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DISCRETE POLYMATROIDS

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Abstract

In this article we give a survey on some recent developments about discrete polymatroids.

Introduction

This article is a survey on the results obtained on the discrete polymatroids, since they were introduced by Herzog and Hibi [7] in 2002. The discrete polymatroid is a multiset analogue of the matroid, closely related to the integral polymatroids. This paper is organized as follows. In Section 1, we give a brief introduction to matroids and polymatroids. In Section 2, the definition and basic combinatorial properties about discrete polymatroids are given, as well as the connection with matroids and integral polymatroids. Since checking whether a finite set is a discrete polymatroid is not easy in general, two techniques for the construction of discrete polymatroids are presented in Section 3.

In Section 4 and 5 we study algebra on discrete polymatroids. More precisely, to a discrete polymatroid P with its set of bases B, and for an arbitrary field K, one can associate the following algebraic objects: $K[B]$, *the base ring* and its toric ideal I_B, the homogeneous semigroup ring $K[P]$, and the polymatroidal ideal $I(B)$. In Section 4, we describe and compare the algebraic properties of $K[P]$ and $K[B]$. Three conjectures related to the base ring $K[B]$ are given, together with the particular case when these are all true. This particular class of discrete polymatroids is the one of discrete polymatroids which satisfy the *strong exchange property.* Since all three conjectures are true in the case of discrete polymatroids with strong exchange property, we study their combinatorial properties in Section 6. Finally, in Section 5 we study the properties of the polymatroidal ideals.

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1 Matroids and Polymatroids

In this section we give the definitions and few basic properties of matroids and polymatroids, together with some examples. The purpose of this section is to give to the reader an idea about this concepts and fix the notations for the rest of the paper. For a detailed material on matroids and polymatroids one can consult [20], [14].

First, we give the definition of a matroid, in the particular case when the finite ground set is $\{1,\ldots,n\}$. However, in the examples which follow the definition, we catch the more general case. For a survey on the current research problems in matroid theory we recommend the survey [15].

Fix an integer $n>0$ and set $[n]=\{1,2,\ldots,n\}$. $2^{[n]}$ is the set of all subsets of $[n]$. For a subset $F\subset[n]$ write $|F|$ for the cardinality of F. The following definition of the matroid is originated in Whitney (1935):

Definition 1.1. *A* matroid *on the ground set* $[n]$ *is a subset* $\mathcal{M}\subset 2^{[n]}$ *satisfying:*

(M1) $\emptyset\in\mathcal{M}$*;*

(M2) if $F_1\in\mathcal{M}$ *and* $F_2\subset F_1$*, then* $F_2\in\mathcal{M}$*;*

(M3) if F_1 *and* F_2 *belong to* $\mathcal{M}$ *and* $|F_1|<|F_2|$*, then there is* $x\in F_2\setminus F_1$ *such that* $F_1\cup\{x\}\in\mathcal{M}$*.*

In particular, if one adds in the definition that the matroid is a nonempty subset, then (M1) can be skipped since it follows from the condition (M2). The conditions (M1) and (M2) say together that $\mathcal{M}$ is an *abstract simplicial complex* on $[n]$. The members of $\mathcal{M}$ are the *independent sets* of $\mathcal{M}$. A *base* of $\mathcal{M}$ is a maximal independent set of $\mathcal{M}$. An easy consequence of (M3) is that any two bases have the same cardinality. The set of bases of $\mathcal{M}$ possesses the following *"exchange property"*:

(E) If B_1 and B_2 are bases of $\mathcal{M}$ and if $x\in B_1\setminus B_2$, then there is $y\in B_2\setminus B_1$ such that $(B_1\setminus\{x\})\cup\{y\}$ is a base of $\mathcal{M}$.

Moreover, the set of bases of $\mathcal{M}$ possesses the following *"symmetric exchange property"*:

(SE) If B_1 and B_2 are bases of $\mathcal{M}$ and if $x\in B_1\setminus B_2$, then there is $y\in B_2\setminus B_1$ such that both $(B_1\setminus\{x\})\cup\{y\}$ and $(B_2\setminus\{y\})\cup\{x\}$ are bases of $\mathcal{M}$.

Alternatively, we can give another definition of matroid in terms of its set of bases. More precisely, given a nonempty set $\mathcal{B}\subset 2^{[n]}$, there exists a matroid $\mathcal{M}$ on the ground set $[n]$ with $\mathcal{B}$ its set of bases if and only if $\mathcal{B}$ possesses the exchange property (E). If we denote the canonical basis vectors of $\mathbb{R}^n$ by $\varepsilon_1, \dots, \varepsilon_n$, then we associate to each $F \subset [n]$ the $(0,1)$-vector $\sum_{i \in F} \varepsilon_i$. In this way a matroid on $[n]$ can now be regarded as a set of $(0,1)$-vectors (see Corollary 2.3). Now we give three important examples of matroids:

Examples 1.2. Vector Matroid: Let V be a vector space and E be a nonempty finite subset of V. We define the matroid $\mathcal{M}$ on the ground set E by taking the independent sets of $\mathcal{M}$ to be the sets of linearly independent elements in E. With linear algebra arguments one can check that the axioms of the matroid are fulfilled.

Cycle Matroid: Let G be a finite graph, with V its set of vertices and E its set of edges. Consider a set of edges independent if and only if it does not contain a simple cycle. Then the set of all these independent sets define a matroid on the ground set E.

Uniform Matroid: Let r and n be nonnegative integers with r no larger than n. Let E be an element set of cardinality n, and let $\mathcal{M}$ be the collection of all subsets of E of cardinality r or less. Then $\mathcal{M}$ is a matroid, called the uniform matroid of rank r on n elements, and it is sometimes denoted by $U_{r,n}$.

For the rest of this section we present the concept of polymatroid and its associated rank function. The concept of polymatroid originated in Edmonds [5], and for further properties the reader can consult [20, Ch. 18], [14].

First, we recall the notations: $[n] = \{1, 2, \dots, n\}$ and $\varepsilon_1, \dots, \varepsilon_n$ are the canonical basis vectors of $\mathbb{R}^n$. Then, we denote by $\mathbb{R}^n_+$ the set of those vectors $u = (u(1), \dots, u(n)) \in \mathbb{R}^n$ with each $u(i) \geq 0$, and $\mathbb{Z}^n_+ = \mathbb{R}^n_+ \cap \mathbb{Z}^n$. For a vector $u = (u(1), \dots, u(n)) \in \mathbb{R}^n_+$ and for a subset $A \subset [n]$, we set

$$u(A) = \sum_{i \in A} u(i).$$

Thus in particular $u(\{i\})$ is the i-th component $u(i)$ of u. The *modulus* of u is

$$|u| = u([n]) = \sum_{i=1}^{n} u(i).$$

Let $u = (u(1), \dots, u(n))$ and $v = (v(1), \dots, v(n))$ be two vectors in $\mathbb{R}^n_+$. We write $u \leq v$ if all components $v(i) - u(i)$ of $v - u$ are nonnegative and, moreover, write $u < v$ if $u \leq v$ and $u \neq v$. We say that u is a *subvector* of v if $u \leq v$. In addition, we set

$$u \vee v = (\max\{u(1), v(1)\}, \dots, \max\{u(n), v(n)\}),$$

$$u \wedge v = (\min\{u(1), v(1)\}, \dots, \min\{u(n), v(n)\}).$$

Thus $u \wedge v \leq u \leq u \vee v$ and $u \wedge v \leq v \leq u \vee v$. We recall:

Definition 1.3. *A* polymatroid *on the ground set* $[n]$ *is a nonempty compact subset* $\mathcal{P}$ *in* $\mathbb{R}_+^n$*, the set of* independent vectors*, such that*

($\mathcal{P}$1) every subvector of an independent vector is independent;

($\mathcal{P}$2) if $u, v \in \mathcal{P}$ *with* $|v| > |u|$*, then there is a vector* $w \in \mathcal{P}$ *such that*

$$u < w \leq u \vee v.$$

A *base* of a polymatroid $\mathcal{P} \subset \mathbb{R}_+^n$ is a maximal independent vector of $\mathcal{P}$, i.e., an independent vector $u \in \mathcal{P}$ with $u < v$ for no $v \in \mathcal{P}$. It follows from ($\mathcal{P}$2) that every base of $\mathcal{P}$ has the same modulus $\mathrm{rank}(\mathcal{P})$, the *rank* of $\mathcal{P}$.

