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CRITERIA FOR SHELLABLE MULTICOMPLEXES*

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Abstract

After [4] the shellability of multicomplexes Γ is given in terms of some special faces of Γ called facets. Here we give a criterion for the shellability in terms of maximal facets. Multigraded pretty clean filtration is the algebraic counterpart of a shellable multicomplex. We give also a criterion for the existence of a multigraded pretty clean filtration.

Introduction

Cleanness is the algebraic counterpart of shellability for simplicial complexes after [2]. A kind of multigraded "sequentially" cleanness the so called pretty cleanness was introduced in [4]. Multigraded pretty cleanness implies sequentially Cohen-Macaulay which remind us a well known result of Stanley [8] saying that shellable simplicial complexes are sequentially Cohen-Macaulay. Pretty cleanness is the algebraic counterpart of shellability of the so called multicomplexes (see [4]). The aim of this paper is to find easy criteria for multigraded pretty cleanness (see Theorem 2.3) or for the shellability of multicomplexes (see Theorem 3.6). The Proposition 3.1 is important in the proof of [4, Proposition (10.1)], where it was a consequence of some results concerning standard pairs given in [9]. Here we give an independent proof. Many useful examples are included. For instance in 3.3 it is given a shellable multicomplex which has a shelling $a_1, \dots, a_r$ which does not satisfy the condition

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$\dim S_1 \geq \ldots \geq \dim S_r$ from [4, Corollary (10.7)] or here 1.13 though certainly there is another shelling for which it holds. Example 3.8 shows that there are shellable multicomplexes which are not maximal shellable.

We express our thanks to J. Herzog especially for some discussions around Theorem 3.6.

1 Preliminaries on pretty clean modules and multicomplexes

Let R be a Noetherian ring, and M a finitely generated R-module. Then it is well known that there exists a so called *prime filtration*

$$\mathcal{F}\colon 0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M$$

that is such that $M_i/M_{i-1} \cong R/P_i$ for some $P_i \in \mathrm{Supp}(M)$. We denote $\mathrm{Supp}(\mathcal{F}) = \{P_1, \ldots, P_r\}$ and r is called the length of $\mathcal{F}$. It follows that

$$\mathrm{Ass}(M) \subset \mathrm{Supp}(\mathcal{F}) \subset \mathrm{Supp}(M).$$

If $\mathrm{Supp}(\mathcal{F}) \subset \mathrm{Min}(M)$ then $\mathcal{F}$ is called *clean*. M is called *clean*, if M admits a clean filtration. R is *clean* if it is a clean module over itself. In particular, if M is clean then $\mathrm{Ass}(M) = \mathrm{Min}(M)$. Let Δ be a simplicial complex on the vertex $\{1, \ldots, n\}$, K a field and $K[\Delta]$ the Stanley-Reisner ring.

Theorem 1.1 (Dress 1991). *Δ is a (non-pure) shellable simplicial complex if and only if $K[\Delta]$ is a clean ring.*

A pure shellable simplicial complex is Cohen-Macaulay. So if I is a reduced monomial ideal of $S = K[x_1, \ldots, x_n]$ such that S/I is clean equidimensional then S/I is Cohen-Macaulay. This result is extended in [4] in a more general frame as we explain bellow.

A prime filtration $\mathcal{F}\colon 0 = M_0 \subset M_1 \subset \ldots \subset M_{r-1} \subset M_r = M$ of M with $M_i/M_{i-1} = R/P_i$ is called *pretty clean*, if for all $i < j$ for which $P_i \subset P_j$ it follows that $P_i = P_j$. This means, roughly speaking, that a proper inclusion $P_i \subset P_j$ is only possible if $i > j$. M is called *pretty clean*, if it has a pretty clean filtration. A ring is called pretty clean if it is a pretty clean module over itself. If $\mathcal{F}$ is pretty clean then $\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}(M)$.

Examples 1.2. Let $S = K[x, y]$ be the polynomial ring over the field K, $I \subset S$ the ideal $I = (x^2, xy)$ and $R = S/I$. Then R is pretty clean but not clean. Indeed, $0 \subset (x) \subset R$ is a pretty clean filtration of R with $(x) = R/(x, y)$, so that $P_1 = (x, y)$ and $P_2 = (x)$. R is not clean since $\mathrm{Ass}(R) \neq \mathrm{Min}(R)$. Note R has a different prime filtration, namely, $\mathcal{G} : 0 \subset (y) \subset (x, y) \subset R$ with factors $(y) = R/(x)$ and $(x, y)/(y) = R/(x, y)$. Hence this filtration is not pretty clean, even though $\mathrm{Supp}(\mathcal{G}) = \mathrm{Ass}(M)$.

