An. Şt. Univ. Ovidius Constanţa Vol. 15(1), 2007, 33–46

REGULARITY FOR CERTAIN CLASSES OF MONOMIAL IDEALS*

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Abstract

We introduce a new class of monomial ideals, called strong Borel type ideals, and we compute the Mumford-Castelnouvo regularity for principal strong Borel type ideals. Also, we describe the $\mathbf{d}$-fixed ideals generated by powers of variables and we compute their regularity.

Introduction.

Let K be an infinite field, and let $S = K[x_1, ..., x_n], n \geq 2$ be the polynomial ring over K. Bayer and Stillman [2] note that a Borel fixed ideal I satisfies the following property $(I : x_j^{\infty}) = (I : (x_1, \ldots, x_j)^{\infty})$ for all $j = 1, \ldots, n$. Herzog, Popescu and Vladoiu state that a monomial ideal is of Borel type if it fulfill the previous condition. We mention that this concept appears also in [3, Definition 1.3] as the so called weakly stable ideal. In fact, Herzog, Popescu and Vladoiu notice that a monomial ideal I is of Borel type, if and only if for any monomial $u \in I$ and for any $1 \leq j < i \leq n$, there exists an integer $t > 0$ such that $x_j^t u / x_i^{\nu_i(u)} \in I$, where $\nu_i(u) > 0$ is the exponent of x_i in u. (See [7, Proposition 1.2].) This property suggest us to define the so called ideals of strong Borel type (Definition 1.1), or simply, (SBT)-ideals. In the first section, we give the explicit form of a principal (SBT)-ideal (Lemma 1.4) and we compute its regularity (Theorem 1.6).

Let $\mathbf{d} : 1 = d_0 | d_1 | \cdots | d_s$ be a strictly increasing sequence of positive integers. We say that $\mathbf{d}$ is a $\mathbf{d}$-sequence. In [4] it was proved that for any

Key Words: p-Borel ideals, Borel type ideals, Mumford-Castelnuovo regularity
2000 Mathematical Subject Classification: Primary: 13P10, Secondary: 13E10
Received: February, 2007
*This paper was supported by the CEEX Program of the Romanian Ministry of Education and Research, Contract CEX05-D11-11/2005 and by the Higher Education Commission of Pakistan

$a \in \mathbb{N}$ there exists a unique sequence of positive integers $a_0, a_1, \ldots, a_s$ such that: $a = \sum_{t=0}^{s} a_t d_t$ and $0 \leq a_t < \frac{d_{t+1}}{d_t}$, for any $0 \leq t < s$. The decomposition $a = \sum_{t=0}^{s} a_t d_t$ is called the **d**-decomposition of a. In particular, if $d_t = p^t$ we get the p-adic decomposition of a. Let $a, b \in \mathbb{N}$ and consider the decompositions $a = \sum_{t=0}^{s} a_t d_t$ and $b = \sum_{t=0}^{s} b_t d_t$. We say that $a \leq_{\mathbf{d}} b$ if $a_t \leq b_t$ for any $0 \leq t \leq s$. We say that a monomial ideal $I \subset S$ is **d**-fixed, if for any monomial $u \in I$ and for any indices $1 \leq j < i \leq n$, if $t \leq_{\mathbf{d}} \nu_i(u)$ then $u \cdot x_j^t / x_i^t \in I$ (see [4, Definition 1.4]).

In [4], it was proved a formula for the regularity of a principal **d**-fixed ideal, i.e the smallest **d**-fixed ideal which contains a given monomial $u \in S$. This formula generalizes the Pardue's formula for the regularity of a principal p-Borel ideal, proved in [1] and [8], and later in [7]. In the section 2, we describe the **d**-fixed ideals generated by powers of variables (Proposition 2.2) and we give a formula for their regularity (Corollary 2.8).

The author is grateful to his adviser Dorin Popescu for his encouragement and valuable suggestions. He owes special thanks to Assistant Professor Alin Ştefan for valuable discussions on Section 2 of this paper. My thanks go also to the School of Mathematical Sciences, GC University, Lahore, Pakistan for supporting and facilitating this research.

1 Monomial ideals of strong Borel type.

Let K be an infinite field, and let $S = K[x_1, ..., x_n], n \geq 2$, be the polynomial ring over K.

Definition 1.1. *We say that a monomial ideal $I \subset S$ is of strong Borel type (SBT) if for any monomial $u \in I$ and for any $1 \leq j < i \leq n$, there exists an integer $0 \leq t \leq \nu_i(u)$ such that $x_j^t u / x_i^{\nu_i(u)} \in I$, where $\nu_i(u) > 0$ is the exponent of x_i in u.*

Remark 1.2. *Obviously, an ideal of strong Borel type is also an ideal of Borel type, but the converse is not true. Take for instance $I = (x_1^3, x_2^2) \subset K[x_1, x_2]$.*

The sum of two ideals of (SBT) is still an ideal of (SBT). The same is true for an intersection or a product of two ideals of (SBT).