Now we give an equivalent description of a polymatroid, very useful in proofs. Let $\mathcal{P} \subset \mathbb{R}_+^n$ be a polymatroid on the ground set $[n]$. The *ground set rank function* of $\mathcal{P}$ is a function $\rho : 2^{[n]} \longrightarrow \mathbb{R}_+$ defined by setting

$$\rho(A) = \max\{v(A) : v \in \mathcal{P}\}$$

for all $\emptyset \neq A \subset [n]$ together with $\rho(\emptyset) = 0$. Then we have

Proposition 1.4. (a) *Let* $\mathcal{P} \subset \mathbb{R}_+^n$ *be a polymatroid on the ground set* $[n]$ *and* ρ *its ground set rank function. Then* ρ *is nondecreasing, i.e., if* $A \subset B \subset [n]$*, then* $\rho(A) \leq \rho(B)$*, and is submodular, i.e.,*

$$\rho(A) + \rho(B) \geq \rho(A \cup B) + \rho(A \cap B)$$

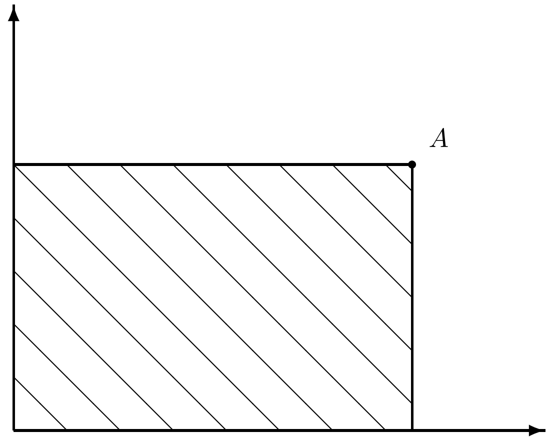
for all $A, B \subset [n]$*. Moreover,* $\mathcal{P}$ *coincides with the compact set*

$$\{x \in \mathbb{R}_+^n : x(A) \leq \rho(A), A \subset [n]\}. \tag{1}$$

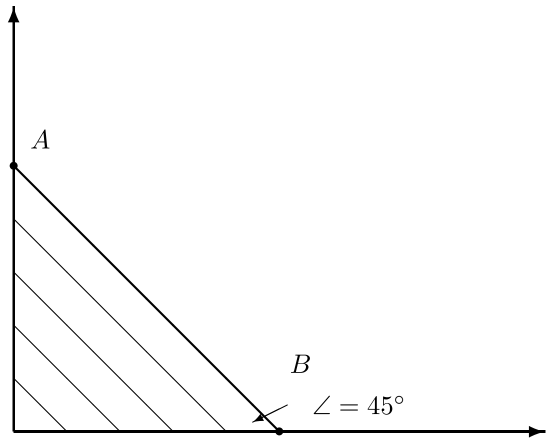
(b) *Conversely, given a nondecreasing and submodular function* $\rho : 2^{[n]} \longrightarrow \mathbb{R}_+$ *with* $\rho(\emptyset) = 0$*, the compact set (1) is a polymatroid on the ground set* $[n]$ *with* ρ *its ground set rank function.*

It follows from Proposition 1.4(a) that a polymatroid $\mathcal{P} \subset \mathbb{R}_+^n$ on the ground set $[n]$ is a convex polytope in $\mathbb{R}^n$. Furthermore a polymatroid is *integral* if and only if its ground set rank function is integer valued. For a detailed material on convex polytopes we refer the reader to [23], [11]. We end this section with an example that shows all possible polymatroids in $\mathbb{R}^2$:

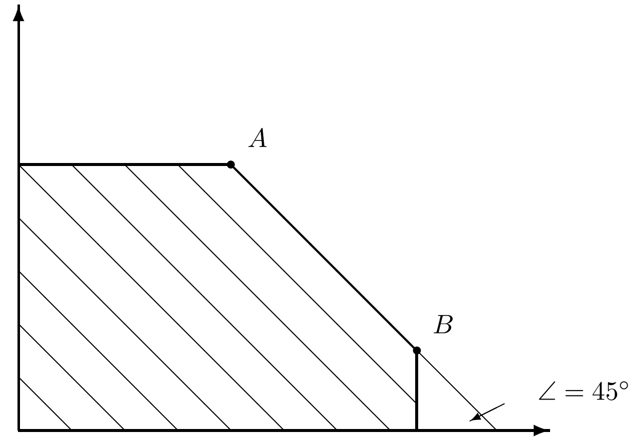
Example 1.5 ([19])**.** Let $\mathcal{P} \subset \mathbb{R}_+^2$ be a polymatroid of rank $d > 0$. Then $\mathcal{P}$ can have one base, i.e. a point A, and therefore $\mathcal{P}$ is a rectangle (see the first picture), or its set of bases is the line $AB : x + y - d = 0$, in which case $\mathcal{P}$ is either a triangle (see the second picture) or a pentagon (see the third picture):



Picture 1



Picture 2



Picture 3

2 Combinatorics on Discrete Polymatroids

In this section we present the basic combinatorial properties of the discrete polymatroids and their connection with matroids and polymatroids. The material is based on the paper of Herzog and Hibi [7]. With the notations introduced in the previous section we can give:

Definition 2.1 ([7]). *Let P be a nonempty finite set of integer vectors in $\mathbb{R}^n_+$, which contains with each $u \in P$ all its integral subvectors. The set P is called a* discrete polymatroid *on the ground set $[n]$ if for all $u, v \in P$ with $|v| > |u|$, there is a vector $w \in P$ such that*

$$u < w \leq u \vee v.$$

A *base* of P is a vector $u \in P$ such that $u < v$ for no $v \in P$. We denote by $B(P)$ the set of bases of a discrete polymatroid P. It follows from the definition that any two bases of P have the same modulus. This common number is called the *rank* of P.

Discrete polymatroids can be characterized in terms of their set of bases as follows:

Theorem 2.2 ([7]). *Let P be a nonempty finite set of integer vectors in $\mathbb{R}^n_+$ which contains with each $u \in P$ all its integral subvectors, and let $B(P)$ be the set of vectors $u \in P$ with $u < v$ for no $v \in P$. The following conditions are equivalent:*

(a) P is a discrete polymatroid;

(b) *if* $u, v \in P$ *with* $|v| > |u|$, *then there is an integer* i *such that* $u + \varepsilon_i \in P$ *and*

$$u + \varepsilon_i \le u \vee v;$$

(c) *(i)* *all* $u \in B(P)$ *have the same modulus,*

(ii) *if* $u, v \in B(P)$ *with* $u(i) > v(i)$, *then there exists* j *with* $u(j) < v(j)$ *such that* $u - \varepsilon_i + \varepsilon_j \in B(P)$.

Condition $(c)(ii)$ from the theorem is also called the *exchange property.* An important consequence of this theorem is that it gives a way to construct discrete polymatroids. According to condition (c), it is enough to give a set of integer vectors of the same modulus, which satisfy the exchange property and then, by taking all its integral subvectors we obtain a discrete polymatroid. The following result, which is obtained from Theorem 2.2 and the definition of matroid, shows that it makes sense to view the discrete polymatroids as generalizations of matroids.

Corollary 2.3 ([7])**.** *Let* B *be a nonempty finite set of integer vectors in* $\mathbb{R}^n_+$. *The following conditions are equivalent:*

(a) B *is the set of bases of a matroid;*

(b) B *is the set of bases of a discrete polymatroid, and for all* $u \in B$ *one has* $u(i) \le 1$ *for* $i = 1, \ldots, n$.

It is useful for induction arguments to work with "new" discrete polymatroids obtained from "old" ones, as it follows from

Lemma 2.4 ([7])**.** *Let* P *be a discrete polymatroid.*

(a) *Let* $d \le \operatorname{rank}(P)$. *Then the set* $P' = \{u \in P : |u| \le d\}$ *is a discrete polymatroid of rank* d *with the set of bases* $\{u \in P : |u| = d\}$.