Example 1.3. Let R be a UFD ring and $t_1, \dots, t_s$ be some irreducible elements in R, even equally some of them. Let P be a prime ideal of R. Then $M := R/I$, $I := t_1 \cdots t_s P$ is a pretty clean module, even clean if P does not contain any of (t_i). Indeed, consider the following filtration on M

$$M = M_{s+1} \supset M_s \supset \ldots \supset M_1 \supset M_0 = (0),$$

where $M_r := (t_r \cdots t_s)/I$ for $1 \le r \le s$. We have

$$M_r/M_{r-1} \cong (t_r \cdots t_s)/(t_{r-1} \cdots t_s) \cong R/(t_{r-1}).$$

and $M_1 \cong R/P$. A special type of this example is given by $R = K[x, y]$ and $I = (x^2, xy)$.

Proposition 1.4 ([4]). *Let M be a pretty clean module. Then all pretty clean filtrations of M have the same length, namely their common length equals $\sum_{\mathfrak{p} \in \mathrm{Ass}(M)} length_M(H^0_{\mathfrak{p}}(M_{\mathfrak{p}}))$, that is this number is bounded by the arithmetic degree of M, which is $\sum_{\mathfrak{p} \in \mathrm{Ass}(M)} length_M(H^0_{\mathfrak{p}}(M_{\mathfrak{p}})) \deg(R/\mathfrak{p})$.*

M is called *sequentially Cohen-Macaulay* if it has a filtration

$$0 = C_0 \subset C_1 \subset C_2 \subset \ldots \subset C_s = M$$

such that C_i/C_{i-1} is Cohen-Macaulay, and

$$\dim(C_1/C_0) < \dim(C_2/C_1) < \ldots < \dim(C_r/C_{r-1}).$$

Theorem 1.5 (Herzog, Popescu [4]). *Let R be a local CM ring admitting a canonical module ω_R, and let M be a pretty clean R-module such that R/P is Cohen-Macaulay for all $P \in \mathrm{Ass}\, M$. Then M is sequentially Cohen-Macaulay. Moreover, if $\dim R/P = \dim M$ for all $P \in \mathrm{Ass}(M)$, then M is clean and Cohen-Macaulay.*

Schenzel introduced in [7] the so called *dimension filtration*

$$\mathcal{F}\colon 0 \subset D_0(M) \subset D_1(M) \subset \cdots \subset D_{d-1}(M) \subset D_d(M) = M$$

of M, which is defined by the property that $D_i(M)$ is the largest submodule of M with $\dim D_i(M) \le i$ for $i = 0, \dots, d = \dim M$.

Theorem 1.6 (Schenzel [7]). *M is sequentially CM, if and only if the factors in the dimension filtration of M are either 0 or CM.*

Theorem 1.7 (Herzog, Popescu [4]). *Let R be a local CM ring admitting a canonical module ω_R, and let M be a finitely generated R-module such that R/P is Cohen-Macaulay for all $P \in \mathrm{Ass}\, M$. Then M is pretty clean if and only if $D_i(M)/D_{i-1}(M)$ is clean for all $1 \le i \le \dim M$.*

So in the above assumptions we may say that pretty clean means sequentially clean. Now we pass to the multigraded case.

Proposition 1.8 (Herzog, Popescu [4])**.** *Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal such that* $\operatorname{Ass} S/I$ *is totally ordered by inclusion. Then S/I is pretty clean.*

A monomial ideal I is called of *Borel type* if $I : (x_1, \dots, x_j)^\infty = I : x_j^\infty$ for all $1 \leq j \leq n$.

Corollary 1.9 (Herzog, Popescu, Vladoiu [5])**.** *If the ideal $I \subset S$ is of Borel type then S/I is pretty clean and in particular sequentially CM.*

Let Δ be a non-pure shellable simplicial complex on the vertex set $\{1, \dots, n\}$ and $F_1, \dots, F_r$ its shelling on the facets of Δ. For $i \geq 2$ we denote by a_i the number of facets of $\langle F_1, \dots, F_{i-1}\rangle \cap \langle F_i \rangle$, and set $a_1 = 0$. Let $P_i = (\{x_j\}_{j \notin F_i})$ be the prime ideal associated to the facet F_i.

Proposition 1.10 ([4])**.** *The filtration*

$$(0) = M_0 \subset M_1 \subset \cdots\cdots M_{r-1} \subset M_r = K[\Delta]$$

with

$$M_i = \bigcap_{j=1}^{r-i} P_j \quad \textit{and} \quad M_i/M_{i-1} \cong S/P_{r-i+1}(-a_{r-i+1})$$

is a clean filtration of S/I_Δ.

Simplicial complexes correspond to the reduced monomial ideals of S. What about general monomial ideals of S?

