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MISCELLANEOUS RESULTS AND CONJECTURES ON THE RING OF COMMUTING MATRICES

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Abstract

Let $X = (x_{ij})$ and $Y = (y_{ij})$ be generic n by n matrices and $Z = XY - YX$. Let $S = k[x_{11}, \ldots, x_{nn}, y_{11}, \ldots, y_{nn}]$, where k is a field, let I be the ideal generated by the entries of Z and let $R = S/I$. We give a survey on results and conjectures on R such as regular sequences in R, the first syzygies of I, the canonical module of R and non-Gorenstein locus. For the case $n = 4$ we give a conjecture on the Betti numbers of I.

1 Introduction

Throughout this article we let R be the ring defined in the abstract. We first give a review of known results for this family of rings and then we give conjectures some of which have not been published before and some that can be found in [12] and [11].

It was shown by Motzkin and Taussky [16] that the variety of commuting matrices in $M_n(k)$ is irreducible of dimension $n^2 + n$. Gerstenhaber [8] also showed that the variety is irreducible. From this it follows that $\mathrm{Rad}(I)$ is prime and that the dimension of R is $n^2 + n$.

It was conjectured by Artin and Hochster that R is Cohen-Macaulay and this has been shown for $n = 3$ in [2] and for $n = 4$ in [10]. In both cases the computer program Macaulay [1] was used to compute a Gröbner basis. It has also been conjectured that R is a domain which follows from the ring being CM (see [21]).

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Simis and Vasconcelos have studied R as the symmetric algebra of the Jacobian module E_n with respect to the x_{ij} variables. They show that the module E_3 has projective dimension 2 and conjecture that this is true for any n, see [19].

In [4] Brennan, Pinto and Vasconcelos show that if the matrices X and Y are symmetric, then R is a complete intersection and a domain. In [3] it is shown that R is normal in this case.

For a general discussion on commuting varieties see [20] chapter 9 and the references cited there.

In [13] we show that for $n \geq 3$ the Koszul dual of the ring is the enveloping algebra of a graded nilpotent Lie-algebra.

Recently, Knutson [14] proved that the off-diagonal elements in $XY - YX$ form a regular sequence.

The Cohen-Macaulayness of the ring may be proved in at least two ways, by finding maximal regular sequences of lenght $n^2 + n$ or by finding a minimal resolution. In this article we give maximal regular sequences that can be verified by a computer for the cases $n = 2, 3$ and 4. The resolution can be computed only in the cases $n = 2$ and $n = 3$. To get an idea on the Betti numbers in other cases we first find the first syzygies and give a general conjecture for these. These first syzygies can then be used to get a conjecture on the canonical module. We can use a computer to partially resolve both R and the canonical module. Splicing together these two and using the Hilbert series we get a conjecture on the Betti numbers in the 4×4 case.

2 A minimal generating set

The generators of I are of the form

$$z_{ij} = \sum_{r=1}^{n} (x_{ir}y_{rj} - y_{ir}x_{rj}) \qquad \text{for } i = 1 \ldots n, j = 1 \ldots n.$$

For $i \neq j$ we see that each monomial occurring in z_{ij} only occurs once so none of these generators can be written as a combination of the others. Among the diagonal entries ($i = j$) there is some mixing of monomials. All the monomials there are of the form $x_{ij}y_{ji}$ and each one of these occurs exactly twice, that is, in z_{ii} and z_{jj}. Since $\mathrm{tr}(XY - YX) = 0$ we have that $z_{11} + \ldots + z_{nn} = 0$ so that this part of I can be generated by $z_{11}, \ldots, z_{n-1n-1}$. In each of these generators we have a monomial that occurs exactly once namely, $x_{in}y_{ni}$ only occurs in z_{ii} (since we have thrown z_{nn} away). Hence z_{ii} can not be written as a combination of the others. So we see that the ideal is minimally generated by $n^2 - n + n - 1 = n^2 - 1$ generators.

