An. Şt. Univ. Ovidius Constanţa Vol. 14(2), 2006, 85–96

A REMARK ON THE HILBERT SERIES OF TRANSVERSAL POLYMATROIDS

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Abstract

In this note we study when the transversal polymatroids presented by $\mathbf{A} = \{A_1, A_2, \ldots, A_m\}$, where all the sets A_i have two elements, have the base ring B_m Gorenstein. Using Worpitzky identity, we show that the numerator of the Hilbert series has the coefficients Eulerian numbers and, from [1], the Hilbert series is unimodal.

1 Introduction

Let K be an infinite field, n and m be positive integers, A_i be some subsets of $[n]$ for $1 \leq i \leq m$, $\mathbf{A} = \{A_1, A_2, \ldots, A_m\}$. Let

$$B_m = K[x_{i_1} x_{i_2} \ldots x_{i_m} : i_j \in A_j, 1 \leq j \leq m]$$

and

$$C = K[x_i y_j : i \in A_j, 1 \leq j \leq m].$$

Obviously $C \subseteq S$, where S is the Segre product of the polynomial rings in n, respectively m, indeterminates,

$$S := K[x_1, x_2, \ldots, x_n] * K[y_1, y_2, \ldots, y_m] = K[x_i y_j : 1 \leq i \leq n, 1 \leq j \leq m].$$

We consider the variables t_{ij}, $1 \leq i \leq n$, $1 \leq j \leq m$ and we define

$$T = K[t_{ij} : 1 \leq i \leq n, 1 \leq j \leq m],$$

$$T(\mathbf{A}) = K[t_{ij} : 1 \leq j \leq m, i \in A_j]$$

Key Words: polymatroids, base ring, Hilbert series
Mathematical Reviews subject classification: 13A02
Received: March, 2006

and the presentations $\phi : T \longrightarrow S$ and $\phi^{'} : T(\mathbf{A}) \longrightarrow C$ defined by $t_{ij} \longrightarrow x_i y_j$.

By [10, Proposition 9.1.2] we know that $\ker(\phi)$ is the ideal $I_2(t)$ of the 2–minors of the $n \times m$ matrix $t = (t_{ij})$ via the map ϕ. The algebras C, $T(\mathbf{A})$, S and T are $\mathbf{Z}^m$ –graded by setting $\deg(x_i y_j) = \deg(t_{ij}) = e_i \in \mathbf{Z}^m$ where e_i, $1 \leq i \leq m$ denote the vectors of the canonical basis of $\mathbf{R}^m$.

By [11, Propositions 4.11 and 8.11] or [10, Proposition 8.1.10] we know that the cycles of the complete bipartite graph $K_{n,m}$ give a universal Gröbner basis of $I_2(t)$.

A cycle of the complete bipartite graph is described by a pair (I, J) of sequences of integers, say

$$I = i_1, i_2, \ldots, i_s,\ \ J = j_1, j_2, \ldots, j_s,$$

with $2 \leq s \leq min(m, n)$, $1 \leq i_k \leq m$, $1 \leq j_k \leq n$ and such that the i_k are distinct and the j_k are distinct. Associated with any such a pair we have a polynomial $F_{(I,J)} = t_{i_1 j_1} \ldots t_{i_s j_s} - t_{i_2 j_1} \ldots t_{i_s j_{s-1}} t_{i_1 j_s}$ which is in $I_2(t)$.

For a $\mathbf{Z}^m$-graded algebra E we denote by E_Δ the direct sum of the graded components of degree $(a, a, \ldots, a) \in \mathbf{Z}^m$. Similarly, for a $\mathbf{Z}^m$–graded E–module M, we denote by M_Δ the direct sum of the graded components of M of degree $(a, a, \ldots, a) \in \mathbf{Z}^m$. Clearly E_Δ is a $\mathbf{Z}$–graded algebra and M_Δ is a $\mathbf{Z}$–graded E_Δ module. Furthermore $-_\Delta$ is exact as a functor on the category of $\mathbf{Z}^m$–graded E–modules with maps of degree 0. Now C_Δ is the K-algebra generated by the elements $x_{i_1} y_1 \ldots x_{i_m} y_m$ with $i_j \in A_j$. Therefore B_m is isomorphic to the algebra C_Δ. Hence we obtain a presentation :

$$0 \longrightarrow J \longrightarrow T(\mathbf{A})_\Delta \longrightarrow\ B_m \longrightarrow 0,$$

where $J = (I_2(t) \bigcap T(\mathbf{A}))_\Delta$.