Let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. For a subset $\Gamma \subset \mathbb{N}_\infty$ denote by $\mathcal{M}(\Gamma)$ the set of all maximal elements of Γ. Let $a \in \Gamma$. Then

$$\operatorname{infpt} a = \{i \colon a(i) = \infty\}$$

is called the *infinite part* of a. A subset $\Gamma \subset \mathbb{N}_\infty^n$ is called a *multicomplex* if

(1) for all $a \in \Gamma$ and all $b \in \mathbb{N}_\infty^n$ with $b \leq a$ it follows that $b \in \Gamma$;

(2) for each $a \in \Gamma$ there exists $m \in \mathcal{M}(\Gamma)$ with $a \leq m$.

The elements of a multicomplex are called *faces* and the elements of $\mathcal{M}(\Gamma)$ are called *maximal facets*. A face $a \in \Gamma$ is called *facet* if for any $m \in \mathcal{M}(\Gamma)$ with $a \leq m$ it holds $\operatorname{infpt} a = \operatorname{infpt} m$.

Consider for example the multicomplex $\Gamma \in \mathbb{N}_\infty^2$ with faces

$$\{a \colon a \leq (0, \infty) \text{ or } a \leq (2, 0)\}.$$

Then $\mathcal{M}(\Gamma) = \{(0, \infty), (2, 0)\}$ and $\mathcal{F}(\Gamma) = \{(0, \infty), (2, 0), (1, 0)\}$. Besides its facets, Γ admits the infinitely many faces $(0, i)$ with $i \in \mathbb{N}$.

Lemma 1.11 ([4]). *Each multicomplex has a finite number of facets.*

Let Γ be a multicomplex, and let $I(\Gamma)$ be the K-subspace in S spanned by all monomials x^a such that $a \notin \Gamma$. This is an ideal in S and the correspondence $\Gamma \to I(\Gamma)$ gives a bijection between the multicomplexes Γ of $\mathbb{N}^n_\infty$ and the monomial ideals I of S.

Let $a \in \mathbb{N}^n$, $m \in \mathbb{N}^n_\infty$ with $m(i) \in \{0, \infty\}$ and $\Gamma(m)$ the multicomplex generated by m, that is the set of all $u \in \mathbb{N}^n_\infty$ with $u \leq m$. Actually given $m_1 \dots, m_r \in \mathbb{N}^n$ we denote by $\Gamma(m_1, \dots, m_r)$ the set of all $u \in \mathbb{N}^n$ with $u \leq m_i$ for some i. The sets of the form $S = a + S^*$, where $S^* = \Gamma(m)$, are called *Stanley sets*. The *dimension of S* is defined to be the dimension of S^*. A multicomplex Γ is *shellable* if the facets of Γ can be ordered $a_1, \dots, a_r$ such that

(1) $S_i = \Gamma(a_i) \setminus \Gamma(a_1, \dots, a_{i-1})$ is a Stanley set for $i = 1, \dots, r$, and

(2) whenever $S_i^* \subset S_j^*$, then $S_i^* = S_j^*$ or $i > j$.

Theorem 1.12 (Herzog, Popescu [4]). *The multicomplex Γ is shellable if and only if $S/I(\Gamma)$ is a multigraded pretty clean ring.*

Note that in the above example the shelling could be $\{(0, \infty), (1, 0), (2, 0)\}$ and so the first Stanley set is given by the axe $\{(0, s) : s \in \mathbb{N}\}$ and the second Stanley set, respectively the third one are the points $(1, 0)$, $(2, 0)$.

Corollary 1.13 ([4]). *A multicomplex Γ is shellable if and only if there exists an order $a_1, \dots, a_r$ of the facets such that for $i = 1, \dots, r$ the sets $S_i = \Gamma(a_i) \setminus \Gamma(a_1, \dots, a_{i-1})$ are Stanley sets with $\dim S_1 \geq \dim S_2 \geq \dots \geq \dim S_r$.*

2 Pretty clean modules

Let R be a Noetherian ring, and M a finitely generated R-module.

Let
$$\mathcal{M} : M_0 = 0 \subset M_1 \subset \dots \subset M_s = M$$
be a filtration of M. Inspired by [2] we call $\mathcal{M}$ *almost clean* if for all $1 \leq i \leq s$ there exists a prime ideal $P \in \operatorname{Ass}_R(M)$ such that $P = (M_{i-1} : x)$ for all $x \in M_i \setminus M_{i-1}$.