Definition 1.3. *Let $\mathcal{A} \subset S$ be a set of monomials. We say that I is the (SBT)-ideal generated by $\mathcal{A}$, if I is the smallest, with respect to inclusion, ideal of (SBT) containing $\mathcal{A}$. We write $I = SBT(\mathcal{A})$.*

In particular, if $\mathcal{A} = \{u\}$, where $u \in S$ is a monomial, we say that I is the principal (SBT)-ideal generated by u, and we write $I = SBT(u)$.

Lemma 1.4. *Let $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ be some integers, $\alpha_1, \ldots, \alpha_r$ be some positive integers and $u = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_r}^{\alpha_r} \in S$. Then, the principal*

(SBT)-ideal generated by u, is:

$$I = SBT(u) = \prod_{q=1}^{r} (\mathbf{m}_q^{[\alpha_q]}),$$

where

$$\mathbf{m}_q = \{x_1, \ldots, x_{i_q}\} \text{ and } \mathbf{m}_q^{[\alpha_q]} = \{x_1^{\alpha_q}, \ldots, x_{i_q}^{\alpha_q}\}.$$

Proof. Denote $I' = \prod_{q=1}^{r} (\mathbf{m}_q^{[\alpha_q]})$. If v is a minimal monomial generator of I', then $v = x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_r}^{\alpha_r}$, for some $1 \leq j_q \leq i_q$, where $1 \leq q \leq r$.
Since

$$v = \frac{x_{j_r}^{\alpha_r}}{x_{i_r}^{\alpha_r}} \cdots \frac{x_{j_2}^{\alpha_2}}{x_{i_2}^{\alpha_2}} \cdot \frac{x_{j_1}^{\alpha_1}}{x_{i_1}^{\alpha_1}} u,$$

and I is of (SBT) it follows that $v \in I$ and thus $I' \subseteq I$. For the converse, simply notice that I' is itself an (SBT)-ideal. $\square$

Remark 1.5. *For any monomial ideal $I \subset S$, we denote $m(I) = \max\{m(u) : u \in G(I)\}$, where $G(I)$ is the set of the minimal generators of I and $m(u) = \max\{i : x_i | u\}$. Also, if M is a graded S-module of finite length, we denote $s(M) = \max\{t : M_t \neq 0\}$.*

Let $I \subset S$ be a Borel type ideal. In [7], it is defined a chains of ideals $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ as follows. We let $I_0 = I$. Suppose I_ℓ is already defined. If $I_\ell = S$ then the chain ends. Otherwise, we let $n_\ell = m(I_\ell)$ and set $I_{\ell+1} = (I_\ell : x_{n_\ell}^\infty)$. Notice that $r \leq n$, since $n_\ell > n_{\ell+1}$ for all $0 \leq \ell < r$. The chain $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ is called the sequential chain *of I. [7, Corollary 2.5] states that*

$$(1) \quad I_{\ell+1}/I_\ell \cong (J_\ell^{sat}/J_\ell)[x_{n_\ell+1}, \ldots, x_n],$$

for all $0 \leq \ell < r$, where $J_\ell \subset S_\ell = K[x_1, \ldots, x_{n_\ell}]$ is the ideal generated by $G(I_\ell)$. Also, [7, Corollary 2.5] gives a formula for the regularity of I, more precisely,

$$(2) \quad reg(I) = \max\{s(J_0^{sat}/J_0), s(J_1^{sat}/J_1), \cdots, s(J_{r-1}^{sat}/J_{r-1})\} + 1.$$

Our next goal is to give a formula for the regularity of a principal (SBT)-ideal. In order to do it, we shall use the previous remark.

Let $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ be some integers, $\alpha_1, \ldots, \alpha_r$ be some positive integers and $u = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_r}^{\alpha_r} \in S$. For each $1 \leq q \leq r$, $1 \leq f \leq q$ with $\alpha_f \leq \alpha_q$ and $1 \leq j \leq i_q$, we define the numbers:

$$\chi_{qj}^{(f)} := \begin{cases} \alpha_j + \alpha_q - 1, & if\ j < q \text{ and } \alpha_j \geq \alpha_f \\ \alpha_f - 1, & otherwise \end{cases},$$

$$\chi_q^{(f)} := \sum_{j=1}^{i_q} \chi_{qj}^{(f)} \text{ and } \chi_q = \max_f \chi_q^{(f)}.$$

Theorem 1.6. *With the above notations, we have* $reg(SBT(u)) = \max_{q=1}^{r} \chi_q + 1$.