$T(\mathbf{A})_\Delta$ is the K–algebra generated by the monomials $t_{1i_1} t_{2i_2} \ldots t_{mi_m}$, with $i_k \in A_k$, that is, $T(\mathbf{A})_\Delta$ is the Segre product $T_1 * T_2 * \ldots * T_m$ of the polynomial rings $T_i = K[t_{ij} : j \in A_i]$. Now we consider the variables s_α with $\alpha \in A := A_1 \times A_2 \times \ldots \times A_m$ Then we get the presentation of the Segre product $T(\mathbf{A})_\Delta$ as a quotient of $K[A]$ by mapping $s_{(j_1,\ldots,j_m)}$ to $t_{1j_1} t_{2j_2} \ldots t_{mj_m}$.

From [5] the defining ideal of $T(\mathbf{A})_\Delta$ is generated by the so-called Hibi relations:

$$s_\alpha s_\beta - s_{(\alpha \vee \beta)} s_{\alpha \wedge \beta)},$$

where

$$\alpha \vee \beta = (\max(\alpha_1, \beta_1), \ldots, \max(\alpha_m, \beta_m)),$$

and

$$\alpha \wedge \beta = (\min(\alpha_1, \beta_1), \ldots, \min(\alpha_m, \beta_m)).$$

Example 1.1. Let $n = 3$ and

$$A_1 = \{1, 2\}, A_2 = \{2, 3\}, A_3 = \{3, 4\}.$$

Then C is the quotient of $K[t_{11}, t_{12}, t_{22}, t_{23}, t_{33}, t_{34}]$ by the zero ideal ($J = 0$ because we don't have cycles). Then

$$B_3 = K[x_1x_2x_3, x_1x_2x_4, x_1x_3^2, x_1x_3x_4, x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4]$$

is the quotient of $K[s_{123}, s_{124}, s_{133}, s_{134}, s_{223}, s_{224}, s_{233}, s_{234}]$ modulo the ideal generated by the Hibi relations:

$$\begin{gathered} s_{123}s_{134} - s_{124}s_{133}\ ,\ s_{123}s_{224} - s_{124}s_{223}, \\ s_{123}s_{234} - s_{124}s_{233}\ ,\ s_{123}s_{233} - s_{133}s_{223}, \\ s_{123}s_{234} - s_{133}s_{224}\ ,\ s_{123}s_{234} - s_{134}s_{223}, \\ s_{124}s_{234} - s_{134}s_{224}\ ,\ s_{133}s_{234} - s_{134}s_{233}, \\ s_{223}s_{234} - s_{224}s_{233}. \end{gathered}$$

Since $K[t_{11}, t_{12}] * K[t_{22}, t_{23}] * K[t_{33}, t_{34}]$ is a Gorenstein ring ([6, Example 7.4]) then B_3 is a Gorenstein ring .

Example 1.2. Let $n = 3$ and

$$A_1 = \{1, 2\}, A_2 = \{2, 3\}, A_3 = \{3, 1\}.$$

Then C is the quotient of $K[t_{11}, t_{12}, t_{22}, t_{23}, t_{33}, t_{31}]$ modulo the ideal generated by the polynomial $t_{11}t_{22}t_{33} - t_{12}t_{23}t_{31}$ (we have one 6-cycles). Then

$$B_3 = K[x_1x_2x_3, x_1^2x_2, x_1x_3^2, x_1^2x_3, x_2^2x_3, x_1x_2^2, x_2x_3^2]$$

is the quotient of $K[s_{123}, s_{121}, s_{133}, s_{131}, s_{223}, s_{221}, s_{233}, s_{231}]$ modulo the ideal generated by the Hibi relations:

$$\begin{gathered} s_{221}s_{233} - s_{223}s_{231},\ s_{131}s_{233} - s_{133}s_{231}, \\ s_{121}s_{233} - s_{123}s_{231},\ s_{131}s_{221} - s_{121}s_{231}, \\ s_{133}s_{221} - s_{123}s_{231},\ s_{131}s_{223} - s_{123}s_{231}, \\ s_{133}s_{223} - s_{233}s_{231},\ s_{121}s_{223} - s_{221}s_{231}, \\ s_{121}s_{133} - s_{131}s_{231}. \end{gathered}$$

and by the linear relation

$$s_{123} - s_{231}$$

Since $K[t_{11}, t_{12}] * K[t_{22}, t_{23}] * K[t_{33}, t_{31}]$ is a Gorenstein ring and $t_{11}t_{22}t_{33} - t_{12}t_{23}t_{31}$ is a regular element in $K[t_{11}, t_{12}] * K[t_{22}, t_{23}] * K[t_{33}, t_{31}]$ then

$$\frac{K[t_{11}, t_{12}] * K[t_{22}, t_{23}] * K[t_{33}, t_{31}]}{(t_{11}t_{22}t_{33} - t_{12}t_{23}t_{31})} \cong B_3$$

is a Gorenstein ring.

Example 1.3. Let $n = 3$ and

$$A_1 = \{1,2\}, A_2 = \{1,2,3\}, A_3 = \{2,3\}.$$

Then C is the quotient of $K[t_{11}, t_{12}, t_{12}t_{22}, t_{23}, t_{23}, t_{33}]$ modulo the ideal generated by the polynomials $t_{11}t_{22} - t_{12}t_{21}$, $t_{22}t_{33} - t_{23}t_{32}$, $t_{11}t_{23}t_{32} - t_{12}t_{21}t_{33}$ (we have two 4-cycles and one 6-cycle),

$$H_C(t) = \frac{1+2t+t^2}{(1-t)^5},$$

and then

$$B_3 = K[x_1^2x_2, x_1^2x_3, x_1x_2x_3, x_1x_2^2, x_1x_3^2, x_2^3, x_2^2x_3, x_2x_3^2]$$

is the quotient of $K[s_{ijk}|(i,j,k) \in \{1,2\} \times \{1,2,3\} \times \{2,3\}]$ modulo the ideal generated by the Hibi relations:

$$\begin{array}{ll}
s_{222}s_{233} - s_{232}s_{223} \ , & s_{212}s_{233} - s_{213}s_{232} \ , \\
s_{212}s_{232} - s_{213}s_{222} \ , & s_{133}s_{232} - s_{213}s_{233} \ , \\
s_{133}s_{222} - s_{213}s_{232} \ , & s_{133}s_{212} - s_{213}s_{132} \ , \\
s_{113}s_{233} - s_{133}s_{213} \ , & s_{113}s_{232} - s_{213}s_{123} \ , \\
s_{113}s_{222} - s_{212}s_{213} \ , & s_{112}s_{233} - s_{213}s_{132} \ , \\
s_{112}s_{232} - s_{212}s_{213} \ , & s_{112}s_{222} - s_{212}s_{122} \ , \\
s_{112}s_{213} - s_{113}s_{212} \ , & s_{112}s_{133} - s_{113}s_{213} \ .
\end{array}$$

and by the linear relations $s_{132} - s_{213}, s_{123} - s_{213}, s_{122} - s_{212}, s_{223} - s_{232}$.

Since B_3 is a domain and the Hilbert series of B_3 is

$$H_{B_3}(t) = \frac{1+5t+t^2}{(1-t)^3},$$

then B_3 is a Gorenstein ring .

2 Hilbert series

Definition 2.1. Let $R = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field K. If M is a finitely generated $\mathbf{N}$-graded R-module, the numerical function:

$$H(M,-) : \mathbf{N} \longrightarrow \mathbf{N}$$

with $H(M,n) = dim_K(M_n)$, for all $n \in \mathbf{N}$, is the Hilbert function and $H_M(t) = \sum_{n \in \mathbf{N}} H(M,n)t^n$ is the Hilbert series of M.

Let n, m be positive integers, A_i be some subsets of $[n]$ such that $|A_i| = l$ for $1 \le i \le m$, $\mathbf{A} = \{A_1, A_2, \ldots, A_m\}$.