Lemma 2.1. *Every finitely generated R-module M has an almost clean filtration.*

Proof. Let $\operatorname{Ass}_R(M) = \{P_1, \dots, P_t\}$ and $(0) = \cap_{i=1}^t N_i$ be an irredundant primary decomposition of (0) in M, where N_i is a P_i-primary submodule of

M. We may suppose the notation such that for all $i < j$ such that $P_i \subset P_j$ it follows $P_i = P_j$. Set $U_j = \cap_{i=1}^{j} N_i$. We get a filtration

$$(0) = U_t \subset \ldots \subset U_1 \subset U_0 = M$$

such that $U_{i-1}/U_i \cong U_{i-1} + N_i/N_i \subset M/N_i$. But $U_{i-1} \neq U_i$ because the primary decomposition is irredundant. Then $\{P_i\} = \mathrm{Ass}_R(M/N_i) = \mathrm{Ass}_R(U_{i-1}/U_i) \neq \emptyset$ and so $\mathrm{Ass}_R(U_{i-1}/U_i) = \{P_i\}$. Thus we reduce to the case $T = U_{i-1}/U_i$. In this case set $V_k = P_i^k T_{P_i} \cap T$ for $k \geq 0$. We get a filtration

$$(0) \subset \ldots \subset V_1 \subset V_0 = T$$

such that there exists an injection $V_{j-1}/V_j \to P_i^{j-1} T_{P_i}/P_i^j T_{P_i}$, the last module being a linear space over the fraction field of R/P_i. Thus V_{j-1}/V_j is torsionless over R/P_i, which is enough.

Corollary 2.2. *If R/P is regular of dimension ≤ 1 for all $P \in \mathrm{Ass}_R(M)$ then M is pretty clean.*

For the proof note only that the quotients V_{j-1}/V_j from the above proof are in this case free and so clean.

Next theorem is an extension of [4, Proposition (5.1)].

Let $S = K[x_1, \ldots, x_n]$ and $I \subset S$ a monomial ideal. For every $P \in \mathrm{Ass}(S/I)$ let $J_P \subset \{1, \ldots, n\}$ be the subset of all j such that $x_j \in P$. Clearly P is monomial and so $\mathrm{height}(P) = |J_P|$ and $\dim P = n - |J_P|$.

Theorem 2.3. *Suppose that for all integer $d > 0$ such that $\mathrm{Ass}^d(S/I) \neq \emptyset$ it holds $|\bigcup_{P \in \mathrm{Ass}^d(S/I)} J_P| \leq n - d + 1$. Then S/I is pretty clean.*

Proof. We follow the proof of Proposition 1.8 given in [4].

Let $I = \bigcap_{P \in \mathrm{Ass}(S/I)} Q_P$ be the irredundant primary monomial decomposition of I, where Q_P is P-primary. Set $d_P = \dim P$ and $d_1 < \ldots < d_r$ be the integers which really appear among (d_P). Set $U_i = \bigcap_{P \in \cup_{j > d_i} \mathrm{Ass}^j(S/I)} Q_P$. By [7] the dimension filtration is given by $D_{d_i}(S/I) = U_i/I$. Using Theorem 1.7 it is enough to show that U_i/U_{i-1} is clean. Let S' be the polynomial ring over K in the variables x_j with $j \in \bigcup_{P \in \mathrm{Ass}^{d_i}(S/I)}$ and set $P' = P \cap S'$, $Q_{P'} = Q_P \cap S'$ for $P \in \mathrm{Ass}^{d_i}(S/I)$. We have $P = P'S$, $Q_P = Q_{P'}S$ and $U_i = U_i'S$ for $U_i' = U_i \cap S'$.. But U_i'/U_{i-1}' is clean by Corollary 2.2 since $\dim S' = n - d_i + 1$ by assumption (that is $\dim S/P \leq 1$ for all $P \in \mathrm{Ass}^{d_i}(S/I)$). Then by base change $U_I/U_{i-1} \cong U_i'/U_{i-1}' \otimes_{S'} S$ is a clean module.

Example 2.4. Let $I = (x_1^2, x_1x_2^2x_3, x_1x_3^2, x_2^2x_4^2, x_2x_3^2x_4) \subset S = K[x_1, \ldots, x_4]$. We have

$$I = (x_1, x_2) \cap (x_1, x_4) \cap (x_1^2, x_2^2, x_3^2) \cap (x_1^2, x_1x_3, x_3^2, x_4^2)$$

and so $\mathrm{Ass}(S/I) = \{(x_1, x_2), (x_1, x_4), (x_1, x_2, x_3), (x_1, x_3, x_4)\}$. An algorithm to find a monomial primary decomposition is given in [10]. Then S/I is pretty clean by Theorem 2.3.

In [4] is given an example of an R-module M which is not pretty clean but has a prime filtration $\mathcal{F}$ with $\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}_R(M)$. The following shows that there are R-modules M for which there exist no prime filtration $\mathcal{F}$ with $\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}_R(M)$. the modules which have a filtration $\mathcal{F}$ with $\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}_R(M)$ are studied in different papers (see e.g. [6]) after Eisenbud's question from [3].

