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GENERALIZED KOSZUL COMPLEXES

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Abstract

This article should be viewed as a survey of generalized Koszul complexes and Koszul bicomplexes with an application to generalized Koszul complexes in projective dimension one. We shall try to give detailed information on the basic definitions and a summary of the main results. Concerning proofs the reader is invited to have a look into [I] or [IV].

Introduction.

We start with the following question: given finite free modules $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ over a noetherian ring R and a complex $\mathcal{F} \xrightarrow{\chi} \mathcal{G} \xrightarrow{\lambda} \mathcal{H}$; in which way does $\operatorname{grade} I_\lambda$ depend on $\operatorname{grade} I_\chi$ and on the ranks of $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$? Here I_χ is the ideal of maximal minors of χ, and $\operatorname{grade} I_\chi$ is the maximal length of a regular sequence contained in I_χ.

If, for example, $\operatorname{rank} \mathcal{F} = 1, \operatorname{rank} \mathcal{G} = n$, and χ is given by a regular sequence $x_1, \ldots, x_n$ in R, i.e. $\chi(1) = (x_1, \ldots, x_n)$, then one knows that $\operatorname{grade} I_\lambda = n$ is possible if and only if $\operatorname{rank} \mathcal{H} = 1$ and n is even. (The if part is trivial: Set $\lambda(a_1, \ldots, a_n) = \sum_{i=1}^{n} (-1)^{i+1} a_i x_{n-i+1}$; for the other direction see [BV2] or Corollary 7 at the end of the article)

Assume that $m = \operatorname{rank} \mathcal{F} \leq n = \operatorname{rank} \mathcal{G}$. Then it turns out that the problem just described is closely connected with the homology of the (generalized) Koszul complex associated with the induced map $\bar{\lambda} : \operatorname{Cok} \chi \to \mathcal{H}$.

1. Classical Koszul complexes. Let G be a module over an arbitrary commutative ring R, and let $\psi : G \to R$ be a linear form. Define

$$\partial_\psi(y_1 \wedge \ldots \wedge y_p) = \sum_{i=1}^{p} (-1)^{i+1} \psi(y_i) y_1 \wedge \ldots \widehat{y_i} \ldots \wedge y_p$$

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for $y_1, \dots, y_p \in G$. The complex

$$\cdots \bigwedge^p G \xrightarrow{\partial_\psi} \bigwedge^{p-1} G \to \cdots \to \bigwedge^1 G \xrightarrow{\psi} R \to 0$$

is called the (classical) *Koszul complex* associated with ψ. We view ψ as an element of $G^* = \operatorname{Hom}_R(G, R)$. Then the map ∂_ψ is the right multiplication of ψ on $\bigwedge G$ within the meaning of the following structure of $\bigwedge G$ as a right $\bigwedge G^*$-module:

$$y_1 \wedge \ldots \wedge y_n \leftharpoondown y_1^* \wedge \ldots \wedge y_p^* = \sum_\sigma \varepsilon(\sigma) \det_{1 \le i,j \le p} (y_j^*(y_{\sigma(i)})) y_{\sigma(p+1)} \wedge \ldots \wedge y_{\sigma(n)}$$

for $y_1, \dots, y_n \in G$ and $y_1^*, \dots, y_p^* \in G^*$, where σ runs through the set $\mathfrak{S}_{n,p}$ of permutations of n elements which are increasing on the intervals $[1, p]$ and $[p+1, n]$.

The notion Koszul complex is also applied to a dual version of the complex considered above. Let $\varphi : R \to G$ be R-linear. The complex we have in mind is

$$0 \to R \xrightarrow{d_\varphi} G \to \cdots \to \bigwedge^{p-1} G \xrightarrow{d_\varphi} \bigwedge^p G \to \cdots$$

where d_φ is the left multiplication by φ on $\bigwedge G$: $\varphi \rightharpoonup y = \varphi(1) \wedge y$ for all $y \in \bigwedge G$.