Let

$$B_m = K[x_{i_1} x_{i_2} \ldots x_{i_m} : i_j \in A_j, 1 \le j \le m]$$

and

$$C = K[x_i y_j : i \in A_j, 1 \le j \le m].$$

From Section 1 we know that B_m is isomorphic to the algebra C_Δ and we have the presentation :

$$0 \longrightarrow J \longrightarrow T(\mathbf{A})_\Delta \longrightarrow\ B_m \longrightarrow 0,$$

where $J = (I_2(t) \bigcap T(\mathbf{A}))_\Delta$.

Now we are interested on the case when $J = (0)$.

Remark 2.2. If $J = (0)$ then B_m is isomorphic to the algebra $(T(\mathbf{A}))_\Delta$.

$J = (0)$ is equivalent with the fact that the bipartite graph presented by $\mathbf{A}$ ($\mathrm{V}_1 = 1, 2, \ldots, m, V_2 = A_1 \cup A_2 \cup, \ldots, \cup A_m$ and an edge from V_1 to V_2 joins $i \in\ V_1$ with $i_j \in\ V_2$ if and only if $i_j \in A_i$) does not have cycles.

If $|A_i| = l$, for $1 \le i \le m$, $|A_i \bigcap A_{i+1}| \le 1$, and $A_j \bigcap A_i = \varnothing$, for $2 \le i \le m$, $j < i - 1$, then the bipartite graph presented by $\mathbf{A}$ does not have cycles, thus the ideal J is zero.

Since $J = (0)$, then B_m is the Segre product of m polynomial rings, each of them in l indeterminates, that is, B_m is a Gorenstein ring (see [6, Example 7.4]); $\dim_K (B_m)_i = \binom{\imath + l - 1}{i}^m$.

In the case $m = 2$ it is known (see [10, proposition 9.1.3]) that the Hilbert series of B_2 is

$$H_{B_2}(t) = \frac{\sum_{k=0}^{l-1} \binom{l-1}{k}^2 t^k}{(1-t)^{2l-1}};\ H(B_2, i) = \dim_k (B_2)_i = \binom{\imath + l - 1}{i}^2.$$

It results that the Krull dimension of B_2 is $\dim_k B_2 = 2l - 1$ and the number of generators of the defining ideal of B_2 (the number of Hibi-relations of B_2) is

$$\mu = \binom{H(B_2, 1) + 1}{2} - H(B_2, 2) = \binom{l^2 + 1}{2} - \binom{l+1}{2}^2 = \binom{l}{2}^2.$$

Remark 2.3. We have the following relation between the Hilbert series of B_{m+1} and B_m :

$$H_{B_{m+1}}(t) = \frac{1}{(l-1)!} \frac{d^{(l-1)}}{dt^{l-1}} (t^{l-1} H_{B_m}(t)).$$

Proof. Since $H_{B_m}(t) = \sum_{i\geq 0}\binom{i+l-1}{i}^m t^i$, then

$$\begin{aligned}
&\frac{1}{(l-1)!}\frac{d^{(l-1)}}{dt^{l-1}}(t^{l-1}H_{B_m}(t)) = \frac{1}{(l-1)!}\frac{d^{(l-1)}}{dt^{l-1}}(t^{l-1}\sum_{i\geq 0}\binom{i+l-1}{i}^m t^i)\\
&= \frac{1}{(l-1)!}\frac{d^{(l-2)}}{dt^{l-2}}(\frac{d}{dt}(t^{l-1}\sum_{i\geq 0}\binom{i+l-1}{i}^m t^i))\\
&= \frac{1}{(l-1)!}\frac{d^{(l-2)}}{dt^{l-2}}((l-1)t^{l-2}\sum_{i\geq 0}\binom{i+l-1}{i}^m t^i + t^{l-2}\sum_{i\geq 0} i\binom{i+l-1}{i}^m t^i)\\
&= \frac{1}{(l-1)!}\frac{d^{(l-2)}}{dt^{l-2}}(\sum_{i\geq 0}\binom{i+l-1}{i}^m (i+l-1)t^i)\\
&= \frac{1}{(l-1)!}\frac{d^{(l-3)}}{dt^{l-3}}(\frac{d}{dt}(t^{l-2}\sum_{i\geq 0}\binom{i+l-1}{i}^m (i+l-1)t^i))\\
&= \frac{1}{(l-1)!}\frac{d^{(l-3)}}{dt^{l-3}}((l-2)t^{l-3}\sum_{i\geq 0}\binom{i+l-1}{i}^m (i+l-1)t^i +\\
&+t^{l-3}\sum_{i\geq 0} i\binom{i+l-1}{i}^m (i+l-1)t^i))\\
&= \frac{1}{(l-1)!}\frac{d^{(l-3)}}{dt^{l-3}}(t^{l-3}\sum_{i\geq 0}\binom{i+l-1}{i}^m (i+l-1)(i+l-2)t^i) = \cdots\\
&= \frac{1}{(l-1)!}\sum_{i\geq 0}\binom{i+l-1}{i}^m (i+l-1)(i+l-2)\cdots(i+2)(i+1)t^i\\
&= \sum_{i\geq 0}\binom{i+l-1}{i}^{m+1} t^i = \; H_{B_{m+1}}(t).
\end{aligned}$$