Example 2.5. Let $S = K[x_1, \ldots, x_4]$, $I = P_1 \cap P_2$ for $P_1 = (x_1, x_2)$, $P_2 = (x_3, x_4)$ and $M = S/I$. Suppose that there exists a filtration $\mathcal{F}$ of M such that $\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}_S(M)$. Then $\mathcal{F}$ is a clean filtration of M because $\mathrm{Ass}_S(M) = \mathrm{Min}(M) = \{P_1, P_2\}$. Note that P_i could appear in $\mathcal{F}$ only of $1 = \ell_{S_{P_i}}(M_{P_i})$-times. Thus $\mathcal{F}$ has the form $(0) \subset N \subset M$, where we may suppose that $M/N \cong S/P_1$ and $N \cong S/P_2$. But this is not possible because then $N = P_1/I$ is not cyclic. Thus M has no filtration $\mathcal{F}$ with $\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}_S(M)$. Note that the hypothesis of Theorem 2.3 do not hold in this frame.

3 Multicomplexes

The following proposition is stated in the proof of [4, Proposition (10.1)] using the standard pairs of [9]. We think that this result deserves a direct proof which we give bellow.

Proposition 3.1. *Let Γ be a multicomplex. The arithmetic degree of $S/I(\Gamma)$ is exactly the number of the facets $\mathcal{F}(\Gamma)$ of Γ.*

Proof. Let φ be the map from Γ to the monomial K-basis of $S/I(\Gamma)$, given by $u \to x^{\tilde{u}}$, where $\tilde{u}(j) = u(j)$ if $j \notin \mathrm{infpt}(u)$ and $\tilde{u}(j) = 0$ otherwise. Let $P \in \mathrm{Ass}(S/I(\Gamma))$ and $\mathcal{F}(\Gamma, P) = \{u \in \mathcal{F}(\Gamma) : P_u = P\}$. We claim that the restriction of φ to $\mathcal{F}(\Gamma, P)$ is injective. Indeed, suppose that we have $\tilde{u} = \tilde{v}$ for some $u, v \in \mathcal{F}(\Gamma, P)$. We have $\mathrm{infpt}(u) = \mathrm{infpt}(v)$ since $P_u = P_v = P$ and so it follows $u = v$.

Now let $u < v$ be two faces of Γ. Then $\mathrm{infpt}(u) \subset \mathrm{infpt}(v)$ and we have equality if and only if $(I(\Gamma(v)) : x^{\tilde{u}})$ is a P_u-primary ideal. Indeed if $\mathrm{infpt}(u) = \mathrm{infpt}(v)$ then $(I(\Gamma(v)) : x^{\tilde{u}}) = (\{x_k^{\tilde{v}(k) - \tilde{u}(k) + 1} : k \notin \mathrm{infpt}(u)\})$, that is a P_u-primary ideal. Conversely, for every $k \notin \mathrm{infpt}(u)$ suppose that a power $x_k^{\alpha_k} \in (I(\Gamma(v)) : x^{\tilde{u}})$. Thus $x_k^{\alpha_k} x^{\tilde{u}} \in I(\Gamma(v))$, that is $\tilde{u} + \alpha_k \varepsilon_k \notin \Gamma(v)$, where ε_k is the k-unitary vector. Then $u(k) + \alpha_k > v(k)$, that is $k \notin \mathrm{infpt}(v)$.

We claim that given a face $u \in \Gamma$ it holds $u \in \mathcal{F}(\Gamma)$ if and only if $\varphi(u) \in H^0_{P_u S_{P_u}}((S/I(\Gamma))_{P_u})$. Indeed, let u be a facet, then for each maximal facet v

with $u \leq v$ it holds $\mathrm{infpt}(u) = \mathrm{infpt}(v)$. This happens if $(I(\Gamma(v)) : x^{\tilde{u}})$ is a P_u-primary ideal for any maximal facet $v \geq u$. It follows

$$\bigcap_{v \in \mathcal{M}(\Gamma), u \leq v} (I(\Gamma(v)) : x^{\tilde{u}})$$

is a P_u-primary ideal. The converse is also true because

$$P_u = \sqrt{(\bigcap_{v \in \mathcal{M}(\Gamma), u \leq v} (I(\Gamma(v)) : x^{\tilde{u}})} \subset \sqrt{I(\Gamma(v)) : x^{\tilde{u}})} = P_v,$$

that is $\mathrm{infpt}(u) = \mathrm{infpt}(v)$. So u is a facet if and only if $(I(\Gamma)S_{P_u} : x^{\tilde{u}})$ is $P_u S_{P_u}$-primary ideal and it follows that there exist exactly $\dim(H^0_{PS_P}((S/I(\Gamma))_P)$-facets u in Γ with $P_u = P$.