3 Regular sequences

To prove that R is Cohen-Macaulay it suffices to show that we have a regular sequence of length $n^2 + n$. In [10] and [2] MACAULAY was used to create a system of parameters using random numbers but it is also possible by extensive guessing to find regular sequences that can be checked by a computer. Below we describe two such sequences that work for the small cases.

3.1 Guess 1

In this section we give a maximal regular sequence for $n = 2$, $n = 3$ and $n = 4$. The ring has dimension $n^2 + n$ by [16].

We start by giving the idea for the 3×3 case. We write the matrices X and Y in the following way

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \\ y_7 & y_8 & y_9 \end{bmatrix}$$

By using MACAULAY we guessed the following regular sequence of length n^2

$$y_1 - x_1, y_2 - x_5, y_3 - x_9, y_4 - x_3, y_5 - x_4, y_6 - x_8, y_7 - x_2, y_8 - x_6, y_9 - x_7$$

Dividing out by it amounts to replacing the matrix Y by the matrix

$$Y' = \begin{bmatrix} x_1 & x_5 & x_9 \\ x_3 & x_4 & x_8 \\ x_2 & x_6 & x_7 \end{bmatrix}$$

Note that the columns of Y' are the rows of X slightly permuted.

In order to generalise this idea to the cases $n = 2$ and $n = 4$ we need a description of the construction of Y':

> We have 9 variables $x_1, \ldots, x_9$. We put x_1 in the left upper corner of the matrix. Then we go to the bottom left corner and put x_2 there. Then we continue upwards and put x_3 above x_2. Now there is no more room in the first column so we go to the next column and put x_4 to the right of x_3. Then continue upwards until there is no more room. Then start at the bottom and move upwards until there is no more room in that column. Move to right to the next column etc.

We have now divided out by a regular sequence of length 9. To get a maximal regular sequence we divide out by 3 variables, e.g. x_9, x_8 and x_1 will do (this was found be guessing).

Now we check if the description for Y' will work for $n = 2$. Here Y' becomes:

$$Y' = \begin{bmatrix} x_1 & x_4 \\ x_2 & x_3 \end{bmatrix}$$

We check using MACAULAY that this is a regular sequence and to get a maximal regular sequence we divide out by x_4 and x_1.

For $n = 4$ we get:

$$Y' = \begin{bmatrix} x_1 & x_6 & x_{11} & x_{16} \\ x_4 & x_5 & x_{10} & x_{15} \\ x_3 & x_8 & x_9 & x_{14} \\ x_2 & x_7 & x_{12} & x_{13} \end{bmatrix}$$

The standard basis of the ideal generated by the entries of $XY' - Y'X$ is too big to be computed. We need 4 more elements to have a maximal regular sequence. By guessing we found that if we divide by the variables x_1, x_9, x_{15} and x_{16} we get a zero dimensional ring having the same Hilbert series as our original ring so we have found a maximal regular sequence.

For 5×5 matrices and bigger we cannot calculate the standard basis so we cannot test if this idea works. It seems however likely that for a general n we can replace the matrix Y by the matrix

$$Y' = \begin{bmatrix} x_1 & x_{n+2} & x_{2n+3} & \cdots & x_{n^2} \\ x_n & x_{n+1} & x_{2n+2} & \cdots & x_{n^2-1} \\ x_{n-1} & x_{2n} & x_{2n+1} & \cdots & x_{n^2-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_2 & x_{n+3} & x_{2n+4} & \cdots & x_{n^2-n+1} \end{bmatrix}$$

To find a maximal regular sequence we have to guess n more elements.