(b) *Let* $x \in P$. *Then the set* $P_x = \{v - x : v \ge x\}$ *is a discrete polymatroid of rank* $d - |x|$.

The exchange property suggests the following definition [7] of a particular class of discrete polymatroids

Definition 2.5 ([7])**.** *A discrete polymatroid* P *satisfies the* strong exchange property, *if for all* $u, v \in B(P)$ *with* $u(i) > v(i)$ *and* $u(j) < v(j)$ *for some* i *and* j, *one has that* $u - \varepsilon_i + \varepsilon_j \in B(P)$.

Examples 2.6. (a) Let B be the set of bases of a discrete polymatroid on the ground set $[n]$ with $n \leq 3$. It is immediate that B satisfies the strong exchange property.

(b) The set $B = \{(2,1,2,1),(1,2,1,2),(1,1,2,2),(2,2,1,1)\}$ is the set of bases of a discrete polymatroid of rank 6. For this just verify condition $(c)(ii)$ of Theorem 2.2. It does not satisfy the strong exchange property. Indeed, if we denote by $u = (2,1,2,1)$ and $v = (1,2,1,2)$ then $u(1) > v(1)$, $u(2) < v(2)$ and $u - \varepsilon_1 + \varepsilon_2 = (1,2,2,1) \notin B$.

(c) The set of all $u \in \mathbb{Z}_+^n$ of modulus d will be denoted by $V_n^{(d)}$, and it is the set of bases of a discrete polymatroid of rank d.

(d) Let $a_1, \ldots, a_n$ and d be nonnegative integers. The set

$$V = \{u : u(i) \text{ is an integer with } 0 \leq u(i) \leq a_i \text{ and } |u| = d\}$$

is the set of bases of a discrete polymatroid of rank d, which satisfies the strong exchange property. Such a discrete polymatroid is also called *polymatroid of Veronese type.*

(e) The set $\{(2,2,0),(1,3,0),(0,4,0),(1,2,1),(0,3,1)\}$ is the set of bases of a discrete polymatroid, which satisfies the strong exchange property, but it is not of Veronese type.

Just as in the case of matroids, we have the symmetric exchange property:

Theorem 2.7 ([7])**.** *If $u = (u(1), \ldots, u(n))$ and $v = (v(1), \ldots, v(n))$ are bases of a discrete polymatroid $P \subset \mathbb{Z}_+^n$, then for each $i \in [n]$ with $u(i) > v(i)$, there is $j \in [n]$ with $u(j) > v(j)$ such that both $u - \varepsilon_i + \varepsilon_j$ and $v - \varepsilon_j + \varepsilon_i$ are bases of P.*

In the proof it is used the rank function of a discrete polymatroid, which we define next. Let $P \subset \mathbb{Z}_+^n$ be a discrete polymatroid and $B(P)$ its set of bases. We define the *rank function* of the discrete polymatroid P to be the function $\rho_P : 2^{[n]} \longrightarrow \mathbb{Z}_+$, by setting

$$\rho_P(A) = \max\{u(A) : u \in P\}$$

for all $\emptyset \neq A \subset [n]$, together with $\rho_P(\emptyset) = 0$. It is easy to check that ρ_P is a nondecreasing function, i.e. if $A \subset B \subset [n]$, then $\rho_P(A) \leq \rho_P(B)$, and from [7] we have that ρ_P is submodular, i.e.

$$\rho_P(A) + \rho_P(B) \geq \rho_P(A \cup B) + \rho_P(A \cap B),$$

for all $A, B \subset [n]$. Conversely, given a nondecreasing and submodular function $\rho : 2^{[n]} \to \mathbb{Z}_+$, then the set $u \in \mathbb{Z}_+^n$ satisfying

$$u(A) \leq \rho(A), \quad \text{for all } A \in 2^{[n]}, \tag{2}$$

is a discrete polymatroid, whose rank function ρ_P equals ρ. In connection to the rank function ρ of a discrete polymatroid P we distinguish two important types of sets. A set $\emptyset \neq A \subset [n]$ is *ρ-closed* if any subset $C \subset [n]$ properly containing A satisfies $\rho(A) < \rho(C)$, and $\emptyset \neq A \subset [n]$ is *ρ-separable* if there exist two nonempty subsets A_1 and A_2 of A with $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$ such that $\rho(A) = \rho(A_1) + \rho(A_2)$. A nonempty subset A of $[n]$ is *ρ-inseparable* if it is not ρ-separable. The following example is intended to give a better view to the construction (2) and the definitions above.

Example 2.8. (a) Let us consider the function $\rho_P : 2^{[3]} \longrightarrow \mathbb{Z}_+$ defined $\rho(\emptyset) = 0$, $\rho(\{1\}) = 1$, $\rho(\{2\}) = 2$, $\rho(\{3\}) = 2$, $\rho(\{1,2\}) = 3$, $\rho(\{1,3\}) = 2$, $\rho(\{2,3\}) = 4$, $\rho(\{1,2,3\}) = 4$. One can easily check that ρ is nondecreasing and submodular. The bases are the integer solutions (u_1, u_2, u_3) of the inequations

$$u_1 \leq 1, u_2 \leq 2, u_3 \leq 2, u_1 + u_2 \leq 3, u_1 + u_3 \leq 2, u_2 + u_3 \leq 4,$$

together with

$$u_1 + u_2 + u_3 = 4,$$

i.e. the vectors $(1,2,1)$ and $(0,2,2)$. Then from the comments made above, taking all subintegral vectors of $(1,2,1)$ and $(0,2,2)$ we obtain the discrete polymatroid P

$$P = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1), (1,0,1), (0,2,0), (0,0,2),$$

$$(0,1,2), (0,2,1), (1,1,1), (1,2,0), (0,2,2), (1,2,1)\}.$$

The ρ-closed subsets of [3] are: $\{1\}$, $\{2\}$, $\{1,2\}$ and $\{1,3\}$. The ρ-inseparable subsets of [3] are: $\{1\}$, $\{2\}$, $\{3\}$ and $\{1,3\}$.

We close this section by stating the result which makes the connection between discrete polymatroids and integral polymatroids.

Theorem 2.9 ([7]). *A nonempty finite set $P \subset \mathbb{Z}_+^n$ is a discrete polymatroid if and only if $\operatorname{conv}(P) \subset \mathbb{R}_+^n$ is an integral polymatroid with $\operatorname{conv}(P) \cap \mathbb{Z}_+^n = P$.*

3 Constructions of Discrete Polymatroids

We have seen in the previous section that verifying whether a certain finite set of integer vectors in $\mathbb{Z}_+^n$ is a discrete polymatroid reduces to the following three steps: first check if for any vector all its subintegral vectors are still in the set, and then if the maximal vectors have the same modulus and satisfy the exchange property. The number of computations being rather big, it would be helpful to have some techniques to construct discrete polymatroids. We present two techniques from [7][Section 8]. The first one shows that a nondecreasing and submodular function defined in a sublattice of $2^{[n]}$ produces a discrete polymatroid. The second one yields the concept of transversal polymatroids.

A sublattice of $2^{[n]}$ is a collection $\mathcal{L}$ of subsets of $[n]$ with $\emptyset \in \mathcal{L}$ and $[n] \in \mathcal{L}$ such that, for all A and B belonging to $\mathcal{L}$, both $A \cap B$ and $A \cup B$ belong to $\mathcal{L}$.

Theorem 3.1 ([7])**.** *Let $\mathcal{L}$ be a sublattice of $2^{[n]}$ and $\mu : \mathcal{L} \longrightarrow \mathbb{R}_+$ an integer valued nondecreasing and submodular function with $\mu(\emptyset) = 0$. Then*

$$P_{(\mathcal{L},\mu)} = \{u \in \mathbb{Z}_+^n : u(A) \leq \mu(A), A \in \mathcal{L}\}$$

is a discrete polymatroid.