If G is free of finite rank n, then both variants are equivalent, which means: there is a (non-canonical) complex isomorphism given by an orientation ν of G:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \bigwedge^n G & \xrightarrow{\partial_\psi} & \bigwedge^{n-1} G \cdots & \longrightarrow & G & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nu_n} & & \downarrow{\scriptstyle \nu_{n-1}} & & \downarrow{\scriptstyle \nu_1} & & \downarrow{\scriptstyle \nu_0} & & \\
0 & \longrightarrow & R & \xrightarrow{d_{\psi^*}} & G^* \ \cdots & \longrightarrow & \bigwedge^{n-1} G^* & \longrightarrow & \bigwedge^n G^* & \longrightarrow & 0
\end{array}$$

(An orientation ν of G is an isomorphism $\nu : \bigwedge^n G \to R$, and $\nu_i : \bigwedge^i G \to \bigwedge^{n-i} G^*$ is defined by $\nu_i(x)(y) = \nu(x \wedge y)$ for all $x \in \bigwedge^i G$ and all $y \in \bigwedge^{n-i} G$.)

2. Generalized Koszul Complexes. Let $\psi : G \to F$ be a linear map of R-modules G and F. By $S(F)$ we denote the symmetric algebra of F. Consider the $S(F)$-linear map $\psi \otimes S(F) : G \otimes S(F) \to S(F)$. The complex $\mathcal{C}(\psi)$ associated with ψ is the classical Koszul complex associated with $\psi \otimes S(F)$. As a complex of R-modules it splits into direct summands $\mathcal{C}(\psi)(t)$

$$0 \to \bigwedge^t G \otimes S_0(F) \xrightarrow{\partial_\psi} \bigwedge^{t-1} G \otimes S_1(F) \to \cdots \to \bigwedge^1 G \otimes S_{t-1}(F) \xrightarrow{\partial_\psi} \bigwedge^0 G \otimes S_t(F) \to 0$$

where we write $\partial_\psi = \partial_{\psi\otimes S(F)}$ for simplicity.

Assume in addition that G and F are free of ranks n and m and that $r = n - m \geq 0$. We choose bases $y_1, \ldots, y_n$ for G and $f_1, \ldots, f_m$ for F. The corresponding dual bases of G^* and F^* are denoted by $y_1^*, \ldots, y_n^*$ and $f_1^*, \ldots, f_m^*$.

Let $\omega : \bigwedge^n G \to R$, $\widetilde{\omega} : \bigwedge^n G^* \to R$ be the orientations of G and G^* given by $\omega(y_1 \wedge \ldots \wedge y_n) = 1$ and $\widetilde{\omega}(y_1^* \wedge \ldots \wedge y_n^*) = 1$. Set $x^* = \psi^*(f_1^*) \wedge \ldots \wedge \psi^*(f_m^*)$ and consider the map

$$\widetilde{\nu}_\psi : \bigwedge^{n-m-t} G^* \to (\bigwedge^t G^*)^*, \quad \widetilde{\nu}_\psi(z^*)(y^*) = \widetilde{\omega}(z^* \wedge y^* \wedge x^*)$$

for all $z^* \in \bigwedge^{n-m-t} G^*$, $y^* \in \bigwedge^t G^*$. One can easily see directly (or use the first diagram below to see) that $\widetilde{\nu}_\psi$ connects the complexes $(\mathcal{C}(\psi)(r-t))^*$ and $\mathcal{C}(\psi)(t)$ to a complex $\widetilde{\mathcal{K}}(\psi)(t)$

$$0 \to (\bigwedge^0 G \otimes S_{r-t}(F))^* \xrightarrow{\partial_\psi^*} \cdots \xrightarrow{\partial_\psi^*} (\bigwedge^{r-t} G \otimes S_0(F))^*$$
$$\xrightarrow{\widetilde{\nu}_\psi} \bigwedge^t G \otimes S_0(F) \xrightarrow{\partial_\psi} \cdots \xrightarrow{\partial_\psi} \bigwedge^0 G \otimes S_t(F) \to 0.$$

Remark 1. The complexes just introduced one can find in [E], Chapter A2.6.1 or in [BV3], 2.C. If $m = 1$, then $\widetilde{\mathcal{K}}(\psi)(t)$ is isomorphic to the classical Koszul complex (for each t). If m is arbitrary and $t = 0$ ($t = 1$), one obtains the complexes of Eagon-Northcott (Buchsbaum-Rim). In general, there is a (non-canonical) complex isomorphism

$$\widetilde{\mathcal{K}}(\psi)(t) \cong \left(\widetilde{\mathcal{K}}(\psi)(n-m-t)\right)^* \tag{1}$$

for all t.