3.2 Guess 2

By examining the generators of I we see that they are sums of 2×2 minors of the matrix $\begin{bmatrix} x_1 & x_2 & \ldots & x_{n^2} \\ y_1 & y_2 & \ldots & y_{n^2} \end{bmatrix}$. Let I_2 be the ideal generated by all 2×2 minors of this matrix. It is known that S/I_2 is CM of dimension $n^2 + 1$ and that there exists a maximal regular sequence that can be decribed by replacing the original matrix by the matrix $\begin{bmatrix} x_1 & x_2 & \ldots & x_{n^2-1} & 0 \\ 0 & x_1 & x_2 & \ldots & x_{n^2-1} \end{bmatrix}$. Inspired by this we checked this regular sequence for the ring of commuting matrices.

For the case $n = 3$ we get the ring R modulo this sequence by replacing the matrices X and Y by the matrices

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 \end{bmatrix}.$$

Forming the commutator of these matrices and calculating the Hilbert series of the corresponding ideal gives that this is a regular sequence. However, it is not maximal as we still have 8 variables and the height of the ideal is 6. We can now find two more nonzerodivisors by testing, e.g. x_8 and $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$ will do.

This idea can easily be used in the 2×2 case, we form the commutator of

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & x_1 \\ x_2 & x_3 \end{bmatrix}.$$

and then mod out by x_3 to get a maximal regular sequence.

For $n = 4$ we have not been able to test this conjecture as the Gröbner basis of the ideal we get from the conjecture is too big to be computed.

4 First syzygies

We restate here a conjecture on the first syzygies that was first given in [12].

Write $I = (f_1, \ldots, f_{n^2})$, with $f_1 = Z_{11}$, $f_2 = Z_{21}$, $\ldots$, $f_{n^2} = Z_{nn}$, where $Z = XY - YX$. A syzygy on I is an n^2-tuple $(a_1, \ldots, a_{n^2})$ such that

$$f_1 a_1 + f_2 a_2 + \cdots + f_{n^2} a_{n^2} = 0. \tag{1}$$

This can be rewritten as

$$\operatorname{tr}\left(\begin{bmatrix} a_1 & \cdots & a_n \\ a_{n+1} & \cdots & \vdots \\ \vdots & & \vdots \\ a_{n^2+n-1} & \ldots & a_{n^2} \end{bmatrix} \begin{bmatrix} f_1 & \cdots & f_{n^2-n+1} \\ f_2 & \cdots & \vdots \\ \vdots & & \vdots \\ f_n & \ldots & f_{n^2} \end{bmatrix} \right) = 0 \tag{2}$$

i.e. as

$$\operatorname{tr}(A(XY - YX)) = 0. \tag{3}$$

So solving (3) for A is equivalent to solving (1) for $(a_1, \ldots, a_{n^2})$.

We can guess a number of solutions to (3):

Degree 0**:** Here we only have one syzygy $A = E$ (the identity matrix), i.e. the ideal is minimally generated by $n^2 - 1$ elements.

Degree 1**:** We have $\mathrm{tr}(X(XY - YX)) = \mathrm{tr}(X^2Y) - \mathrm{tr}(XYX) = 0$ so $A = X$ is a solution and similarly we get that $A = Y$ is a solution. The two syzygies we get are obviously independent over k as they have the bidegrees $(1, 0)$ and $(0, 1)$. In [13] we proved that these are the only ones of degree 1.

Degree 2**:** We see that $A = X^2$ and $A = Y^2$ are solutions. The only other monomials in X and Y are XY and YX and neither of those is a solution. We have

$$\begin{aligned}
&\mathrm{tr}((XY + YX)(XY - YX)) \\
&\quad = \mathrm{tr}(XYXY) - \mathrm{tr}(XYYX) + \mathrm{tr}(YXXY) - \mathrm{tr}(YXYX) \\
&\quad = \mathrm{tr}(XYXY) - \mathrm{tr}(X^2Y^2) + \mathrm{tr}(X^2Y^2) - \mathrm{tr}(XYXY) \\
&\quad = 0
\end{aligned}$$

so $A = XY + YX$ gives a syzygy. We thus have syzygies of bidegrees $(2, 0), (1, 1), (0, 2)$.