Example 3.2 ([7])**.** Let $\mathcal{L}$ be a chain of length n of $2^{[n]}$, say

$$\mathcal{L} = \{\emptyset, \{n\}, \{n-1, n\}, \ldots, \{1, \ldots, n\}\} \subset 2^{[n]}.$$

Given nonnegative integers $a_1, \ldots, a_n$, define $\mu : \mathcal{L} \longrightarrow \mathbb{R}_+$ by

$$\mu(\{i, i+1, \ldots, n\}) = a_i + a_{i+1} + \cdots + a_n, \quad 1 \leq i \leq n$$

together with $\mu(\emptyset) = 0$. Then the discrete polymatroid $P_{(\mathcal{L},\mu)} \subset \mathbb{Z}_+^n$ is

$$P_{(\mathcal{L},\mu)} = \{u = (u_1, \ldots, u_n) \in \mathbb{Z}_+^n : \sum_{j=i}^{n} u_j \leq \sum_{j=i}^{n} a_j, \quad 1 \leq i \leq n\}.$$

For the second result about construction of discrete poymatroids, first we need to fix some notations. Let $\mathcal{A} = (A_1, \ldots, A_d)$ be a family of nonempty subsets of $[n]$. It is not required that $A_i \neq A_j$ if $i \neq j$. Let

$$B_{\mathcal{A}} = \{\varepsilon_{i_1} + \cdots + \varepsilon_{i_d} : i_k \in A_k, \quad 1 \leq k \leq d\} \subset \mathbb{Z}_+^n$$

and define the integer valued nondecreasing function $\rho_{\mathcal{A}} : 2^{[n]} \longrightarrow \mathbb{R}_+$ by setting

$$\rho_{\mathcal{A}}(X) = |\{k : A_k \cap X \neq \emptyset\}|, \quad X \subset [n].$$

Now we can state

Theorem 3.3 ([7])**.** *The function $\rho_{\mathcal{A}}$ is submodular and $B_{\mathcal{A}}$ is the set of bases of the discrete polymatroid $P_{\mathcal{A}} \subset \mathbb{Z}_+^n$ arising from $\rho_{\mathcal{A}}$.*

The discrete polymatroid $P_{\mathcal{A}} \subset \mathbb{Z}_+^n$ is called the *transversal polymatroid* presented by $\mathcal{A}$. Observe that $\operatorname{rank}(P) = d$. The following examples, given by Herzog and Hibi, show that Example 3.2 is a transversal polymatroid and that not all discrete polymatroids are transversal.

Examples 3.4 ([7])**.** (a) Let $r_1, \ldots, r_d \in [n]$ and set $A_k = [r_k] = \{1, \ldots, r_k\}$, $1 \leq k \leq d$. Let $\min(X)$ denote the smallest integer belonging to X, where $\emptyset \neq X \subset [n]$. If $\mathcal{A} = (A_1, \ldots, A_d)$, then

$$\rho_{\mathcal{A}}(X) = \rho_{\mathcal{A}}(\{\min X\}) = |\{k : \min(X) \leq r_k\}|.$$

If $\emptyset \neq X \subset [n]$ is $\rho_{\mathcal{A}}$-closed, then $X = \{\min(X), \min(X)+1, \ldots, n\}$. Let

$$a_i = |\{k : r_k = i\}|, \quad 1 \leq i \leq n.$$

Thus

$$\rho_{\mathcal{A}}(\{i, i+1, \ldots, n\}) = a_i + a_{i+1} + \cdots + a_n, \quad 1 \leq i \leq n.$$

The transversal polymatroid $P_{\mathcal{A}} \subset \mathbb{Z}_+^n$ presented by $\mathcal{A}$ is

$$P_{\mathcal{A}} = \{u = (u_1, \ldots, u_n) \in \mathbb{Z}_+^n : \sum_{j=i}^{n} u_j \leq \sum_{j=i}^{n} a_j, \quad 1 \leq i \leq n\}.$$

Thus $P_{\mathcal{A}}$ coincides with the discrete polymatroid $P_{(\mathcal{L},\mu)}$ in Example 3.2.

(b) Let $P \subset \mathbb{Z}_+^4$ denote the discrete polymatroid of rank 3 consisting of those $u = (u_1, u_2, u_3, u_4) \in \mathbb{Z}_+^4$ with $u_i \leq 2$ for $1 \leq i \leq 4$ and with $|u| \leq 3$. Then P is not transversal. Suppose, on the contrary, that P is the transversal polymatroid presented by $\mathcal{A} = (A_1, A_2, A_3)$ with each $A_k \subset [4]$. Since $(2,1,0,0), (2,0,1,0), (2,0,0,1) \in P$ and $(3,0,0,0) \notin P$, we assume that $1 \in A_1$, $1 \in A_2$ and $A_3 = \{2,3,4\}$. Since $(1,2,0,0), (0,2,1,0), (0,2,0,1) \in P$ and $(0,3,0,0) \notin P$, we assume that $2 \in A_1$ and $A_2 = \{1,3,4\}$. Since $(0,0,2,1) \in P$ and $(0,0,3,0) \notin P$, one has $4 \in A_1$. Hence $(0,0,0,3) \in P$, a contradiction.

4 The Ehrhart Ring and the Base Ring of a Discrete Polymatroid

In this section we follow the notations from [7]. Let K be a field and let $x_1, \ldots, x_n$ and s be indeterminates over K. If $u = (u_1, \ldots, u_n) \in \mathbb{Z}_+^n$, then we denote by x^u the monomial ${x_1}^{u_1} \cdots {x_n}^{u_n}$. Let P be a discrete polymatroid of rank d on the ground set $[n]$ with set of bases B. The toric ring $K[B]$ generated over K by the monomials x^u, where $u = (u(1), \ldots, u(n)) \in B$, is called the *base ring* of P. Since P is the set of integer vectors of an integral polymatroid $\mathcal{P}$ (see Theorem 2.9), we may study the *Ehrhart ring of* $\mathcal{P}$. For this, one considers the cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ with $\mathcal{C} = \mathbb{R}_+\{(p,1) : p \in \mathcal{P}\}$. Then $Q = \mathcal{C} \cap \mathbb{Z}_{n+1}$ is a subsemigroup of $\mathbb{Z}^{n+1}$, and the Ehrhart ring of $\mathcal{P}$ is defined to be the toric ring $K[\mathcal{P}] \subset K[x_1, \ldots, x_n, s]$ generated over K by the monomials $x^u s^i$, $(u, i) \in Q$. By Gordan's Lemma ([2, Proposition 6.1.2]), $K[\mathcal{P}]$ is normal.

Notice that $K[\mathcal{P}]$ is naturally graded if we assign to $x^u s^i$ the degree i. We denote by $K[P]$ the K-subalgebra of $K[\mathcal{P}]$ which is generated over K by the elements of degree 1 in $K[\mathcal{P}]$. Since $P = \mathcal{P} \cap \mathbb{Z}^n$ it follows that $K[P] = K[x^u s : u \in P]$. Observe that $K[B]$ may be identified with the subalgebra $K[x^u s : u \in B]$ of $K[P]$.

The base ring $K[B]$ was introduced in 1977 by N. White, in the particular case when B is the set of bases of a matroid, and he showed that for every matroid, the ring $K[B]$ is normal (see [21]) and thus Cohen-Macaulay. It is natural to ask whether the same holds for the base ring of any discrete polymatroid. Herzog and Hibi showed that:

Theorem 4.1 ([7])**.** *$K[P] = K[\mathcal{P}]$. In particular, $K[P]$ is normal.*

As a corollary they obtain also

Corollary 4.2 ([7])**.** *$K[B]$ is normal.*

Furthermore, if both $K[P]$ and $K[B]$ are Cohen-Macaulay, it is natural to ask when these rings are Gorenstein. The following example shows that, in general, for a given discrete polymatroid P with the set of bases B, there is not necessarily a relation between Gorenstein property for $K[P]$ and $K[B]$.

Example 4.3 ([7])**.** (a) Let $P \subset \mathbb{Z}_+^3$ be the discrete polymatroid consisting of all integer vectors $u \in \mathbb{Z}_+^3$ with $|u| \leq 3$. Then the base ring $K[B]$ is the Veronese subring $K[x, y, z]^{(3)}$, the subring of $K[x, y, z]$ generated by all monomials of degree 3. Thus $K[B]$ is Gorenstein. On the other hand, since the Hilbert series of the Ehrhart ring $K[P]$ is $(1+16t+10t^2)/(1-t)^4$, it follows that $K[P]$ is not Gorenstein.