Let G be arbitrary and F as above, i.e. equipped with a basis $f_1, \ldots, f_m$ the dual basis of which is $f_1^*, \ldots, f_m^*$. From a structural point of view there is a smoother (and more general) version of $\widetilde{\mathcal{K}}(\psi)(t)$ if one uses the canonical right $\bigwedge G^* \otimes S(F)$-module structure of $\bigwedge G \otimes S(F)^*$. Here $S(F)^* = \oplus S_p(F)^*$ is the so called *graded dual* of $S(F)$. We consider the complexes $\mathcal{K}(\psi)(t)$

$$\cdots \to \bigwedge^{t+m+p} G \otimes S_p(F)^* \xrightarrow{\partial_\psi} \cdots \xrightarrow{\partial_\psi} \bigwedge^{t+m} G \otimes S_0(F)^*$$
$$\xrightarrow{\nu_\psi} \bigwedge^t G \otimes S_0(F) \xrightarrow{\partial_\psi} \cdots \xrightarrow{\partial_\psi} \bigwedge^0 G \otimes S_t(F) \to 0$$

where

(a) ν_ψ is the right multiplication on $\bigwedge G$ by $x^* = \psi^*(f_1^*) \wedge \ldots \wedge \psi^*(f_m^*)$,

(b) the differential ∂_ψ on the right of ν_ψ is the right multiplication by $\psi \otimes S(F)$ on $\bigwedge G \otimes S(F)$ as above,

(c) and the differential ∂_ψ on the left of ν_ψ is defined by means of the right $\bigwedge G^* \otimes S(F)$-module structure of $\bigwedge G \otimes S(F)^*$ as follows: we view $\psi \in \operatorname{Hom}(G, F)$ as an element of $G^* \otimes F \subset \bigwedge G^* \otimes S(F)$; then $\psi = \sum_i \psi^*(f_i^*) \otimes f_i$ and

$$\begin{aligned}\partial_\psi(y_1 \wedge \ldots \wedge y_p \otimes z) &= y_1 \wedge \ldots \wedge y_p \otimes z \leftharpoonup \sum_i \psi(f_i^*) \otimes f_i \\ &= \sum_i \big(y_1 \wedge \ldots \wedge y_p \leftharpoonup \psi^*(f_i^*)\big) \otimes z \cdot f_i.\end{aligned}$$

for all $y_j \in G$ and $z \in S(F)^*$.

So all complex maps in $\mathcal{K}(\psi)(t)$ are certain right multiplications, a fact which facilitates computations in the following.

If G is free of rank n and ω is the orientation of G from above, then we obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & (\bigwedge^0 \otimes S_{r-t})^* \arrow[r, "\partial_\psi^*"] & \cdots \arrow[r, "\partial_\psi^*"] & (\bigwedge^{r-t})^* \arrow[dr, "\tilde{\nu}_\psi"] & & & & \\
 & & & & \bigwedge^t \arrow[r, "\partial_\psi"] & \cdots \arrow[r, "\partial_\psi"] & S_t \arrow[r] & 0 \\
0 \arrow[r] & \bigwedge^n \otimes S_{r-t}^* \arrow[r, "\partial_\psi"'] \arrow[uu, "\pm\omega_n \otimes 1"] & \cdots \arrow[r, "\partial_\psi"'] & \bigwedge^{m+t} \arrow[ur, "\nu_\psi"'] \arrow[uu, "\pm\omega_{m+t}"] & & & &
\end{tikzcd}$$

where $S_p = S_p(F)$, $S_p^* = S_p(F)^*$, $\bigwedge^p = \bigwedge^p G$, and $r = n - m$ as above. So, in particular,