□

Let $A(t) := \sum_i a_i t^i$ and $B(t) := \sum_i b_i t^i$ be two power series in $\mathbf{Z}$[[t]]. Then we denote by $Had(A,B) := \sum_i (a_i b_i)t^i$ the Hadamard product of A and B ([8]).

Definition 2.4. ([8]) Let $A(t)$ be the Hilbert series of a standard $k-$algebra S. $ri(A)$ (or $ri(S)$) is the regularity index of A (or of S), i.e. the first integer r such that for every $s \geq r$ the Hilbert function of S takes the same values as the Hilbert polynomial of S.

Remark 2.5. $ri(S) = a(S) + 1$, where $a(S)$ is the $a-$invariant of S.

Proposition 2.6 ([8]). *Let* $A(t) := \frac{P(t)}{(1-t)^a}$ *and* $B(t) := \frac{Q(t)}{(1-t)^b}$, *where* $p := deg(P)$, $q := deg(Q)$, $P(1) \neq 0$, $Q(1) \neq 0$, *and assume that* $A(t)$ *and* $B(t)$ *are the Hilbert series of standard* $k-$*algebras. Then*

1) $ri(A) = p - a + 1$ *and* $ri(B) = q - b + 1$;

2) $ri(Had(A, B)) \leq \max(ri(A), ri(B))$;

3) $Had(A, B) = \frac{R(t)}{(1-t)^{a+b-1}}$, *with* $R(1) \neq 0$;

4) $\deg(R) \leq \max(ri(A), ri(B)) + (a + b - 1) - 1$;

Theorem 2.7 ([8]). *Let* S_1 *and* S_2 *be two standard* $k-$*algebras with the Hilbert series* H_{S_1}, H_{S_2}. *Then the Hilbert series of Segre product of* S_1 *and* S_2 *is*

$$H_{S_1 * S_2} = Had(H_{S_1}, H_{S_2}).$$

Definition 2.8. Let $R = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field K and M be a finitely generated $\mathbf{N}$-graded R-module. The *difference operator* Δ on the set of numerical functions $H(M, -)$ is

$$(\Delta H(M, -))(n) = H(M, n + 1) - H(M, n),$$

where $H(M, -)$ is the Hilbert function.

The $m - times$ *iterated* Δ *operator* ("$m - difference\ of\ H(M, n)$") will be denoted by Δ^m.

Proposition 2.9. *If* $|A_i| = 2$, *for* $1 \leq i \leq m$, $|A_i \bigcap A_{i+1}| \leq 1$ *and* $A_j \bigcap A_i = \varnothing$, *for* $1 \leq i \leq m - 1$, $1 \leq j < i - 1$, *then the Hilbert series of* B_m *is*

$$H_{B_m}(t) = \frac{\sum_{k=0}^{m-1} A(m, k + 1)t^k}{(1 - t)^{m+1}},$$

where

$$A(m, k) = kA(m - 1, k) + (m - k + 1)A(m - 1, k - 1),$$

with $A(m, 1) = A(m, m) = 1$ *and* $2 \leq k \leq m - 1$.