(b) Let $P \subset \mathbb{Z}_+^3$ be the discrete polymatroid consisting of all integer vectors $u \in \mathbb{Z}_+^3$ with $|u| \leq 4$. Then $K[B] = K[x, y, z]^{(4)}$ is not Gorenstein. On the other hand, the Hilbert series of the Ehrhart ring $K[P]$ is

$$(1 + 31t + 31t^2 + t^3)/(1 - t)^4;$$

thus $K[P]$ is Gorenstein.

(c) Let $P \subset \mathbb{Z}_+^2$ be the discrete polymatroid with $B = \{(1, 2), (2, 1)\}$ its set of bases. Then both $K[P]$ and $K[B]$ are Gorenstein.

However in [7], Herzog and Hibi give a combinatorial criterion for $K[P]$ to be Gorenstein

Theorem 4.4 ([7])**.** *Let $P \subset \mathbb{Z}_+^n$ be a discrete polymatroid and suppose that the canonical basis vectors $\varepsilon_1, \ldots, \varepsilon_n$ of $\mathbb{R}^n$ belong to P. Let ρ denote the* *ground set rank function of the integral polymatroid $\mathcal{P} = \mathrm{conv}(P) \subset \mathbb{R}^n$. Then the Ehrhart ring $K[P]$ of P is Gorenstein if and only if there exists an integer $\delta \geq 1$ such that*

$$\rho(A) = \frac{1}{\delta}(|A| + 1)$$

for all ρ-closed and ρ-inseparable subsets A of $[n]$.

In particular we have the following nice example ([7])

Example 4.5 ([7])**.** (a) Let $P_n^{(d)} \subset \mathbb{Z}_+^n$ be the discrete polymatroid consisting of all integer vectors $u \in \mathbb{Z}_+^n$ with $|u| \leq d$ and $B_n^{(d)}$ the set of bases of $P_n^{(d)}$. Let ρ denote the ground set rank function of the integral polymatroid $\mathrm{conv}(P) \subset \mathbb{R}_+^n$. Then $\rho(A) = d$ for all $\emptyset \neq A \subset [n]$. Thus $[n]$ is the only ρ-closed and ρ-inseparable subset of $[n]$. Hence the Ehrhart ring $K[P_n^{(d)}]$ is Gorenstein if and only if d divides $n+1$. On the other hand, the base ring $K[B_n^{(d)}]$ is the Veronese subring $K[x_1, \ldots, x_n]^{(d)}$. Thus $K[B_n^{(d)}]$ is Gorenstein if and only if d divides n. (Note, in fact, that $K[P_n^{(d)}]$ is just the Veronese subring $K[x_1, \ldots, x_n, s]^{(d)}$.)

(b) Let $\rho : 2^{[n]} \longrightarrow \mathbb{Z}_+$ denote the nondecreasing function defined by $\rho(A) = |A| + 1$ for all $\emptyset \neq A \subset [n]$ together with $\rho(\emptyset) = 0$. Then ρ is submodular and all nonempty subsets of $[n]$ are ρ-closed and ρ-inseparable. Let $P \subset \mathbb{Z}_+^n$ be the discrete polymatroid of rank $n+1$ arising from ρ. Then the Ehrhart ring $K[P]$ is Gorenstein ($\delta = 1$). Moreover since the set of bases of P is $B = \{(2, 1, \ldots, 1), \ldots, (1, \ldots, 1, 2)\}$, the base ring $K[B]$ is isomorphic to the polynomial ring in n variables; thus $K[B]$ is Gorenstein.

We now turn to the problem when the base ring of a discrete polymatroid is Gorenstein. A complete answer is not given so far. However, there are some particular classes for which there is a complete description. For example, in [4] there is a classification of the Gorenstein rings belonging to the class of algebras of Veronese type, a distinguished class of discrete polymatroids (2.6(c)). Herzog and Hibi, in [7][Theorem 7.6.] find a characterization for the base ring of a generic discrete polymatroid to be Gorenstein.

For the rest of this section we shall discuss three conjectures related to the base ring. We recall that for a discrete polymatroid P with the set of bases B and for a field K, $K[B]$ is the algebra generated over K by the monomials x^u, where $u = (u(1), \ldots, u(n)) \in B$. If $S = K[y_u : u \in B]$ is the polynomial ring in the indeterminates y_u, with $u \in B$, denote by I_B, the toric ideal of the base ring $K[B]$, i.e. the kernel of the K-algebra homomorphism

$$S = K[y_u : u \in B] \longrightarrow K[B]$$

$$y_u \longmapsto x^u.$$

I_B contains some obvious elements, namely, those arising from symmetric exchange: let $u, v \in B$ with $u(i > v(i)$ and $u(j) < v(j)$, and such that $u - \varepsilon_i + \varepsilon_j$ and $v - \varepsilon_j + \varepsilon_i$ belong to B. Then the binomial $y_u y_v - y_{u-\varepsilon_i+\varepsilon_j} y_{v-\varepsilon_j+\varepsilon_i} \in I_B$. Following the notations of Herzog and Hibi([7]), we call such a relation a *symmetric exchange relation.*

- White conjectured ([22]) that for a matroid the symmetric exchange relations generate I_B.

Since the discrete polymatroid is a natural generalization of the matroid (see Corollary 2.3), Herzog and Hibi ([7]) conjectured that this also holds for a discrete polymatroid. They showed in [7],Theorem 5.3.(a) that the two conjectures are equivalent:

Theorem 4.6 ([7])**.** *Suppose that each matroid has the property that the toric ideal of its base ring is generated by symmetric exchange relations, then this is also true for each discrete polymatroid.*

Several commutative algebraists, for example Herzog and Hibi [7], asked whether two succesively stronger assertions hold:

- Is the base ring $K[B]$ a Koszul algebra?
- Does the toric ideal I_B of a discrete polymatroid possess a quadratic Gröbner basis?

Example 4.7. Let P be the discrete polymatroid of rank 4 whose set of bases is $B = \{u_1, \ldots, u_6\}$, where $u_1 = (1,1,0,1,1,0)$, $u_2 = (1,1,0,1,0,1)$, $u_3 = (0,1,1,1,1,0)$, $u_4 = (0,1,1,1,0,1)$, $u_5 = (1,0,1,1,1,0)$, $u_6 = (1,0,1,1,0,1)$. We denote by φ the homomorphism

$$\varphi : S = K[y_{u_1}, \ldots, y_{u_6}] \longrightarrow K[B] = K[x^{u_1}, \ldots, x^{u_6}] \subset K[x_1, \ldots, x_6],$$

given by

$$\varphi(y_{u_i}) = x^{u_i}, 1 \leq i \leq 6.$$

Then, the toric ideal I_B is the ideal

$$I_B = (y_{u_4}y_{u_5} - y_{u_3}y_{u_6}, y_{u_1}y_{u_4} - y_{u_2}y_{u_3}, y_{u_1}y_{u_6} - y_{u_2}y_{u_5}).$$

Since $u_3 = u_4 - \varepsilon_6 + \varepsilon_5$, $u_6 = u_5 - \varepsilon_5 + \varepsilon_6$, $u_2 = u_1 - \varepsilon_5 + \varepsilon_6$, $u_5 = u_6 - \varepsilon_6 + \varepsilon_5$, the binomials which generate I_B are

$$y_{u_4}y_{u_5} - y_{u_4-\varepsilon_6+\varepsilon_5} y_{u_5-\varepsilon_5+\varepsilon_6}, y_{u_1}y_{u_4} - y_{u_1-\varepsilon_5+\varepsilon_6} y_{u_4-\varepsilon_6+\varepsilon_5},$$

$$y_{u_1}y_{u_6} - y_{u_1-\varepsilon_5+\varepsilon_6} y_{u_6-\varepsilon_6+\varepsilon_5},$$

hence the conjecture of White holds in this case. Moreover the same binomials are a Groebner basis for the ideal I_B, so the other two conjectures are true in this case.