$$\mathcal{K}(\psi)(t) \cong \widetilde{\mathcal{K}}(\psi)(t)$$

in this case, and in view of the isomorphism (1) we obtain a complex isomorphism

$$\mathcal{K}(\psi)(t) \cong \big(\mathcal{K}(\psi)(n-m-t)\big)^*. \tag{2}$$

The dual version of the classical Koszul complex has a similar generalization. Let $\varphi : H \to G$ be R-linear where H is free of rank l (and G is arbitrary). By $D(H)$ we denote the divided power algebra of H. We view φ as an element of $H^* \otimes G \subset S(H^*) \otimes \bigwedge G$ and use the canonical structure of $D(H) \otimes \bigwedge G$ as a left $S(H^*) \otimes \bigwedge G$-module. Then we define d_φ to be the left multiplication by φ on $D(H) \otimes \bigwedge G$ (i.e.

$$\varphi \rightharpoonup x_1^{(k_1)} \ldots x_p^{(k_p)} \otimes y = \sum_j x_1^{(k_1)} \ldots x_j^{(k_j - 1)} \ldots x_p^{(k_p)} \otimes \varphi(x_j) \wedge y$$

for $x_i \in H$ and $y \in \bigwedge G$) and on $S(H^*) \otimes \bigwedge G$ (in an obvious way). We obtain complexes $\mathcal{L}(\varphi)(t)$

$$0 \to D_t(H) \otimes \bigwedge^0 G \xrightarrow{d_\varphi} \cdots \xrightarrow{d_\varphi} D_0(H) \otimes \bigwedge^t G$$
$$\xrightarrow{\nu^\varphi} S_0(H^*) \otimes \bigwedge^{t+l} G \xrightarrow{d_\varphi} \cdots \xrightarrow{d_\varphi} S_p(H^*) \otimes \bigwedge^{t+l+p} G \to \cdots$$

where the connection map ν^φ is the left multiplication on $\bigwedge G$ by $\varphi(h_1) \wedge \ldots \wedge \varphi(h_l)$, $h_1, \ldots, h_l$ being a basis of H. It is immediately clear that for $l = 1$ (and all t) one gets the dual version of the classical Koszul complex.

If G is free of rank $n \geq l$ and ω is the orientation of G from above, then the complexes $\mathcal{L}(\varphi)(t)$ are via ω isomorphic with the complexes $\mathcal{D}_t(\varphi)$ in [BV1]. In fact $\mathcal{D}_t(\varphi)$ is built by connecting the complex

$$0 \to D_t(H) \otimes \bigwedge^0 G \xrightarrow{d_\varphi} \cdots \xrightarrow{d_\varphi} D_0(H) \otimes \bigwedge^t G \to 0$$

with the complex

$$0 \to (D_0(H) \otimes \bigwedge^{n-l-t} G)^* \xrightarrow{d_\varphi^*} \cdots \xrightarrow{d_\varphi^*} (D_{n-l-t}(H) \otimes \bigwedge^0 G)^* \to 0$$

through the map

$$\widetilde{\nu}^\varphi : \bigwedge^t G \to (\bigwedge^{n-l-t} G)^*, \quad \widetilde{\nu}^\varphi(z)(y) = \omega(\varphi(h_1) \wedge \ldots \wedge \varphi(h_l) \wedge y \wedge z)$$

where $z \in \bigwedge^t G,\ y \in \bigwedge^{n-l-t} G$, and there is a commutative diagram

$$\begin{tikzcd}
 & & & & \bigwedge^{t+l} \arrow[r, "d_\varphi"] \arrow[dd, "\pm\omega_{t+l}"] & \cdots \arrow[r, "d_\varphi"] & S^*_{s-t} \otimes \bigwedge^n \arrow[r] \arrow[dd, "\pm 1\otimes\omega_n"] & 0 \\
0 \arrow[r] & D_t \arrow[r, "d_\varphi"] & \cdots \arrow[r, "d_\varphi"] & \bigwedge^t \arrow[ur, "\nu^\varphi"] \arrow[dr, "\widetilde{\nu}^\varphi"'] & & & & \\
 & & & & (\bigwedge^{s-t})^* \arrow[r, "d^*_\varphi"'] & \cdots \arrow[r, "d^*_\varphi"'] & (D_{s-t})^* \otimes \bigwedge^0 G \arrow[r] & 0
\end{tikzcd}$$

where $D_p = D_p(H)$, $S^*_p = S_p(H^*)$, $\bigwedge^p = \bigwedge^p G$, and $s = n - l$.