Proof. Firstly, we describe the sequential chain of I.
Since $I_r := I = \prod_{q=1}^{r}(\mathbf{m}_q^{[\alpha_q]})$, it follows that $I_{r-1} := (I_r : x_{i_r}^{\infty}) = \prod_{q=1}^{r-1}(\mathbf{m}_q^{[\alpha_q]})$. Analogously, we get $I_q := (I_{q+1} : x_{i_{q+1}}^{\infty}) = \prod_{e=1}^{q}(\mathbf{m}_e^{[\alpha_e]})$, for all $0 \leq q < r$. Therefore, the sequential chain of I is

$$I = I_r \subset I_{r-1} \subset \cdots \subset I_1 \subset I_0 = S.$$

Let J_q be the ideal of $S_q = K[x_1, \ldots, x_{i_q}]$ generated by $G(I_q)$, for $1 \leq q \leq r$. Denoting $s_q = s(J_q^{sat}/J_q)$, (2) from Remark 1.5 implies $reg(I) = \max\{s_q : 1 \leq q \leq r\}$, so, in order to compute the regularity of I, we must determine the numbers s_q. We claim that $s_q = \chi_q$.

First of all, note that $J_q = I_q \cap S_q$ and $J_q^{sat} = I_{q-1} \cap S_q$. Let $1 \leq f \leq q$ with $\alpha_f \leq \alpha_q$ and $w = x_1^{\chi_{q1}^{(f)}} \cdots x_{i_q}^{\chi_{q,i_q}^{(f)}}$. Since $\chi_{qe}^{(f)} \geq \alpha_e$ for any $1 \leq e \leq q-1$ we get $x_1^{\chi_{q1}^{(f)}} \cdots x_{q-1}^{\chi_{q,q-1}^{(f)}} \in J_q^{sat} = \prod_{e=1}^{q-1}(\mathbf{m}_e^{[\alpha_e]})S_q$, therefore $w \in J_q^{sat}$. On the other hand, one can easily see that $w \notin J_q$, so w is a nonzero element in J_q^{sat}/J_q with $\deg(w) = \chi_q$, thus $s_q \geq \chi_q$.

In order to prove the converse inequality, we consider a monomial $u \in J_q^{sat}$ with $\deg(u) \geq \chi_q + 1$ and we show that $u \in J_q$. Assume, by contradiction, that $u \notin J_q$. Since $u \in J_q^{sat}$, it follows that $u = x_{j_1}^{\alpha_1} \cdots x_{j_{q-1}}^{\alpha_{q-1}} \cdot x_1^{\beta_1} \cdots x_{i_q}^{\beta_{i_q}}$, where $1 \leq j_e \leq i_e$ for $1 \leq e \leq q-1$ and $\beta_1 + \cdots + \beta_{i_q} \geq \chi_q - \sum_{e=1}^{q-1} \alpha_e$. Let $A = \{1, \ldots, i_q\} \setminus \{j_1, \ldots, j_{q-1}\}$. Since $u \notin J_q$ and $x_{j_1}^{\alpha_1} \cdots x_{j_{q-1}}^{\alpha_{q-1}} \in J_q^{sat}$ it follows $\beta_j \leq \alpha_q - 1$ for all $j \in A$.

Write $\{1, \ldots, q-1\} = \cup_{i=1}^{m} E_i$, where $E_i = \{e_{i1}, \ldots, e_{ik_i}\}$, such that $j_{e_{ik}} = j_{e_i}$ for all $1 \leq k \leq k_i$ and $E_i \cap E_{i'} = \emptyset$ whenever $i \neq i'$. With these notations,

$$u = x_{j_{e_1}}^{\alpha_{e_{11}} + \cdots + \alpha_{e_{1k_1}} + \beta_{j_{e_1}}} \cdots x_{j_{e_m}}^{\alpha_{e_{m1}} + \cdots + \alpha_{e_{mk_m}} + \beta_{j_{e_m}}} \cdot \prod_{j \in A} x_j^{\beta_j}.$$

Let $1 \leq f \leq q$ be such that $\alpha_f \leq \alpha_q$, $\beta_j < \alpha_f$ for all $j \in A$ and α_f be the largest integer among all the $\alpha_{f'}$, with f' satisfying the above conditions. Suppose that there exist some $1 \leq i \leq m$ and $1 \leq k \leq k_i$ such that $\alpha_{e_{ik}} < \alpha_q$. It follows that $\beta_{j_{e_i}} \leq \alpha_f - \alpha_{e_{ik}} - 1$, otherwise $u \in J_q$. One can immediately conclude that $\sum_{e=1}^{q-1} \alpha_e + \sum_{j=1}^{i_q} \beta_j \leq \chi_q^{(f)}$. □