Degree 3**:** Here we get at least the monomial solutions X^3, Y^3, XYX, YXY and the binomial solutions $X^2Y + YX^2$, $XY^2 + Y^2X$. Macaulay calculations indicate that it is enough to take one syzygy of each bidegree i.e. X^3, Y^3,XYX, YXY will do.

Degree 4**:** X^4, Y^4, $X^3Y + YX^3$, $Y^3X + XY^3$,$X^2YX + XYX^2$, $Y^2XY + YXY^2$ and $XY^2X - YX^2Y$.

Degree 5**:** X^5, Y^5, X^2YX^2, Y^2XY^2, $X^4Y + YX^4$, $XY^4 + Y^4X$, $XYX^2Y + YX^2YX$, $YXY^2X + XY^2XY$.

We can check this for small values of n. For $n = 3$ we get the following Betti numbers:

```
% betti s3
total:   8    33
-----------------
    2:   8     2
    3:   -    31
```

As expected we get 2 linear first syzygies. There are 31 first syzygies of degree 2, $\binom{8}{2} = 28$ of those are the trivial syzygies (Koszul relations) and the 3 nontrivial ones correspond to $A = X^2$, $A = Y^2$ and $A = XY + YX$.
Considering $n = 4$ and $n = 5, 6, 7$ (partial computation) we give the conjecture below on the first Betti numbers. We use the notation of Macaulay 2 to display the Betti numbers, i.e. the number in column i row j (starting with column 0, row 0) is $\beta_{i,i+j}$.

$$
\begin{array}{rccc}
\mathit{total}: & 1 & n^2-1 & \binom{n^2-1}{2} + \binom{n+1}{2} - 1 \\
0: & 1 & . & . \\
1: & . & n^2-1 & 2 \\
2: & . & . & \binom{n^2-1}{2} + 3 \\
3: & . & . & 4 \\
4: & . & . & 5 \\
. & . & . & . \\
. & . & . & . \\
. & . & . & . \\
n-1: & . & . & n \\
n: & . & . & .
\end{array}
$$

The $\binom{n^2-1}{2}$ syzygies of degree 2 are the Koszul relations and we conjecture that the other first syzygies are given by polynomials in X and Y, one of each possible bidegree.

The characteristic polynomial gives us certain information on the first syzygies. For a generic $n \times n$ matrix X we have that the smallest degree of a polynomial p such that $p(X) = 0$ is n so that the smallest power of X, that can be written as a linear combination of smaller powers, is n. Because the syzygies corresponding to $X^2, X^3, \ldots, X^{n-1}$ have $y-$degree zero they cannot be written as linear combinations of any syzygies involving y-variables. A similar result holds for syzygies that are given by powers of Y so we have at least 2 syzygies of each degree $1, \ldots, n-1$.

5 Canonical module

If R is Cohen-Macaulay (which is known for the cases $n = 2, 3, 4$) then its canonical module is defined as

$$\omega_R := \mathrm{Ext}^d_S(S/I, S)$$

where $d = n^2 - n$ is the height of I. Let $J = j_1, \ldots, j_{n^2-n}$ be the subideal of I generated by the off-diagonal elements in $XY - YX$. The generators of J

form a regular sequence ([14]) and we get

$$\mathrm{Ext}^d_S(S/I,S) \cong \mathrm{Ext}^d_{S/j_1}(S/I,S/j_1) \cong \cdots\cdots \cong \mathrm{Hom}_{S/J}(S/I,S/J) \cong (J:I)/J$$

We can use the first syzygies to compute the ideal quotient $(J:I)$. For the cases $n=2,3$ everything can be computed using Macaulay but for the case $n=4$ we can not compute the Gröbner basis of J. By studying the structure of $(J:I)$ for $n=2,3$ we make a conjecture on $(J:I)$ for $n=4$. We can partially check this conjecture by comparing with the Hilbert series.