We record a result concerning the connection between the generalized Koszul complex and its dual.

Proposition 2. *Let $\psi : G \to F$ be a homomorphism of an (arbitrary) R-module G into a finite free R-module F. Then the canonical map $\theta : \bigwedge G^* \to (\bigwedge G)^*$ induces a natural complex morphism*

$$\tau : \ \mathcal{L}(\psi^*)(t) \to \big(\mathcal{K}(\psi)(t)\big)^*$$

where the connection homomorphisms ν^{ψ^*} and ν_ψ are defined with respect to the same basis of F. If θ is an isomorphism (for example, if G is finitely generated and projective), then τ is a complex isomorphism.

3. Koszul Bicomplexes. Assume now that F, G, H are free of ranks m, n, l and that $H \xrightarrow{\varphi} G \xrightarrow{\psi} F$ is a complex. Then there is a rather simple associativity formula concerning the various right and left multiplications, we considered before, which allows us to assemble the complexes $\mathcal{K}(\psi)(t)$ and $\mathcal{L}(\varphi)(t)$ to the *Koszul bicomplexes* $\mathcal{B}(t)$:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots H \otimes \bigwedge^{t+m} G \otimes F^* \arrow[r] \arrow[d] & \bigwedge^{t+m+1} G \otimes F^* \arrow[r, "\pm\nu^{\varphi}"] \arrow[d, "\partial_\psi"] & \bigwedge^{t+l+m+1} G \otimes F^* \arrow[r] \arrow[d, "\partial_\psi"] & H^* \otimes \bigwedge^{t+l+m+2} G \otimes F^* \cdots \arrow[d] & \\
\cdots H \otimes \bigwedge^{t+m-1} G \arrow[r, "d_\varphi"] \arrow[d, "\pm\nu_\psi"] & \bigwedge^{t+m} G \arrow[r, "\pm\nu^{\varphi}"] \arrow[d, "\pm\nu_\psi"] & \bigwedge^{t+l+m} G \arrow[r, "d_\varphi"] \arrow[d, "\pm\nu_\psi"] & H^* \otimes \bigwedge^{t+l+m+1} G \cdots \arrow[d, "\pm\nu_\psi"] & \\
\cdots H \otimes \bigwedge^{t-1} G \arrow[r, "d_\varphi"] \arrow[d] & \bigwedge^{t} G \arrow[r, "\pm\nu^{\varphi}"] \arrow[d, "\partial_\psi"] & \bigwedge^{t+l} G \arrow[r, "d_\varphi"] \arrow[d, "\partial_\psi"] & H^* \otimes \bigwedge^{t+l+1} G \cdots \arrow[d] & \\
\cdots H \otimes \bigwedge^{t-2} G \otimes F \arrow[r] \arrow[d] & \bigwedge^{t-1} G \otimes F \arrow[r, "\pm\nu^{\varphi}"] \arrow[d] & \bigwedge^{t+l-1} G \otimes F \arrow[r] \arrow[d] & H^* \otimes \bigwedge^{t+l} G \otimes F \cdots \arrow[d] & \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The rows in the upper half arise from $\mathcal{L}(\varphi)(t+m+j)$ tensored with $S_(F)^*$, $j = 0, 1, \ldots$, while the rows below are built from $\mathcal{L}(\varphi)(t-j)$ tensored with

