 54], [7]) A. V. Kuznetsov established the theorem of completeness according to which we can build a finite set of closed with respect to expressibility classes of functions in the k-valued logics such that any system of functions of this logic is complete if and only if it is not included in any of these classes. In 1965 [19] I. Rosenberg established the criterion of completeness in the k-valued logics formulated in terms of pre-complete classes of functions, i.e. in terms of maximal, incomplete and closed classes of functions.

Continuing the ideas of K. Gödel (1933), A. Tarski (1938), P. Novikov (1977) about the embedding of the intuitionistic logic into the modal logic

Key Words: Functional expressibility; m-completeness; Provability logic.
Received: 28 February, 2006

with the goal of the subsequent interpretation of the modality "provability" as a formal deduction in the Peano's Arithmetic, R.M.Solovay [20] and A.V.Kuznetsov, A.I.Muravitski [10] introduces into consideration the propositional provability calculus G.

The formulas of the propositional provability calculus G are built, as usual, of propositional variables $p, q, r, \dots$, which may have indices, of logic connectives & (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\neg$ (negation), and Δ (Gŏdel's provability), of the auxiliary symbols of left **(** and right **)** parantheses, and comma **,** [20], [10]. The propositional provability calculus G is defined by axioms of the classic propositional calculus and its rules of inference, additional three axioms

$$\Delta p \supset \Delta\Delta p,\ \Delta(\Delta p \supset p) \supset \Delta p,\ \Delta(p \supset q) \supset (\Delta p \supset \Delta q),$$

and an additional rule of inference of *necessitation* $A/\Delta A$. The propositional provability logic GL is defined as the set of all formulas, which can be deduced in the calculus G.

In the following we will discuss only formulas in the propositional provability calculus G mentioned above.

Definition 1. As usual, a set of formulas is called *an extension of the propositional provability logic GL* if it contains all axioms of the calculus G and it is closed with respect to all rules of inference of G.

In the sequel we will discuss only about extensions of the logic GL. Let L_1 and L_2 be two extensions of the logic GL.

Definition 2. The logic L_2 is called *an extension of the logic* L_1 if $L_1 \subseteq L_2$.

As in the case of the superintuitionistic logics [5] the set of all extensions of the logic GL can be divided into layers [1], called sometimes the Maximova-Hosoi's layers. In [13] it was shown that a logic L belongs to the layer n if and only if $\vdash_L \Delta^n(p\&\neg p)$ and $\not\vdash_L \Delta^{n-1}(p\&\neg p)$. The logic L belongs to the infinite layer if for any $n = 0, 1, \dots$ we have $\not\vdash_L \Delta^n(p\&\neg p)$.

Definition 3. In [13], the logic L is called *locally tabular* if for any non-negative integer n there exists at most a finite number of formulas of the same n variables, which are not two by two equivalent. Obviously, any tabular extension is also local tabular.

Next lemma gives us necessary and sufficient conditions of local tabularity of a logic.

Lemma 1. *([13],[1]) An arbitrary extension of the propositional provability logic GL is locally tabular if and only if it belongs to a finite layer.*

It is known [12], [11], [2] that diagonalizable algebras serve as algebraic models for the provability logic and its extensions.

Definition 4. The system

$$\mathfrak{D} =< E; \&, \vee, \supset, \neg, \Delta >$$

is said to be *a diagonalizable algebra* [12], if $< E; \&, \vee, \supset, \neg >$ is a Boolean algebra, and operation Δ satisfies the identities:

$$\Delta(x\&y) = (\Delta x\&\Delta y),\ \Delta(\Delta x \supset x) = \Delta x,\ \Delta(x \supset x) = (x \supset x).$$

As usual for Boolean algebras we shall denote the unit and zero elements of $\mathfrak{D}$ by 1 and 0 correspondingly.

Definition 5. A formula F is said to be *valid on the algebra* $\mathfrak{D}$ if its value interpreted on $\mathfrak{D}$ is allways 1.

Definition 6. The set of all valid formulas on the algebra $\mathfrak{D}$ is reffered to as *the logic of the algebra* $\mathfrak{D}$, and is denoted by $L\mathfrak{D}$.

Definition 7. [8], [9] The formula F is said to be *expressible in the logic L via a system of formulas* Σ, if F can be obtained from propositional variables and formulas of Σ applying a finite number of times: a) the rule of substitution of equivalent formulas in the logic L, and b) the rule of weak substitution, which allows, being given formulas A and B, to substitute one of them in another instead of a given corresponding propositional varuiable.

Definition 8. [8],[9] The system of formulas Σ is said to be *(functionally) complete (with respect to the expressibility) in the logic* L, if all formulas of the calculus G are expressibile in the logic L via formulas of Σ.