For $n=3$ the nontrivial syzygies on I are given by $A \in \{E,X,Y,X^2,Y^2,XY+YX\}$. The ideal I is generated by $(f_1,\dots,f_9)$ where f_1, f_5 and f_9 are from the diagonal of $XY-YX$ and $J=(f_2,f_3,f_4,f_6,f_7,f_8)$. Pick 3 different syzygies, A, B and C. Then

$$\begin{aligned}
a_1f_1+a_5f_5+a_9f_9 &= a_2f_2+a_3f_3+a_4f_4+a_6f_6+a_7f_7+a_8f_8\\
b_1f_1+b_5f_5+b_9f_9 &= b_2f_2+b_3f_3+b_4f_4+b_6f_6+b_7f_7+b_8f_8\\
c_1f_1+c_5f_5+c_9f_9 &= c_2f_2+c_3f_3+c_4f_4+c_6f_6+c_7f_7+c_8f_8
\end{aligned}$$

so

$$\det\begin{bmatrix} a_1 & b_1 & c_1\\ a_5 & b_5 & c_5\\ a_9 & b_9 & c_9\end{bmatrix}\cdot f_i \in J \quad \text{for i=1, 5, 9.}$$

Direct calculations using Macaulay give that it suffices to take the generators of J and the elements given by

$$(A,B,C) \in \{(E,X,Y),(E,X,X^2),(E,X,Y^2),(E,Y,Y^2),(E,Y,X^2)\}$$

to get a minimal generating set for $(J:I)$. The bidegrees of these additional generators are $(1,1)$, $(3,0)$, $(2,1)$, $(1,2)$ and $(0,3)$ so it seems that it suffices to use enough triples of syzygies to give one generator of each bidegree.

Similarly we construct $J:I$ for the case $n=4$ (for details and a conjecture on the general case see [11]). Since we cannot compute the standard basis of J we cannot test whether this conjecture is true. We partially resolve $(J:I)/J$ using Macaulay and get the following Betti numbers:

```
% 1% 2% betti cp
total:      14   200   660  3821
--------------------------------
    4:       3     -     -     -
    5:       4   110   256    90
    6:       7    90   908  3656
    7:       -     -     6    75
```

This gives us the Betti numbers of the tail of the resolution of I (see e.g. cor. 3.3.9 in [5]). So we can compare this with the Hilbert series of S/I:

$$\begin{aligned} h_{S/I}(t) &= (1 - 15t^2 + 2t^3 + 108t^4 - 26t^5 - 562t^6 + 466t^7 + 1613t^8 - 2742t^9 \\ &\quad -1078t^{10} + 5994t^{11} - 4367t^{12} - 2262t^{13} + 5630t^{14} - 3650t^{15} \\ &\quad +818t^{16} + 166t^{17} - 103t^{18} + 4t^{19} + 3t^{20})/(1-t)^{32} \end{aligned}$$

We see that our conjecture fits with the (last 6) coefficients of the polynomial in the numerator. Partially computing the resolution of I we get the Betti numbers:

```
o18 = total: 1 16 115 595 2127 2791 848 60 5
          0: 1  .   .   .    .    .   .  . .
          1: . 15   2   .    .    .   .  . .
          2: .  . 108  30    3    .   .  . .
          3: .  .   4 565  466   45   4  . .
          4: .  .   .   . 1658 2746 844 60 5
```

Splicing together these two Betti tables and using the Hilbert series we get the following conjecture on the Betti numbers (Table 1), where $-d + c = -2262$ (from the Hilbert series). The boldfaced numbers are the two earlier Macaulay computations and the others are based on the Hilbert series.