$S_j(F)$, $j = 0, 1, \ldots$; we abbreviate $d_\varphi \otimes 1_{S(F^*)}$ and $d_\varphi \otimes 1_{S(F)}$ to d_φ, and correspondingly $\nu^\varphi \otimes 1_{S(F^*)}$ and $\nu^\varphi \otimes 1_{S(F)}$ to ν^φ. The columns are obtained analogously: in western direction we have to tensorize $D_i(H)$ with $\mathcal{K}(\psi)(t-i)$, $i = 0, 1, \ldots$, while going east we must tensorize $S_i(H^*)$ with $\mathcal{K}(\psi)(t+l+i)$, $i = 0, 1, \ldots$; as before we shorten the complex maps to ∂_ψ and ν_ψ. The signs of ν^φ and ν_ψ are determined by the associativity formula. We point out that for $m = (\operatorname{rank} F =)1$ the bicomplexes $\mathcal{B}(t)$ have predecessors in [HM] and in [BV2], [BV3].

Consider the cokernel of the complex morphism between the first two row complexes in the diagram which we denote by $\mathcal{M}(t)$, that is

$$\mathcal{M}(t) = \operatorname{Cok}\left(\mathcal{L}(\varphi)(t+m+1) \xrightarrow{\partial_\psi} \mathcal{L}(\varphi)(t+m)\right).$$

Similarly we set

$$\mathcal{N}(t) = \operatorname{Ker}\left(\mathcal{L}(\varphi)(t) \xrightarrow{\partial_\psi} \mathcal{L}(\varphi)(t-1)\right).$$

Clearly ν_ψ induces a complex map

$$\mathcal{M}(t) \xrightarrow{\nu} \mathcal{N}(t).$$

We shall now establish a connection with the setup in the beginning. Starting from an R-module M which has a presentation

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ \chi\ } \mathcal{G} \longrightarrow M \longrightarrow 0$$

where $\mathcal{F}$, $\mathcal{G}$ are free modules of ranks m and n, we consider an R-homomorphism $\bar{\lambda} : M \to \mathcal{H}$ into a finite free R-module $\mathcal{H}$ of rank $l \le n$. By λ we denote the corresponding lifted map $\mathcal{G} \to \mathcal{H}$. Dualizing $\mathcal{F} \xrightarrow{\chi} \mathcal{G} \xrightarrow{\lambda} \mathcal{H}$ we are back in the situation previously studied. So we set $F = \mathcal{F}^*$, $G = \mathcal{G}^*$, $H = \mathcal{H}^*$, $\psi = \chi^*$, $\varphi = \lambda^*$, and consider the complex

$$H \xrightarrow{\varphi} G \xrightarrow{\psi} F.$$

As above we set $s = n - l$ and $r = n - m$. In addition let $\rho = n - m - l$. Using Proposition 2 and the isomorphism (2), it is an easy exercise in linear algebra to see that

$$\mathcal{M}(t) \cong \mathcal{K}(\bar{\lambda})(\rho - t).$$

So the map ν from above induces a complex map

$$\mathcal{K}(\bar{\lambda})(\rho - t) \xrightarrow{\mu} \mathcal{N}(t).$$

As we shall see in the next section, the map μ enables us in some nice cases to compute the homology of the complex on the left hand side from the homology on the right. The following remark is immediately clear.

Remark 3. If I_χ $(= I_\psi)$ $= R$, then all columns of the bicomplex $\mathcal{B}(t)$ are split exact. Consequently the maps ν and μ are isomorphisms.

4. Grade sensitivity. In the following we shall work with the assumptions and the notation of the previous section. Additionally we shall assume that the ring R *is noetherian.*

It is well known that the complexes $\widetilde{\mathcal{K}}(t)$ and $\widetilde{\mathcal{L}}(t)$ (s. section 2) are *grade sensitive* which means that the homology $H^i(\widetilde{\mathcal{K}}(t))$ $(H^i(\widetilde{\mathcal{L}}(t)))$ vanishes for $i <$ grade I_ψ $(i <$ grade $I_\varphi)$. We exploit this fact (and the complex isomorphisms pictured in the diagrams of section 2) to investigate the homology of $\mathcal{N}(t)$. It is not surprising that the most satisfactory results are obtained in the case in which the grade of I_χ $(= I_{\psi^*})$ has the greatest possible value $r + 1$. For the sake of clarity we shall restrict our report to this case.