Let $\Gamma \subset \mathbb{N}^n_\infty$ be a multicomplex and $a, b \in \Gamma$. We call a a *lower neighbour* of b if there exists an integer k, $1 \leq k \leq n$ such that

(i) $a(i) = b(i)$ for $i \neq k$,

(ii) either $a(k) + 1 = b(k) < \infty$, or $a(k) < \infty$ and $b(k) = \infty$.

It is easy to see that the multicomplex Γ is *shellable* if the facets of Γ can be ordered $a_1, \ldots, a_r$ such that

1. $a_1 \in \{0, \infty\}^n$,
2. for $i = 2, \ldots, r$ the maximal facets of $\langle a_1, \ldots, a_{i-1} \rangle \cap \langle a_i \rangle$ are lower neighbours of a_i;
3. for each $k \notin \mathrm{supp}\ a_i$, $1 \leq k \leq n$ such that $a_i(k) > 0$ there exists a maximal facet w of $\Gamma(a_1, \ldots, a_{i-1}) \cap \Gamma(a_i)$ such that $w(k) < a_i(k)$.
4. for all $1 \leq j < i \leq r$ such that $\mathrm{supp}\, a_j \subset \mathrm{supp}\, a_i$, it follows that $\mathrm{supp}\, a_j = \mathrm{supp}\, a_i$.

Actually Γ satisfies (a) above for any $i > 1$ if and only if it satisfies (2),(3) above, and it satisfies (a) for $i = 1$ if and only if (1) holds. Also Γ satisfies (b) above if and only if it satisfies (4) above. There are orders of the facets of some shellable multicomplexes which satisfy (1)-(3) but not (4) as shows the following:

Example 3.2. Let $a = (\infty, 0, \infty, \infty)$, $b = (1, 1, \infty, 0)$,$c = (0, 2, \infty, \infty)$ and $\Gamma = \langle a, b, c\rangle$. Then $\mathcal{F}(\Gamma) = \{a, b, c, (o, 1, \infty, \infty)\}$. We may order these facets in the following way $u_1 = a$,$u_2 = (0, 1, \infty, \infty)$, $u_3 = b$, $u_4 = c$. Note that $\langle u_1\rangle \cap \langle u_2\rangle$ has just one maximal facet $(0, 0, \infty, \infty)$ which is a neighbour of u_2. Also $\langle u_1, u_2\rangle \cap \langle u_3\rangle$ has two maximal facets $(1, 0, \infty, 0)$, $(0, 1, \infty, 0)$, both being neighbours of u_3. Finally note that $\langle u_1, u_2, u_3\rangle \cap \langle u_4\rangle$ has just one maximal facet u_2 which is a neighbour of u_4. So it is easy to see that this order satisfies (1)-(3), but not (4) because $P_{u_3} = (x_1, x_2, x_4) \supset (x_1, x_2) = P_{u_4}$. Actually, Γ is shellable because of Theorem 2.3 or [4, Proposition 5.1].

Also there are shellable multicomplexes Γ which have shellings $u_1, \dots u_r$ for which the Stanley sets $S_i = \Gamma(a_i) \setminus \Gamma(a_1, \dots, a_{i-1})$ do not satisfy $\dim S_1 \geq \dim S_2 \geq \dots \geq \dim S_r$ as shows the following:

Example 3.3. Let $a = (0, \infty, 1, \infty)$, $b = (0, 0, 2, \infty)$, $c = (\infty, \infty, 1, 0)$ and $\Gamma = \langle a, b, c\rangle$. Γ has apart of the maximal faces a, b, c and the following facets: $d = (\infty, \infty, 0, 0)$, $e = (0, \infty, 0, \infty)$. Choose the order $u_1 = d$, $u_2 = e$, $u_3 = a$, $u_4 = b$, $u_5 = c$. Note that $\langle u_1\rangle \cap \langle u_2\rangle$ has just one maximal facet $(0, \infty, 0, 0)$ which is a neighbour of u_2 and $\langle u_1, u_2\rangle \cap \langle u_3\rangle$ has one maximal facet $(0, \infty, 0, \infty)$ a neighbour of u_3. Also note that $\langle u_1, u_2, u_3\rangle \cap \langle u_4\rangle$ has just one maximal facet $(0, 0, 1, \infty)$ which is a neighbour of u_4 and $\langle u_1, u_2, u_3, u_4\rangle \cap \langle u_5\rangle$ has two maximal facets $(\infty, \infty, 0, 0)$, $(0, \infty, 1, 0)$ both being neighbours of u_5. We have $P_{u_1} = (x_3, x_4) = P_{u_5}$, $P_{u_2} = (x_1, x_3) = P_{u_3}$, $P_{u_4} = (x_1, x_2, x_3)$. Since $\dim P_{u_5} > \dim P_{u_4}$ the filtration does not satisfy the dimension condition above though it is pretty clean.