total :	1	15	115	595	2127	4713	6902	4432+	5710+	3821	1170	200	14
0 :	**1**	.	.	.	.	.	.	.	.	.	.	.	.
1 :	.	**15**	**2**	.	.	.	.	.	.	.	.	.	.
2 :	.	.	**108**	**30**	**3**	.	.	.	.	.	.	.	.
3 :	.	.	**4**	**565**	**466**	**45**	**4**	.	.	.	.	.	.
4 :	.	.	.	.	**1658**	**2746**	**844**	**60**	**5**	.	.	.	.
5 :	.	.	.	.	.	1922	6054	4372	c	**75**	**6**	.	.
6 :	.	.	.	.	.	.	.	d	5705	**3656**	**908**	**90**	**7**
7 :	.	.	.	.	.	.	.	.	.	90	**256**	**110**	4
8 :	.	.	.	.	.	.	.	.	.	.	.	.	**3**

Table 1

6 Resolution

By viewing the Betti table above we see that there is a certain multiplicative pattern on the "top staircase", i.e. we have in the second row 15 and 2, the last 2 numbers in the third row are 30 and 3, the last 2 numbers in the fourth row are 45 and 4 etc. Checking partial computation for $n = 5$ and $n = 6$ we get even more Betti numbers that are multiples of previous Betti numbers.

```
n=5
o9 = total: 1 25 291 2486 561 72
         0: 1  .   1    .   .  .
         1: . 25   2    .   .  .
         2: .  . 279   48   3  .
         3: .  .   4 2096 558 72
         4: .  .   5  342
```

```
n=6
o14 = total: 1 36 605 6720 1199 105 4
          0: 1  .   1    .    .   . .
          1: . 36   2    .    .   . .
          2: .  . 598   70    3   . .
          3: .  .   4 6650 1196 105 4
```

So up to a certain row (probably row $n-1$) the generators and the first syzygies seem to generate everything (and the "multiplication" is nonzero). Our conjecture is that we have the following Betti numbers for a general n (for the sake of space the first Betti number given is $\beta_{1,2}$): see Table 2, where p means products of earlier entries. The numbers M, s and k are based on a conjecture on the canonical module in the general case which can be found in [11].

It is known that the Koszul dual of a ring A is the enveloping algebra of a Lie-algebra, called the Lie algebra associated to A. In [13] we proved for R that this Lie-algebra is nilpotent of index 3. We also showed that the dimension of the Lie algebra in degree 3 is 2 which gives (by [15]) that the number of independent linear first syzygies is 2. Fröberg and Löfwall give in [7] a theorem relating kernels of multiplication on Koszul homology and the associated Lie algebra. In this case we get that we always have at least the boldfaced Betti numbers in the table.

	hd 1	hd 2	hd 3	hd 4	hd 5	hd 6	hd 7	hd 8	$\cdots$	n^2-n-1	n^2-n
1:	$\mathbf{n^2-1}$	$\mathbf{2}$	-	-	-	-	-	-	$\cdots$	-	-
2:	-	$\binom{\mathbf{n^2-1}}{\mathbf{2}}+3$	$\mathbf{2(n^2-1)}$	$\mathbf{3}$	-	-	-	-	$\cdots$	-	-
3:	-	4	p	p	$3\cdot(n^2-1)$	4	-	-	$\cdots$	-	-
4:	-	5	p	p	p	p	$\ddots$	$\ddots$	$\cdots$	-	-
$\vdots$	-	-	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	-	-
$n-2$	-	$n-1$	p	p	p	p	p	p	$\cdots$	-	-
$n-1$	-	n	?	?	?	?	?	?	$\cdots$	-	-
$\vdots$	-	-	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	-	-
$\frac{n(n-1)}{2}$	-	-	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\frac{n(n-1)}{2}(n^2-1)$	$\frac{n(n-1)}{2}+1$
$\vdots$	-	-	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
M:	-	-	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$s(k-s+1)+1$
$M+1$:	-	-	-	-	-	-	-	-	-	-	-