Theorem 4. *Let t be a non-negative integer. Set $C = \operatorname{Cok}\psi$ and $h = \operatorname{grade} \mathrm{I}_\varphi$. With the assumptions just made, for the homology $\bar{H}^{\cdot}$ of $\mathcal{N}(t)$ the following holds:*

(a) $\bar{H}^i = 0$ *for* $i = 0, \dots, \min(2, h-1)$;

(b)

$$\bar{H}^i = \begin{cases} D_{t-\frac{i-1}{2}}(H) \otimes S_{\frac{i-1}{2}}(C) & \text{if} \quad 3 \le i < \min(h, 2t+3),\ i \not\equiv 0\ (2), \\ 0 & \text{if} \quad 3 \le i < \min(h, 2t+3),\ i \equiv 0\ (2); \end{cases}$$

(c) $\bar{H}^i = 0$ *for* $2t + 3 \le i < \min(h, 2(t+1) + l)$;

(d)

$$\bar{H}^i = \begin{cases} S_{\frac{i-l}{2}-t-1}(H) \otimes S_{\frac{i+l}{2}-1}(C) & \text{if} \ \ 2(t+1) + l \le i < h,\ i \equiv l\ (2), \\ 0 & \text{if} \ \ 2(t+1) + l \le i < h,\ i \not\equiv l\ (2). \end{cases}$$

There are similar statements in case t is a negative integer (s. [I] or [IV] for details). Via $\mu : \mathcal{K}(\bar{\lambda})(\rho - t) \to \mathcal{N}(t)$ these results supply information on the homology of $\mathcal{K}(\bar{\lambda})(t)$ since one can easily show that

μ_i is an isomorphism for $i > 0$ and injective for $i = 0$.

Theorem 5. *With notation as in Theorem 4 set $S_0(C) = R/I_\chi$. Equip $\mathcal{K}(\bar{\lambda})(t)$ with the graduation induced by the complex morphism $\mu : \mathcal{K}(\bar{\lambda})(t) \to \mathcal{N}(\rho - t)$. Then for the homology $H^{\cdot}$ of $\mathcal{K}(\bar{\lambda})(t)$ the following holds:*

(a) *in case* $t \leq \frac{\rho}{2}$,

$$H^i = \begin{cases} D_{\rho-t-\frac{i-1}{2}}(\mathcal{H}^*) \otimes S_{\frac{i-1}{2}}(C) & \text{if } 0 \leq i < h,\ i \not\equiv 0\ (2), \\ 0 & \text{if } 0 \leq i < h,\ i \equiv 0\ (2); \end{cases}$$

(b) *in case* $\frac{\rho}{2} < t \leq \rho$,

$$H^i = \begin{cases} D_{\rho-t-\frac{i-1}{2}}(\mathcal{H}^*) \otimes S_{\frac{i-1}{2}}(C) & \text{if } 0 \leq i < \min(h, 2(\rho-t+1)),\ i \not\equiv 0\ (2), \\ S_{\frac{i-l}{2}-\rho+t-1}(\mathcal{H}) \otimes S_{\frac{i+l}{2}-1}(C) & \text{if } 2(\rho-t+1)+l \leq i < h,\ i-l \equiv 0\ (2), \\ 0 & \text{otherwise if } 0 \leq i < h; \end{cases}$$

(c) *in case* $\rho < t < r$,

$$H^i = \begin{cases} S_{\frac{i-r+t-1}{2}}(\mathcal{H}) \otimes S_{\frac{i+r-t-1}{2}}(C) & \text{if } r-t+1 \leq i < h,\ i+r-t \not\equiv 0\ (2), \\ 0 & \text{otherwise if } 0 \leq i < h; \end{cases}$$

(d) *in case* $r \leq t$,

$$H^i = \begin{cases} S_{\frac{i-1}{2}+t-r}(\mathcal{H}) \otimes S_{\frac{i-1}{2}}(C) & \text{if } 0 \leq i < h,\ i \not\equiv 0\ (2), \\ 0 & \text{if } 0 \leq i < h,\ i \equiv 0\ (2). \end{cases}$$