With the help of computer algebra packages many people tried to disprove these conjectures. Since no counterexample was given so far, it seems that there is a good chance that these conjectures are true. A positive answer for the three conjectures is given by Herzog and Hibi in [7],Theorem 5.3.(b), in the particular case of discrete polymatroids which satisfy the strong exchange property:

Theorem 4.8 ([7])**.** *Let P be a discrete polymatroid whose set of bases B satisfies the strong exchange property. Then:*

(a) I_B has a quadratic Gröbner basis and $K[B]$ is Koszul,

(b) I_B is generated by symmetric exchange relations.

Since the above conjectures about $K[B]$ could be true what can be said about $K[P]$? More precisely, what can be said about the algebraic properties of $K[B]$ compared to those of $K[P]$? Following the notations of [7] we recall that a K-algebra A has a quadratic Gröbner basis if the defining ideal of A has this property for some term order. Then, Herzog and Hibi proved that

Theorem 4.9 ([7])**.** (a) *Suppose that $K[P]$ has quadratic relations, or a quadratic Gröbner basis or is Koszul, then $K[B]$ has these properties, too.*

(b) *Given a property $\mathcal{E}$. Suppose that $K[B(P)]$ satisfies $\mathcal{E}$ for all discrete polymatroids P. Then also $K[P]$ satisfies $\mathcal{E}$ for all discrete polymatroids P.*

5 Polymatroidal Ideals

In this section we shall review some of the properties of another algebraic object, which can be associated to a discrete polymatroid, namely the *polymatroidal ideal*.

Let P be a discrete polymatroid on the ground set $[n]$, with the set of bases B. If we denote by $R := K[x_1, \ldots, x_n]$, the polynomial ring in n variables over a field K, then the monomial ideal $I(B)$ of R generated by all monomials t^u with $u \in B$, is called the *polymatroidal ideal* associated to the discrete polymatroid P. It follows from Theorem 2.2, that a polymatroidal ideal can be equivalently characterized as a monomial ideal I, whose generators have the same degree and satisfy the following exchange property:

for all $u, v \in G(I)$ and all i with $\nu_i(u) > \nu_i(v)$, there exists an integer j with

$$\nu_j(u) < \nu_j(v) \text{ such that } x_j(u/x_i) \in G(I),$$

where by $G(I)$ we denote the unique minimal set of monomial generators of I, and for a monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ we set $\nu_i(u) = a_i$.

A monomial ideal I is said to have *linear quotients* if for some order $u_1, \dots, u_m$ of the elements of $G(I)$ and all $j = 1, \dots, m$ the colon ideals $(u_1, \dots, u_{j-1}) : u_j$ are generated by a subset of $\{x_1, \dots, x_n\}$.

Examples 5.1 ([3])**.** (a) The ideal

$$J = (x_1^2x_2, x_1x_2x_3, x_2x_3x_4, x_3x_4^2) \subset K[x_1, \dots, x_4]$$

has linear quotients, the successive colons being:

$$(0), \quad (x_1), \quad (x_1), \quad (x_2).$$

(b) The ideal J above and the ideal $I = (x_2, x_3)$ are both ideals of $R = K[x_1, \dots, x_4]$, with linear quotients. The resolution of IJ is

$$0 \to R(-8) \to R^3(-6) \oplus R^2(-7) \to R^{10}(-5) \oplus R(-6) \to R^8(-4) \to IJ \to 0,$$

which is not linear, and therefore, according to Remark 5.3, IJ does not have linear quotients. Hence, in general the product of ideals with linear quotients may not have linear quotients.

(c) Powers of ideals with linear quotients do not have in general linear quotients. For example the ideal

$$I = (x_1^2x_2, x_1^2x_3, x_1x_3^2, x_2x_3^2, x_1x_3x_4) \subset K[x_1, \dots, x_4]$$

has linear quotients, the quotients being

$$(0), \quad (x_2), \quad (x_1), \quad (x_1), \quad (x_1, x_3).$$

But I^2 has not linear quotients since the minimal resolution of I^2 is

$$0 \to R^2(-9) \oplus R(-10) \to R^{12}(-8) \oplus R^2(-9) \to$$

$$\to R^{24}(-7) \oplus R(-8) \to R^{15}(-6) \to I^2 \to 0,$$

and is not linear.

It has been shown by Herzog and Takayama that

Proposition 5.2 ([10])**.** *A polymatroidal ideal I has linear quotients with respect to the reverse lexicographic order of the generators.*

Remark 5.3. It is known, see for example [3, Lemma 4.1], that if a monomial ideal I has all its minimal generators of the same degree and it has linear quotients, then I has a linear resolution. Therefore, we have that any polymatroidal ideal has a linear resolution.

As a consequence of the fact that a polymatroidal ideal has linear quotients one can compute the length of the minimal free resolution of R/I over R. More precisely, if I is a polymatroidal ideal and if $u_1, \ldots, u_m$ are the monomials belonging to $G(I)$ ordered by the reverse lexicographic order $<_{rev}$ induced by the ordering $x_1 > x_2 > \ldots > x_n$, i.e. $u_s <_{rev} \ldots <_{rev} u_2 <_{rev} u_1$, then the colon ideals $(u_1, \ldots, u_{j-1}) : u_j$ are generated by a subset of $\{x_1, \ldots, x_n\}$, for all $j = 1 \ldots, m$. Following the notation from [8], we denote by $q_j(I)$ the number of variables which generates the colon ideal $(u_1, \ldots, u_{j-1}) : u_j$. Then, if we denote by $q(I) = \max_{2\leq j\leq m} q_j(I)$, via [10, Corollary 1.6.], the following formula is obtained

- $\operatorname{proj}\dim_R(R/I) = q(I) + 1$.

Hence, using Auslander-Buchsbaum Theorem, the formula for depth is:

- $\operatorname{depth}(R/I) = n - q(I) - 1$.

For the computation of the dimension of R/I, one only needs that I is a monomial ideal, and therefore

- $\dim(R/I) = n - \operatorname{height}(I)$,

where $\operatorname{height}(I)$ is the minimal cardinality of the vertex covers of I. We recall that for a monomial ideal I, a *vertex cover* of I is a subset V of $\{x_1, \ldots, x_n\}$ such that each $u \in G(I)$ is divisible by some $x_i \in V$. A vertex cover V is called *minimal* if no proper subset of V is a vertex cover of I.

Since there are formulae for computing $\operatorname{depth}(R/I)$ and $\dim(R/I)$, when I is a polymatroidal ideal it is natural to ask when I is Cohen-Macaulay, i.e. the quotient ring R/I is Cohen-Macaulay. Using the above formulae and the fact that polymatroidal ideal has linear quotients, Herzog and Hibi proved that:

Theorem 5.4 ([8]). *A polymatroidal ideal I is Cohen-Macaulay if and only if I is*

(a) a principal ideal,

(b) a Veronese ideal, or

(c) a squarefree Veronese ideal.

Furthermore, what can be said about the product of two polymatroidal ideals? Since the product of ideals with linear quotients is not necessarily an ideal with linear quotients, as the Examples 5.1 (b,c) show, it is somehow surprising that we have

Proposition 5.5 ([3]). *Let I and J be polymatroidal ideals. Then the product IJ is polymatroidal.*

In particular, products and powers of polymatroidal ideals have linear quotients and linear resolutions. For the case of matroidal ideals, i.e. squarefree polymatroidal ideals, Conca and Herzog show also that another operation, namely the *squarefree product* behaves "well":

Theorem 5.6 ([3]). *Let I and J be matroidal ideals. Then $I * J$ is matroidal.*

We recall that for matroidal ideals I and J, the squarefree product, denoted by $I * J$, is the ideal generated by all monomials uv, with $u \in G(I)$ and $v \in G(J)$ such that uv is squarefree. For example if we consider the polymatroidal ideals $I = (x_1x_3, x_1x_2, x_2x_4, x_3x_4)$ and $J = (x_1x_3, x_2x_3, x_2x_4, x_1x_4)$ of $K[x_1, x_2, x_3, x_4]$, then $I * J = (x_1x_2x_3x_4)$.