Table 2

7 The not-Gorenstein and not-complete intersection loci

In this section we consider the above loci for the cases $n = 2$, $n = 3$ and $n = 4$. We can see by looking at the Hilbert series that R is not Gorenstein. The not-Gorenstein locus of R is defined as

$$\{p \in \operatorname{Spec}(R) \mid R_p \text{ is not Gorenstein}\}.$$

Since we know that R is Cohen-Macaulay the not-Gorenstein locus is given by (see [18])

$$\{p \in \operatorname{Spec}(R) \mid \mu((\operatorname{Ext}_S^h(R,S))_p) > 1\} = V(F_1(\operatorname{Ext}_S^h(R,S)))$$

where $\mu(M)$ is the minimal number of generators of M, $h = \operatorname{ht}(I)$ and $\operatorname{Ext}_S^h(R,S)$ is the canonical module of R. The *ith Fitting invariant* of a module M is computed from its presentation i.e. suppose $s^m \xrightarrow{N} s^n \to M \to 0$ is a presentation of M then $F_i(M)$ is the ideal generated by the $(n-i)$ minors of N.

For $n = 2$ we get that the not-Gorenstein locus is $V(ng)$ where ng is the ideal generated by $x_1 - x_4, x_2, x_3, y_1 - y_4, y_2, y_3$. This ideal contains I and has height 4 in $R = S/I$.

For $n = 3$ the presentation of the canonical module is given by a 5×32 matrix, the ideal of its 4×4 minors is minimally generated by 4332 generators of degrees 2 and 3. This ideal has height 4 in R.

For the case $n = 4$ the resolution is not possible to compute. From our conjecture on the canonical module we get a presentation given by a 14×200 matrix of which we need to compute 13×13 minors. This is not possible so we can not compute the not-Gorenstein locus in this case.

For the cases $n = 2, 3$ we see that R_p is Gorenstein for any $p \in \operatorname{Spec}(R)$ with $\operatorname{ht}(p) \leq 3$ so it seems plausible that this is true in general.

The not complete intersection locus is defined as

$$\{p \in \operatorname{Spec}(R) \mid R_p \text{ is not c.i.}\}.$$

Since R is Cohen-Macaulay the not complete intersection locus becomes $V(nc)$ where nc is the ideal generated by the $n-1$ minors of the module of first syzygies of I (see [18]).

We can calculate the ideal nc for $n = 2$ and $n = 3$ and we get that R_p is a complete intersection for any $p \in \operatorname{Spec}(R)$ with $\operatorname{ht}(p) \leq 3$.

8 Poincare series

In [17] a conjecture on the Poincare series is given for the case $n = 3$. The conjecture says that

$$P_R(x, y)^{-1} = (1 + 1/x)/A(xy) - H_R(-xy)/x$$

where $A(xy)$ is the Hilbert series of the Koszul dual.

For the case $n = 2$ the ideal has a quadratic Gröbner basis and hence is a Koszul algebra so we have $P_R(xy) = A(xy) = 1/H_R(-xy)$. In this case we get that the formula is trivially true.

From [13] we have that $A(xy) = \frac{(1+xy)^{2n^2}(1+x^3y^3)^2}{(1-x^2y^2)^{n^2-1}}$ for any $n \geq 3$ and for $n = 4$ we can compute the Gröbner basis (see [10]) and thus the Hilbert series and part of the resolution of the field k over the quotient ring S/I. A partial computation gives the Betti numbers

```
% 1% 2% betti p
total:      1    32   511   5449  43680+
-------------------------------------
    0:      1    32   511   5442  43584
    1:      -     -     -      3     96
    2:      -     -     -      4      ?
```

If we compute the right hand side in the formula above using the series we have and compare the result with the Betti table above we see that it does not give the Poincare series for the case $n = 4$ as there is no term corresponding to the 4 in the table. It seems however plausible that the formula might be adapted to the 4×4 case.

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