Definition 9. [8],[9] The system of formulas Σ is said to be *pre-complete in the logic* L, if it is incomplete in L and for any formula F, which is not expressible in L via Σ, the system $\Sigma \cup \{F\}$ is already complete in L.

Definition 10. [21],[3] *The formula $F(p_1, \ldots, p_n)$ is a model for the Boolean function* $f(x_1, \ldots, x_n)$, if for any ordered set $(\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \{0, 1\}$, $i = 1, \ldots, n$, we have $F(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_n)$, where logical connectors from F are interpreted in a natural way on the two-valued Boolean algebra.

Definition 11. [4] The system of formulas Σ is called *m-complete in the logic* L, if at least a model for every Boolean function is expressible via Σ in the logic L. Analogously is defined the notion of the *m-pre-completeness.*

Iu. N. Tolstova [21] obtained the criteria of m-completeness in the class of all 3-valued functions. O. I. Covalgiu [3] solved the analogous problem for the class of all functions of 3-valued Δ-pseudo-Boolean algebra Z_3 in terms of seven m-pre-complete classes.

We consider the diagonalizable algebra $\mathfrak{M}^*$ of binary infinite sequences $\mu = (\mu_1, \mu_2, \dots)$, $\mu_i \in \{0, 1\}$, $i = 1, 2, \dots$, generated by its zero element $(0, 0, \dots)$, where Boolean operations over elements of $\mathfrak{M}^*$ are defined componentwise, and $\Delta\mu = (1, \gamma_1, \gamma_2, \gamma_3, \dots)$, and $\gamma_i = \mu_1 \& \dots \& \mu_i$ [17]. Consider now the logic $L\mathfrak{M}^*$, which is an extension of the propositional provability logic GL.

Definition 12. *The formula $F(p_1, \dots, p_n)$ preserves the relation $R(x_1, \dots, x_m)$ on the algebra $\mathfrak{A}$*, if for any elements β_{ij} of $\mathfrak{A}$, $(i = 1, \dots, m;\ j = 1, \dots, n)$ the relations

$$R(\beta_{11}, \dots, \beta_{m1}), \dots, R(\beta_{1n}, \dots, \beta_{mn}))$$

imply the relations

$$R(F(\beta_{11}, \dots, \beta_{1n}), \dots, F(\beta_{m1,\dots,\beta_{mn}})).$$

Definition 13. An extension L of the provability logic is called *tabular*, if there exists a finite diagonalizable algebra $\mathfrak{D}$ such that $L = L\mathfrak{D}$.

It is known that pre-complete classes are important in establishing criteria for completeness in the given logic. Next definitions make the difference among them.

Definition 14. ([16]) A pre-complete with respect to functional expressibility class of formulas P in the logic L is *tabular*, if there exists a tabular extension L_1 of the logic L, in which P is also pre-complete. Analogously, an m-pre-complete class P of formulas of the logic L is said to be *tabular* in the logic L if there is a tabular extension of L in which it is also m-pre-complete.

For example, the pre-complete classes of formulas $K_1, K_2, \dots$ of the collection that was built in [17] are tabular in the provability logic, and the classes of functions which are in the arbitrary closed class K of functions of the algebra Z_3 are tabular [4].

Definition 15. Let $Form$ be the set of all formulas of the calculus G. The mapping $f : \mathfrak{D} \to Form$, in [17], [16], is *a formula realization of the diagonalizable algebra $\mathfrak{D}$ into the logic L* if it is an isomorphism between $\mathfrak{D}$ and some subalgebra of the Lindenbaum's algebra of the logic L when formulas are considered up to equivalent counterparts.

In [17] was built a formula realization f according to the definition above. Let us remember that $\vdash f(1) \sim (p \supset p)$. Other useful properties of f are mentioned in the following two lemmas.

Lemma 2. *[17] For any formula $F(p_1, \dots, p_n)$ of the propositional provability logic GL and for any elements $\beta_1, \dots, \beta_n$ of the algebra $\mathfrak{M}^*$, the next general statement holds:*

$$\vdash f(F[p_1/\beta_1, \dots, p_n/\beta_n]) \sim F[p_1/f(\beta_1), \dots, p_n/f(\beta_n)].$$

Lemma 3. *[18] Let f be the above mentioned formula realization of the algebra $\mathfrak{M}^*$ into the logic GL and β be an arbitrary element of $\mathfrak{M}^*$. Then the formula realization f puts into correspondence to the element β the unary formula $f(\beta)$, such that, for any element γ of the same algebra $\mathfrak{M}^*$, the following equality takes place*

$$f(\beta)[\gamma] = \beta. \tag{1}$$

Now we are ready to state our results. We consider the class K of formulas that preserves the relation $x \neq 1$ on the algebra $\mathfrak{M}^*$.