Remark 2.10. The sequence in k , $A(m, k)$ with $1 \leq k \leq m$ is symmetric for any $m \geq 2$. Indeed, if $m = 2$ then $A(2, 1)$=$A(2, 2)$=1. If $m > 2$ then

$$A(m, k) = k\ A(m - 1, k) + (m - k + 1)\ A(m - 1, k - 1) =$$

$$= kA(m - 1, m - k) + (m - k + 1)A(m - 1, m - k + 1) = A(m, m - k + 1).$$

Proof. We know that $B_m = T_1 * T_2 * \ldots * T_m$, where $T_i = K[t_{ij} : j \in A_i]$ is Segre product of m polynomial rings in two indeterminates and $\dim_k(B_m)_i = \binom{i+2-1}{i}^m = (i+1)^m$.

We show that the Krull dimension of B_m is $\dim B_m = m+1$ and the Hilbert series of B_m, $H_{B_m}(t) = \frac{R(t)}{(1-t)^{m+1}}$, with $\deg(R) \leq m-1$.

We proceed by induction on m. *The case* $m=1$ is clear. Suppose $m \geq 2$. For every $1 \leq i \leq m$ we have $ri(H_{T_i}) = -1$, thus $ri(H_{B_m}) = -1$. Since $B_{m+1} = B_m * T_{m+1}$ we have

$$H_{B_{m+1}}(t) = Had(H_{B_m}, T_{m+1}) = \frac{R(t)}{(1-t)^{(m+1)+2-1}} = \frac{R(t)}{(1-t)^{m+2}};$$

$$\deg(R) \leq \max(ri(H_{B_m}), ri(H_{T_{m+1}})) + ((m+1)+2-1) - 1 = m.$$

If $R(t) := \sum_{k=0}^{m-1} r_i t^i$, then we need to find the coefficients $r_i's$. We may compute the first m values of $H(B_m, i)$. Then it suffices to take the $(m+1)^{th}$ difference of these first m values and we get the required $r_i's$. For this it suffices to go backward in the algorithm which determines the numerators of the Hilbert series and to obtain $H(B_m, i) = \dim_k(B)_i$ for all i.

We define

$$A_0(m,k) = r_k = A(m,k),$$
$$A_i(m,1) = 1, A_i(m,k) = A_i(m,k-1) + A_{i-1}(m,k),$$

for $i \geq 1$ and $2 \leq k \leq m$. For $m \geq 2$ and $2 \leq k \leq m$ fixed we want to prove that

$$A_t(m,k) = \sum_{s=1}^{k} A(m,s) \binom{t+k-s-1}{k-s}$$

for any $t \geq 1$ (with the convention that the binomial coefficient $\binom{m}{n}$ is zero if $m < n$).

We proceed by induction on t.

Case $t = 1$.

Since for any $m \geq 2$ and fixed $2 \leq k \leq m$, we have

$$\begin{aligned}
&A_1(m,k) = A_1(m,k-1) + A(m,k),\\
&A_1(m,k-1) = A_1(m,k-2) + A(m,k-1),\\
&A_1(m,k-2) = A_1(m,k-3) + A(m,k-2),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&A_1(m,3) = A_1(m,2) + A(m,3),\\
&A_1(m,2) = A_1(m,1) + A(m,2),\\
&A_1(m,1) = 1 = A(m,1),
\end{aligned}$$

we obtain:

$$A_1(m,k) = \sum_{s=1}^{k} A(m,s).$$

Case $t > 1$: From

$$\begin{aligned}
&A_t(m,k+1) = A_t(m,k) + A_{t-1}(m,k+1),\\
&A_{t-1}(m,k+1) = A_{t-1}(m,k) + A_{t-2}(m,k+1),\\
&A_{t-2}(m,k+1) = A_{t-2}(m,k) + A_{t-3}(m,k+1),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&A_3(m,k+1) = A_3(m,k) + A_2(m,k+1),\\
&A_2(m,k+1) = A_2(m,k) + A_1(m,k+1),\\
&A_1(m,k+1) = A_1(m,k) + A(m,k+1),
\end{aligned}$$