From 5 (or directly from 4) one can derive some partial answers to the question asked in the beginning. To make it plausible we look at the complex $\mathcal{K}(\bar{\lambda})(t)$:

$$\cdots \to \textstyle\bigwedge^r M \otimes S_{\rho-t}(\mathcal{H})^* \xrightarrow{\partial_{\bar{\lambda}}} \cdots \xrightarrow{\partial_{\bar{\lambda}}} \bigwedge^{t+l} M \xrightarrow{\nu_{\bar{\lambda}}} \bigwedge^t M \xrightarrow{\partial_{\bar{\lambda}}} \cdots \xrightarrow{\partial_{\bar{\lambda}}} S_t(\mathcal{H}) \to 0.$$

Let $t \leq \frac{\rho}{2}$. Then H^0 is the homology of $\mathcal{K}(\bar{\lambda})(t)$ at $\bigwedge^r M \otimes S_{\rho-t}(\mathcal{H})^*$. If we take $t = -l-1$, then $H^{r+1} = 0$ since H^{r+1} is the homology of $\mathcal{K}(\bar{\lambda})(-l-1)$ at $\bigwedge^{-1} M = 0$. On the other hand, if r were even, and $\operatorname{grade} I_\lambda \geq r+1$, then some playing with the homology of the bicomplex $\mathcal{B}(t)$ would yield an exact sequence

$$0 \to S_{\frac{r}{2}}(C) \otimes S_{\frac{r}{2}+1}(\mathcal{H})^* \to H^{r+1}$$

which is obviously impossible. The proof of the following theorem works with similar arguments.

Theorem 6. *Suppose that* $\operatorname{grade} I_\chi = r+1$. *Then* $I_\lambda \subset I_\chi$, *and in particular* $\operatorname{grade} I_\lambda \leq r+1$. *Set* $\rho = r - l$.

(a) *If there is a* $\bar{\lambda}$ *such that* $\operatorname{grade} I_\lambda > |\,\rho\,| + 1$, *then* $l = 1$ *and* r *is odd.*

(b) *Suppose in addition that* χ *is minimal. Then the following are equivalent:*

(1) *There is a* $\bar{\lambda}$ *such that* $\operatorname{grade} I_\lambda > |\,\rho\,| + 1$*;*

(2) $l = 1$ *and* (i) $r = 1$ *or* (ii) $m = 1$ *and* $r \geq 3$ *is odd.*

(c) *The following conditions are equivalent:*

(1') $\operatorname{grade} I_\lambda > |\,\rho\,| + 1$*;*

(2') $I_\lambda = I_\chi$.

Corollary 7. *Suppose that* $\operatorname{grade} I_\chi = r + 1$*. Then the following conditions are equivalent:*

(1) *there is a* $\bar{\lambda}$ *with* $\operatorname{grade} I_\lambda = n - l + 1$*;*

(2) $l = 1$*,* $m = 1$ *and* $r \geq 1$ *is odd.*

To avoid misunderstanding: a homomorphism of finite free R-modules is called *minimal* if the entries of a representing matrix generate a proper ideal of R.

For the implication (2) $\Rightarrow$ (1) we refer to the example in the beginning. To give at least one complete *proof* we will show (1) $\Rightarrow$ (2) now.

Since we assumed that $n - l \geq 0$, we have $\operatorname{grade} I_\lambda \geq 1$. So λ^* is injective and $\operatorname{rank} \operatorname{Im} \lambda^* = l$. In the same way $n - m \geq 0$ and $\operatorname{grade} I_\chi \geq 1$ imply that $\operatorname{Ker} \chi^* = (\operatorname{Cok} \chi)^*$ has rank $n - m = r$. From $\operatorname{Im} \lambda^* \subset \operatorname{Ker} \chi^*$ we conclude $r - l \geq 0$. By assumption $\operatorname{grade} I_\lambda = n - l + 1 > r - l + 1 = \rho + 1$, and by Theorem 6,(c) we obtain $I_\lambda = I_\chi$, in particular $n - l = r = n - m$, so $l = m$. Part (a) in the same Theorem yields that $l = 1$ and $r \geq 1$ is odd.

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