Another interesting result was obtained by Villarreal about the Rees algebra $\mathcal{R}(I)$,

$$\mathcal{R}(I) = \bigoplus_{j \geq 0} I^j t^j \subset R[t],$$

of a polymatroidal ideal I. More precisely, he shows that:

Proposition 5.7 ([18]). *If I is a polymatroidal ideal, then the Rees algebra $\mathcal{R}(I)$ is a normal ring.*

In the end of this section, we present the concept of ideal of fiber type, which was introduced by Herzog, Hibi and Vladoiu [9], together with some examples, among them being the polymatroidal ideals.

Let K be a field, $R = K[x_1, \dots, x_n]$ the polynomial ring, $I \subset S$ an equigenerated graded ideal, that is, a graded ideal whose generators $f_1, \dots, f_m$ are all of same degree. Then the Rees ring

$$\mathcal{R}(I) = \bigoplus_{j \geq 0} I^j t^j = R[f_1t, \dots, f_mt] \subset R[t]$$

is naturally bigraded with $\deg(x_i) = (1, 0)$ for $i = 1, \dots, n$ and $\deg(f_it) = (0, 1)$ for $i = 1, \dots, m$.

Let $T = R[y_1, \dots, y_m]$ be the polynomial ring over R in the variables $y_1, \dots, y_m$. Then we define the natural surjective homomorphism of bigraded K-algebras $\varphi\colon T \to \mathcal{R}(I)$ with

$$\varphi(x_i) = x_i, \text{ for } i = 1, \dots, n$$

and
$$\varphi(y_j) = f_j t, \text{ for } j = 1, \dots, m,$$
where we consider on T the bigrading induced by setting $\deg(x_i) = (1, 0)$, for $i = 1, \dots, n$, and $\deg(y_j) = (0, 1)$, for $j = 1, \dots, m$.

If $\alpha = (a_{ij})_{\substack{i=1,\dots,r \\ j=1,\dots,m}}$ is the relation matrix of I, then for $i = 1, \dots, r$, the bihomogeneous polynomials $g_i = \sum_{j=1}^{m} a_{ij} y_j$ belong to $\mathrm{Ker}(\varphi)$, and T/L, with $L = (g_1, \dots, g_r)$, is isomorphic to the symmetric algebra $S(I)$ of I. The generators g_i of L are all linear in the variables y_j.

Let $\mathfrak{m} = (x_1, \dots, x_n)$ be the graded maximal ideal of R. The K-algebra $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ is called the *fiber ring of* I.

Note that the standard graded subalgebra $\mathcal{R}(I)_{(0,*)} = \bigoplus_{j \geq 0} \mathcal{R}(I)_{(0,j)}$ of $\mathcal{R}(I)$ is isomorphic to $K[f_1, \dots, f_m] \subset R$, and that the composition of the natural K-algebra homomorphisms $\mathcal{R}(I)_{(0,*)} \to \mathcal{R}(I) \to \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ is an isomorphism. Therefore the fiber ring of I is isomorphic to $K[f_1, \dots, f_m]$.

The homomorphism $\varphi\colon T \to \mathcal{R}(I)$ induces a surjective K-algebra homomorphism
$$\varphi'\colon K[y_1, \dots, y_m] = T/\mathfrak{m}T \to \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I) = K[f_1, \dots, f_m].$$
The elements in $\mathrm{Ker}(\varphi')$ are called the *fiber relations*. We note that
$$\varphi' = \varphi_{(0,*)}\colon T_{(0,*)} = K[y_1, \dots, y_m] \longrightarrow \mathcal{R}(I)_{(0,*)} = K[f_1, \dots, f_m].$$
Therefore $\mathrm{Ker}(\varphi') \subset \mathrm{Ker}(\varphi)$. If we set $J = \mathrm{Ker}(\varphi')$, then $K[f_1, \dots, f_m] = K[y_1, \dots, y_m]/J$.

The natural map $\psi\colon S(I) \to \mathcal{R}(I)$ is a surjective homomorphism of bigraded K-algebras. Recall that I is called of *linear type*, if ψ is an isomorphism, that is, if $\mathrm{Ker}(\varphi) = L$. The next best situation is given by

Definition 5.8 ([9])**.** *The ideal I is called of* fiber type*, if* $\mathrm{Ker}(\varphi) = (L, JT)$.

Note that I is of fiber type if and only if $\mathrm{Ker}(\varphi)$ is generated by elements of bidegree $(*, 1)$ and $(0, *)$.

Examples 5.9. (a) The ideal $I = (x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3) \subset K[x_1, x_2]$ is of fiber type. Indeed, with the help of `Singular` [6], the kernel of the homomorphism φ, described above, is
$$\mathrm{Ker}(\varphi) = (y_3^2 - y_2 y_4, y_2 y_3 - y_1 y_4, x_2 y_3 - x_1 y_4, y_2^2 - y_1 y_3, x_2 y_2 - x_1 y_3, x_2 y_1 - x_1 y_2).$$

(b) This example is due to Villarreal ([17, Theorem 8.2.1]). Let $f, g \in R$ be monomials. We denote by $[f, g]$ the least common multiple of f and g. Let $f_1, \dots, f_m \in R$. If $\alpha = (i_1, \dots, i_s)$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_s \leq m$ we set $f_\alpha = f_{i_1} f_{i_2} \cdots f_{i_s}$.

Theorem 5.10 ([17]). *Suppose $I = (f_1, \dots, f_m)$ is an equigenerated monomial ideal satisfying:*

(∗) for all non-decreasing sequences $\alpha = (i_1, \dots, i_s)$ and $\beta = (j_1, \dots, j_s)$ with $i_k, j_k \in [m]$ for $k = 1, \dots, s$ for which $f_\alpha \neq f_\beta$, there exist integers r and t such that $f_{i_r}(f_\alpha/f_{i_t})$ divides $[f_\alpha, f_\beta]$.

Then I is of fiber type.

Moreover, condition (∗) is satisfied if I is squarefree ideal generated in degree 2. In degree 3 it is not true as the example below shows.

(c) This example is due to Villarreal ([17, Example 8.2.2]). The ideal $I = (x_1x_2x_3, x_2x_4x_5, x_5x_6x_7, x_3x_6x_7) \subset K[x_1, \dots, x_7]$ is not of fiber type. Indeed, using `Singular`, we obtain that the kernel of the morphism φ is minimally generated by

$$x_3y_3-x_5y_4, x_6x_7y_2-x_2x_4y_3, x_6x_7y_1-x_1x_2y_4, x_4x_5y_1-x_1x_3y_2, x_4y_1y_3-x_1y_2y_4,$$

and the generator $x_4y_1y_3 - x_1y_2y_4$ has bidegree $(1, 2)$.

Finally we have

Theorem 5.11 ([9]). *Let $I \subset R$ be a polymatroidal ideal. Then I is of fiber type.*

6 Discrete Polymatroids which Satisfy the Strong Exchange Property

In this section we present a structure theorem for these discrete polymatroids, and some of their properties, based on a joint paper with Herzog and Hibi [9]. For the beginning we give a motivation for studying this class of discrete polymatroids. We have seen in Section 4 three conjectures related to the base ring $K[B]$ and its toric ideal, which are proved to be true (see Theorem 4.8) in the special case of discrete polymatroids which satisfy strong exchange property. Therefore, it would be interesting to see what special combinatorial properties share this class of discrete polymatroids. An example of discrete polymatroid which satisfies the strong exchange property was given in Example 2.6 (d), by the polymatroid of Veronese type. We recall that the discrete polymatroid of Veronese type is the discrete polymatroid whose set of bases $B \subset V_n^{(d)}$ is given as follows: for $i = 1, \dots, n$ there exist integers $a_i \geq 1$ such that $u \in V_n^{(d)}$ belongs to B if and only if $u(i) \leq a_i$, for $i = 1, \dots, n$.