we obtain

$$A_t(m,k+1) = \sum_{j=1}^{t} A_j(m,k) + A(m,k+1).$$

For $t > 1$,

$$\begin{aligned}
A_t(m,k+1) &= \sum_{j=1}^{t} A_j(m,k) + A(m,k+1)\\
&= \sum_{j=1}^{t}(\sum_{s=1}^{k} A(m,s)\binom{j+k-s-1}{k-s}) + A(m,k+1)\\
&= \sum_{s=1}^{k}(\sum_{j=1}^{t}\binom{j+k-s-1}{k-s})A(m,s) + A(m,k+1)\\
&= \sum_{s=1}^{k}\binom{t+k-s}{k-s+1}A(m,s) + A(m,k+1)\\
&= \sum_{s=1}^{k+1} A(m,s)\binom{t+k-s}{k-s+1},
\end{aligned}$$

since

$$\sum_{j=1}^{t}\binom{j+k-s-1}{k-s} = \binom{t+k-s}{k-s+1}.$$

Now we want to prove that $A_{m+1}(m,k) = k^m$.

From [7] or [12] we mention the Worpitzky identity :

$$k^m = \sum_{s=1}^{m} A(m,s)\binom{k+s-1}{m}.$$

We know that

$$A_{m+1}(m,k) = \sum_{s=1}^{k} A(m,s)\binom{m+k-s}{k-s} = \sum_{s=1}^{k} A(m,s)\binom{m+k-s}{m}.$$

Thus

$$k^m = \sum_{s=1}^{m} A(m,s)\binom{k+s-1}{m} = A(m,m)\binom{k+m-1}{m} + A(m,m-1)$$
$$\binom{k+m-1-1}{m} + \ldots + A(m,m-k+2)\binom{m+1}{m} + A(m,m-k+1)\binom{m}{m}$$
$$= A(m,1)\binom{k+m-1}{m} + A(m,2)\binom{k+m-2}{m} + \ldots$$
$$\ldots + A(m,k-1)\binom{k+m-k+1}{m} + A(m,k)\binom{k+m-k}{m}$$
$$= \sum_{s=1}^{k} A(m,s)\binom{m+k-s}{m} = A_{m+1}(m,k).$$

Thus the $r_k = A(m,k+1)$ for $0 \le k \le m-1$.

$\square$

Example 2.11. *We compute the Hilbert series for*

$$\mathbf{A} = \{A_1 = \{1,2\}, A_2 = \{2,3\}, A_3 = \{3,4\}, A_4 = \{4,5\}\}.$$

k	1	2	3	4
$k^4 = A_5(4,k)$	1	16	81	256
$A_4(4,k)$	1	15	65	175
$A_3(4,k)$	1	14	50	110
$A_2(4,k)$	1	13	36	60
$A_1(4,k)$	1	12	23	24
$A_0(4,k)$	1	11	11	1

The last row of the table contains the coefficients of the numerator of $H_{B_4}(t)$. Thus the Hilbert series of B_4 is

$$H_{B_4}(t) = \frac{1 + 11t + 11t^2 + t^3}{(1-t)^5}.$$

Corollary 2.12. *The number of generators of the defining ideal of B_m (the number of Hibi-relations of B_m) is*

$$\mu = \binom{H(B_m,1)+1}{2} - H(B_m,2) = \binom{2^m+1}{2} - 3^m = 2^{2m-1} + 2^{m-1} - 3^m.$$

Corollary 2.13. *The h-vector of the Hilbert series associated to the transversal polymatroid presented by $\mathbf{A} = \{A_1, A_2, \ldots, A_m\}$, such that $|A_i| = 2$, for $1 \le i \le m$, $|A_i \cap A_{i+1}| \le 1$ and $A_j \bigcap A_i = \varnothing$, for $1 \le i \le m-1$, $1 \le j < i-1$, is unimodal .*

Proof. From [1], we know that $A(m,k)$ is log-concave sequence in k, for all m, thus is unimodal. $\square$

Acknowledgments. I would like to thank for financial support to University of Genova and CNCSIS program during one month visit to University of Genova. I am grateful to Professor Aldo Conca for helpful discussions and hospitality. Also I wish to express my thanks to Professor Dorin Popescu for support, helpful discussions and hints. This work was also partially supported by the CEEX Program of the Romanian Ministry of Education and Research, contract CEx05-D11-11/2005.

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