Lemma 4. *Let f be the above mentioned formula realization of the algebra $\mathfrak{M}^*$ into the logic GL. Then for any element β of this algebra $\mathfrak{M}^*$ the formula $f(\beta)$ belongs to the class of formulas K if and only if the following condition is fulfilled*

$$\beta \neq 1. \tag{2}$$

Proof. (If) Let element β satisfies the condition (2). We have to show that the formula $f(\beta)$ conserves the relation $x \neq 1$ on the algebra $\mathfrak{M}^*$. Consider an arbitrary element $\gamma \in \mathfrak{M}^*$ such that $\gamma \neq 1$. Then, using (2) and the already proved in lemma 3 equality $f(\beta)[\gamma] = \beta$ we found out the relation $f(\beta)[\gamma] \neq 1$, which means that the formula $f(\beta)$ conserves the relation $x \neq 1$. Thus we obtain that $f(\beta) \in K$.

(Only-if) Let the formula $f(\beta)$ belongs th the class K, but element β does not satisfies the condition (2), i.e. $\beta = 1$. Then, using again the above mentioned equality (1) from lemma 3 and the equality $\beta = 1$, we obtain

$$f(\beta)[\gamma] = 1,$$

which means that the formula $f(\beta)$ does not conserve the relation $x \neq 1$. Subsequently, the formula $f(\beta)$ does not belong to the class K. The last thing together with initial supposition that $f(\beta) \in K$ give rise to a contradiction. So, lemma is proved.

Lemma 5. *Let the logic L satisfy the conditions*

$$GL \subseteq L \subseteq L\mathfrak{M}^*.$$

Then the class K of formulas is m-pre-complete in L.

Proof. First of all let us observe that the class K is m-incomplete. Indeed, since Boolean function $\neg x$ does not preserve the relation $x \neq 1$, so does any model of $\neg x$. Now, let us start to prove that K is m-pre-complete.

Let $B(p_1, \dots, p_n)$ be an arbitrary formula, that does not belong to the class K. Then, by the force of the definition of K, there exist such elements $\beta_1, \dots, \beta_n$ of the algebra $\mathfrak{M}^*$ such that for every $i = 1, \dots, n$

$$\beta_i \neq 1$$

holds, together with the following:

$$B[\beta_1, \dots, \beta_n] = 1.$$

In view of the above mentioned formula realization f of the algebra $\mathfrak{M}^*$ into the logic GL, the last equality leads us to the relation

$$\vdash f(B[\beta_1, \dots, \beta_n]) \sim f(1).$$

Applying the lema 2 to the left hand side of this equivalence, and, taking into consideration the properties of f, we obtain, that

$$\vdash B[f(\beta_1), \dots, f(\beta_n)]) \sim (p \supset p). \tag{3}$$

Note that, since the elements β_i for $i = 1, \dots, n$ satisfy the relation $x \neq 1$, the formulas $f(\beta_i)$, according to lemma 4, also belong to the class K. That is why the deduction (3) shows, that the formula $(p \supset p)$ is expressible in the logic GL via formula B and formulas of the class K. Observe that: a) $\{(p\&q), (\neg p\&q)\} \subseteq K$, and b) the formula $p\&q$ is a model for the Boolean function $x\&y$. It remains to prove that via the system of formulas $K \cup \{p \supset p\}$ we can obtain a model for the Boolean function $\neg x$. Finally, note that $\vdash (\neg p\&(p \supset p)) \sim \neg p$. Therefore the formula $\neg p$ is an obvious model for $\neg x$. This implies that K is m-pre-complete in L, which has been to be proved.

The next theorem establishes the role of the non-tabular extensions of the provability logic GL when dealing with the problem of completeness in GL, since there exist non-tabular pre-complete classes of formulas.

Theorem 1. *Let L be GL, or $L\mathfrak{M}^*$, or any other intermediary between them logic. Then there exists an m-pre-complete in the logic L class of formulas which is not tabular.*

Proof. The fact that the class K of formulas defined above is m-pre-complete follows from lemma 5. It remains to prove that K is not tabular, i.e. to prove that it is m-complete in any tabular extension of L. It is easy to check that

$$\{p\&q, \neg p\&q, \Delta p, 0, \Delta 0, \Delta^2 0, \dots\} \subseteq K. \tag{4}$$

According to lemma 1 and the fact that any tabular extension L_1 of GL is also local tabular, we have that for L_1 there exists a non-negative integer n such that $\vdash_{L_1} \Delta^n \sim (p \supset p)$. Note also that

$$\vdash_L (\neg p \& (p \supset p)) \sim \neg p. \tag{5}$$

In view of (4) and (5), we have that the system K of formulas is m-complete in any tabular extension of GL. We just proved the theorem.

Acknowledgment. We express here our gratitude and thanks to prof. Mefodie Ratsa from the Institute of Matematics and Computer Science of the Academy of Sciences of Moldova for his attention to our scientific evolution during the years.

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