It was proved in [9, Theorem 1.1] that the discrete polymatroids satisfying the strong exchange property are essentially of Veronese type. More precisely, two sets $A, B \in \mathbb{R}^n$ are *isomorphic*, if there exists an affinity $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(A) = B$. Then, the theorem is:

Theorem 6.1 ([9]). *Let P be a discrete polymatroid which satisfies the strong exchange property. Then $B(P)$ is isomorphic to the bases of a polymatroid of Veronese type.*

Example 6.2. To see how this theorem works let us consider the Example 2.6 (e). The set $B(P) = \{(2,2,0),(1,3,0),(0,4,0),(1,2,1),(0,3,1)\}$ is isomorphic to $B' = \{(2,0,0),(1,1,0),(0,2,0),(1,0,1),(0,1,1)\}$ by the translation $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$,

$$\varphi(u) = u - (0,2,0).$$

It is straightforward to see that B' is the set of bases of the polymatroid of Veronese type given by $d = 2$, $a_1 = 2$, $a_2 = 2$ and $a_3 = 1$.

Let $u, v \in V_n^{(d)}$. We recall from [9] that the set

$$[u,v] = \{w \in V_n^{(d)} \colon \min\{u(i), v(i)\} \le w(i) \le \max\{u(i), v(i)\} \quad \text{for all } i\}$$

is called the *interval* between u and v. Then, it is given in [9, Lemma 1.2] an equivalent characterization of discrete polymatroids satisfying the strong exchange property:

Proposition 6.3 ([9]). *Suppose that B is a set of integer vectors u in $\mathbb{R}^n$ with $u \ge 0$ and $u([n]) = d$. Then B is the set of bases of a discrete polymatroid which satisfies the strong exchange property if and only if $B = \bigcup_{u,v\in B}[u,v]$.*

As a consequence of this equivalent description, it is given in [9, Remark 1.3.] an algorithmic method to construct the smallest set of bases of a discrete polymatroid with strong exchange property which contains a finite set $A := \{u_1, u_2, \dots, u_k\}$ of integer vectors of the same modulus d. The procedure goes like this: Denote $A_1 := A$ and let $A_2 := \bigcup_{1\le i<j\le k}[u_i, u_j]$. If $A_2 = A_1$, then the previous lemma implies that A_1 is the set we want. If $A_2 \ne A_1$ then we take $A_3 := \bigcup_{u,v\in A_2}[u,v]$. Assuming that we have defined A_i and $A_i \ne A_{i-1}$, then we consider $A_{i+1} := \bigcup_{u,v\in A_i}[u,v]$. If $A_{i+1} = A_i$ then A_i is the set we want, otherwise we continue this procedure. Because we have $|A_i| < |A_{i+1}|$ and $A_i \subseteq V_n^{(d)}$ for any i and $|V_n^{(d)}|$ is finite, then after a finite number of steps we obtain the desired set of bases.

As a consequence of Theorem 6.1, Herzog, Hibi and Vladoiu obtain that discrete polymatroids satisfying the strong exchange property, are "locally" nothing but uniform matroids. More precisely, they prove

Theorem 6.4 ([9]). *Let $u = (u(1), u(2), \dots, u(n))$ be a given point in $\mathbb{R}^n \setminus \mathbb{Z}^n$ with $u([n]) \in \mathbb{N}$, and such that $u \ge 0$, and let $I = \{i \in [n] \colon u(i) \notin \mathbb{Z}\}$. Then, with respect to inclusion, there exists a unique smallest discrete polymatroid P_u of rank $d = u([n])$ with $u \in \operatorname{conv}(B(P_u))$ satisfying the strong exchange property. Moreover the set of bases $B(P_u)$ of P_u is isomorphic to the set of bases of the uniform matroid $U_{k,m}$ where $k = \sum_{i\in I}(u(i) - \lfloor u(i)\rfloor)$ and $m = |I|$.*

Example 6.5. Let $u = (1,3;2,1;0;3,7;5;0,9;2) \in \mathbb{R}^7 \setminus \mathbb{Z}^7$ with $u([7]) = 15$. It is shown in the proof of the theorem that the set of bases $B(P_u)$ of P_u is

$$B' = \{v \in \mathbb{Z}^7 \colon v(i) \in \{\lfloor u(i) \rfloor, \lceil u(i) \rceil\} \text{ for each } i \in [7] \text{ and } v([7]) = 15\},$$

where $\lfloor x \rfloor$ is the biggest integer $\leq x$ and $\lceil x \rceil$ is the smallest integer $\geq x$. Therefore $v(3) = 0$, $v(5) = 5$, $v(7) = 2$, $v(1) \in \{1,2\}$, $v(2) \in \{2,3\}$, $v(4) \in \{3,4\}$, $v(6) \in \{0,1\}$. After an easy computation we obtain that $B(P_u)$ is the set of the following vectors

$$(1,2,0,4,5,1,2), \quad (1,3,0,3,5,1,2), \quad (1,3,0,4,5,0,2),$$

$$(2,2,0,3,5,1,2), \quad (2,2,0,4,5,0,2), \quad (2,3,0,3,5,1,2).$$

Moreover, $m = 4$, and $k = 15 - (1+2+0+3+5+0+2) = 2$. The sets of bases $B(P_u)$ and $B(U_{2,4})$ is given by the affinity $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$, where

$$\varphi(v) = v - (1,2,0,3,5,0,2), \text{ for all } v \in \mathbb{R}^n.$$

The theorem gives the following nice properties of the discrete polymatroids which satisfy the strong exchange property:

Corollary 6.6 ([9])**.** *Let $B(P)$ be the set of bases of a discrete polymatroid P. Then the following conditions are equivalent:*

(a) P satisfies the strong exchange property,

(b) For all $u \in \operatorname{conv}(B(P))$ we have $P_u \subset P$.

Corollary 6.7 ([9])**.** *Let $\mathcal{P}$ be a set of discrete polymatroids all of the same rank, satisfying the strong exchange property. Then the following conditions are equivalent:*

(a) $\bigcap_{P \in \mathcal{P}} B(P) \neq \emptyset$,

(b) $\bigcap_{P \in \mathcal{P}} \operatorname{conv}(B(P)) \neq \emptyset$.

Corollary 6.8 ([9])**.** *Let $\mathcal{P}$ be a set of discrete polymatroids all of the same rank, satisfying the strong exchange property and the equivalent conditions of Corollary 6.7, then*

$$\bigcap_{P \in \mathcal{P}} B(P) = B(\bigcap_{P \in \mathcal{P}} P) \quad \textit{and} \quad \operatorname{conv}(\bigcap_{P \in \mathcal{P}} B(P)) = \bigcap_{P \in \mathcal{P}} \operatorname{conv}(B(P)).$$

In particular, Corollary 6.8 says that the intersection of discrete polymatroids, all of the same rank, which satisfy strong exchange property is a discrete polymatroid with strong exchange property, provided that the intersection of their set of bases is non-empty. The following example shows that all the hypotheses in Corollary 6.7 are needed.

Examples 6.9 ([9]). (a) The intersection of discrete polymatroids is in general not a discrete polymatroid, even if they have the same rank and the intersection of their set of bases is non-empty. Consider the discrete polymatroids P_1 and P_2, whose sets of bases are:

$$B(P_1) = \{(1,0,1,0),(0,1,1,0),(0,1,0,1),(0,0,1,1)\},$$

$$B(P_2) = \{(1,0,1,0),(1,1,0,0),(0,1,0,1),(1,0,0,1)\}.$$

Then $B(P_1) \cap B(P_2) = \{(1,0,1,0),(0,1,0,1)\}$ does not satisfy the exchange property, so it is not the set of bases of a discrete polymatroid.

(b) The condition $\bigcap_{P\in\mathcal{P}} B(P) \neq \emptyset$ is essential, even if all $P \in \mathcal{P}$ satisfy the strong exchange property. Let P_1, P_2, P_3 be the discrete polymatroids, whose sets of bases are:

$$B(P_1) = \{(2,0,2),(3,0,1),(2,1,1)\},$$

$$B(P_2) = \{(2,1,1),(1,1,2),(1,2,1)\},$$

$$B(P_3) = \{(0,2,2),(0,3,1),(1,2,1)\}.$$

Then P_1, P_2 and P_3 satisfy the strong exchange property but

$$P_1 \cap P_2 \cap P_3 = \\ \{(1,1,1),(1,1,0),(1,0,1),(0,1,1),(0,0,2),(1,0,0),(0,1,0),(0,0,1),(0,0,0)\},$$

is not a discrete polymatroid.

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