Let $\Gamma \subset \mathbb{N}^n_\infty$ be a multicomplex and $u_1, \dots, u_r$ its maximal facets so $\Gamma = \langle u_1, \dots, u_r\rangle$.

Lemma 3.4. *If* $\operatorname{infpt} u_j = \operatorname{infpt} u_1$ *for all* $j \geq 1$, *that is* $I(\Gamma)$ *is primary ideal, then* Γ *is shellable and* $S/I(\Gamma)$ *is clean.*

Proof. This lemma is a consequence of Theorem 2.3 or of [4, Proposition 5.1] but we prefer to give here the proof since it is elementary. Note that a monomial primary ideal Q can be seen as the extension of a primary ideal Q' associated to a maximal ideal in a polynomial ring S' in fewer variables which enter really in the generators of Q. Then S'/Q' is a clean module and so by base change S/Q is too. $\square$

Γ is *maximal shellable* if the maximal facets of Γ can be ordered $u_1, \dots, u_r$ and there exists s, $1 \leq s \leq r$ such that

1. $\operatorname{infpt} u_1 = \operatorname{infpt} u_j$, for all $1 \leq j \leq s$

2. for $i = s+1, \dots, r$ the maximal facets of $\langle u_1, \dots, u_{i-1}\rangle \cap \langle u_i\rangle$ differ from u_i, only in one component,

3. for all $s \le j < i \le r$ such that infpt $u_j \subset$ infpt u_i it follows infpt $u_j =$ infpt u_i.

Suppose Γ satisfies the above conditions, so it is maximal shellable. Fix an $i = s+1, \dots r$ and let $w_{i1}, \dots, w_{ic}$ be the maximal facets of $\Gamma(u_1, \dots, u_{i-1}) \cap \Gamma(u_i)$. Thus for each $1 \le j \le c$ there exists just one λ_j, $1 \le \lambda_j \le n$ such that $w_{ij}(\lambda_j) < u_i(\lambda_j)$ and so $w_{ij}(\lambda_j) \in \mathbb{N}$. Set $f_i = \Pi_{j=1, w_{ij}(\lambda_j)<\infty}^{c} x_{\lambda_j}^{w_{ij}(\lambda_j)+1}$. We claim that

$$\bigcap_{s=1}^{i-1} I(\Gamma(u_s)) + I(\Gamma(u_i)) = I(\Gamma(u_i)) + (f_i).$$

The monomial f_i has the form $f_i = x^{t_i}$ for some $t_i \in \mathbb{N}^n$. By definition of f_i we have $t_i \le u_i$, that is $t_i \in \Gamma(u_i)$ and $t_i \notin \Gamma(w_{ij})$ for all $1 \le j \le c$. Thus $t_i \in \Gamma(u_i) \setminus \Gamma(w_{i1}, \dots, w_{ic})$. Since $\Gamma(w_{i1}, \dots, w_{ic}) = \Gamma(u_1, \dots, u_{i-1}) \cap \Gamma(u_i)$ it follows $t_i \notin \Gamma(u_1, \dots, u_{i-1})$, that is

$$f_i = x^{t_i} \in I(\Gamma(u_1, \dots, u_{i-1})) = \bigcap_{s=1}^{i-1} I(\Gamma(u_s)).$$

Conversely, let $x^q \in I(\Gamma(u_1, \dots, u_{i-1})) \setminus I(\Gamma(u_i))$, that is $q \notin \Gamma(u_1, \dots, u_{i-1})$ and $q \in \Gamma(u_i)$. Then $q \notin \Gamma(w_{i1}, \dots, w_{ic}) = \cup_{j=1}^{c} \Gamma(w_{ij})$ and $q \le u_i$. Thus $u_i(k) \ge q(k) > w_{ij}(k)$ for at least one k and so $k = \lambda_j$ and it follows that $x^q \in (f_i)$.

Set $a_i = deg\ f_i$ for $i > 1$ and $a_1 = 0$. We obtain the following isomorphisms of grade S-modules

$$(\bigcap_{s=1}^{i-1} I(\Gamma(u_s)))/(\bigcap_{s=1}^{i} I(\Gamma(u_s))) \cong (\bigcap_{s=1}^{i-1} I(\Gamma(u_s))) + I(\Gamma(u_i)))/I(\Gamma(u_i)) =$$

$$(I(\Gamma(u_i)) + (f_i))/I(\Gamma(u_i)) \cong (f_i)/f_i I(\Gamma(u_i)) \cong S/(I(\Gamma(u_i)) : f_i)(-a_i).$$

By construction of f_i we see that $x_{\lambda_j}^{u_i(\lambda_j)-w_{ij}(\lambda_j)} f_i \in I(\Gamma(u_i))$ if $\lambda_j \notin$ infpt u_i. If $0 \le u_i(k) < \infty$ and k is not a λ_j then x_k does not enter in f_i and so x_k enters in $(I(\Gamma(u_i)) : f_i)$ only at the power he had in $I(\Gamma(u_i))$. However we see that $(I(\Gamma) : f_i)$ is a irreducible P_{u_i}-primary ideal. Finally note that the condition (3) says that for all $i > j$ such that $P_{u_i} \subset P_{u_j}$ it follows $P_{u_i} = P_{u_j}$. Thus we have shown:

Proposition 3.5. *Let $\Gamma \subset \mathbb{N}^n_\infty$ be a maximal shellable multicomplex and $u_1, \ldots, u_r$ its shelling. Then there exists a filtration of $S/I(\Gamma)$*
$(0) = M_0 \subset M_1 \subset \cdots\cdots M_{r-s} \subset M_{r-s+1} = S/I(\Gamma)$ *with*

$$M_i = \bigcap_{j=1}^{r-i} I(\Gamma(u_j)) \quad \textit{and} \quad M_i/M_{i-1} \cong S/J_i(-a_{r-i+1})$$

for some irreducible ideals J_i associated to $P_{u_{r+1-i}}$, for $i \le r-s$ and the primary ideal $J_{r-s+1} = I(\Gamma(u_1, \ldots, u_s))$.

Theorem 3.6. *If the multicomplex $\Gamma \subset \mathbb{N}^n_\infty$ is maximal shellable then Γ is shellable in particular $S/I(\Gamma)$ is a pretty clean ring.*

Proof. By the above proposition it is enough to see that that the factors from the above filtration are clean. This follows by the Lemma 3.4.

We end this section with some examples.

Example 3.7. Let $J = (x_1^2, x_2^2, x_3, x_4) \cap (x_1, x_2, x_3^2, x_4^2)$ and $T = (x_1, x_2, x_5^2, x_6^2)$ be primary ideals in $S = K[x_1, \ldots, x_6]$. Set $I = J \cap T$. Then S/I is not Cohen-Macaulay since from the exact sequence

$$0 \to S/I \to S/J \oplus S/T \to S/J+T \to 0$$

we get $\operatorname{depth}(S/I) = 1$. It follows that S/I is not pretty clean too and so $\Gamma(I)$ cannot be maximal shellable. This is indeed the case since one can take $s = 2, r = 3$, $u_1 = (1,1,0,0,\infty,\infty)$, $u_2 = (0,0,1,1,\infty,\infty)$ and $u_3 = (0,0,\infty,\infty,1,1)$. For this order one can see that $\Gamma(u_1, u_2, u_3)$ is not maximal shellable since the condition (2) does not hold.

The following example shows that there exist shellable multicomplexes which are not maximal shellable.

Example 3.8. Let $J = (x_1^2, x_2^2, x_3, x_4) \cap (x_1, x_2, x_3^2, x_4^2)$ and $L = (x_1^2, x_2, x_3, x_5^2)$ be primary ideals in $S = K[x_1, \ldots, x_5]$. Set $I = J \cap L$. Then S/I is Cohen-Macaulay but $\Gamma(I)$ is not maximal shellable. This is indeed the case since one can take $s = 2, r = 3$, $u_1 = (1,1,0,0,\infty)$, $u_2 = (0,0,1,1,\infty)$ and $u_3 = (1,0,0,\infty,1)$. The maximal facets of $\Gamma(u_1, u_2) \cap \Gamma(u_3)$ are $w_1 = (1,0,0,0,1)$ and $w_2 = (0,0,0,1,1)$. Clearly w_1 satisfies the condition (2) but w_2 not. Also note that $J = I + Sx_4^2$ and so the factors of the filtration $(0) \subset J/I \subset S/I$ are all cyclic given by primary ideals. By Lemma 3.4 we see that S/I is pretty clean and so shellable.

Example 3.9. Let $J = (x_1^2, x_2^2, x_3, x_4) \cap (x_1, x_2, x_3^2, x_4^2)$ and $T = (x_1, x_2, x_3, x_5^2)$ be primary ideals in $S = K[x_1, \ldots, x_5]$. Set $I = J \cap T$. Then S/I is Cohen-Macaulay because $\Gamma(I)$ is maximal shellable. This is indeed the case since one can take $s = 2, r = 3$, $u_1 = (1, 1, 0, 0, \infty)$, $u_2 = (0, 0, 1, 1, \infty)$ and $u_3 = (0, 0, 0, \infty, 1)$. The only maximal facet of $\Gamma(u_1, u_2) \cap \Gamma(u_3)$ is $w = (0, 0, 0, 1, 1)$. Clearly w satisfies the condition (2).

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