Example 1.7. Let $u = x_2^6 x_3^7 \in S = K[x_1, x_2, x_3]$. From Lemma 1.4 it follows that $I = SBT(u) = (x_1^6, x_2^6)(x_1^7, x_2^7, x_3^7)$. With the notations of 1.5 and 1.6, we have $J_1 = (x_1^6, x_2^6) \subset K[x_1, x_2]$ and $J_2 = I$. Also, $J_1^{sat} = K[x_1, x_2]$ and $J_2^{sat} = (x_1^6, x_2^6) \subset S$. Obviously, $\chi_1 = \chi_1^{(1)} = 2 \cdot 5 = 10$, i.e. $s(J_1^{sat}/J_1) = s(K[x_1, x_2]/(x_1^6, x_2^6)) = 10$. We have $\chi_2^{(1)} = (6 + 7 - 1) + 2 \cdot 5 = 23$ and $\chi_2^{(2)} = 3 \cdot 6 = 18$, therefore $\chi_2 = 23$ and thus $reg(I) = \max\{10, 23\} + 1 = 24$.

In the end of this section, we mention the following result, which generalizes a result of Eisenbud-Reeves-Totaro (see [6, Proposition 12]).

Proposition 1.8. *[5, Corollary 8] If I is a Borel type ideal, then*

$$reg(I) = \min\{e : \ e \geq \deg(I), \ I_{\geq e} \textit{ is stable}\},$$

where $\deg(I)$ *is the maximal degree of a minimal monomial generator of I.*

In particular, this holds for (SBT)-ideals, and thus we get the following corollary.

Corollary 1.9. *With the notations of Theorem* 1.5*, if $I = SBT(u)$ and $e \geq \max_{q=1}^{r} \chi_q + 1$, then $I_{\geq e}$ is stable.*

Remark 1.10. *Note also that the regularity of an (SBT)-ideal, $I \subset S$, is upper bounded by $n(\deg(I) - 1) + 1$, (see [9, Theorem 2.2]). In fact,* $\deg(I)$ *is the maximum degree of a minimal generator of I as an (SBT)-ideal!*

2 d-fixed ideals generated by powers of variables.

Let us fix some notations. Let $u_1, \ldots, u_m \in S$ be some monomials. We say that I is the **d**-fixed ideal generated by $u_1, \ldots, u_m$, if I is the smallest **d**-fixed ideal , w.r.t inclusion, which contains $u_1, \ldots, u_m$, and we write

$$I = < u_1, \ldots, u_m >_{\mathbf{d}} .$$

In particular, if $m = 1$, we say that I is the principal **d**-fixed ideal generated by $u = u_1$ and we write $I = < u >_{\mathbf{d}}$.

In the case when I is a principal **d**-fixed ideal , [4, Theorem 3.1] gives a formula for the Castelnuovo-Mumford regularity of I. Using similar techniques as in [4], we shall compute the regularity for **d**-fixed ideals generated by powers of variables. We recall some results proved in [4] which are useful. Let α be a positive integer and let $I = < x_n^{\alpha} >_{\mathbf{d}} \subset S = K[x_1, \ldots, x_n]$. Suppose $\alpha = \sum_{t=0}^{s} \alpha_t d_t$ with $\alpha_s \neq 0$. Then:

- $I = \prod_{t=0}^{s} (\mathbf{m}^{[d_t]})^{\alpha_t}$, where $\mathbf{m} = \{x_1, \ldots, x_n\}$ and $\mathbf{m}^{[d]} = \{x_1^d, \ldots, x_n^d\}$ [4, 1.6].

- $Soc(S/I) = (J + I)/I$, with

$$J = \sum_{t=0}^{s} (x_1 \cdots x_n)^{d_t - 1} (\mathbf{m}^{[d_t]})^{\alpha_t - 1} \prod_{j>t} (\mathbf{m}^{[d_j]})^{\alpha_j} [4, 2.1].$$

- $reg(I) = \max\{e : ((J + I)/I)_e \neq 0\} = \alpha_s d_s + (n-1)(d_s - 1)$ (see [4, 3.1]).

- If $e \geq reg(I)$ then $I_{\geq e}$ is stable (see [4, 3.6] or apply Proposition 1.8, since any d-fixed ideal is of Borel type, see [4, 1.11]).

Lemma 2.1. *If $1 \leq j \leq j' \leq n$ and $\alpha \geq \beta$ are positive integers, then $< x_j^\alpha > \subset < x_{j'}^\beta >$.*

Proof. Indeed, using [4, 1.7] it is enough to notice that $< x_j^\alpha > \subset < x_{j'}^\alpha >$, since $x_j^\alpha \in < x_{j'}^\alpha >$. □

Our next goal is to give the set of the minimal generators of a **d**-fixed ideal generated by some powers of variables. Using the previous lemma, we had reduced to the next case:

Proposition 2.2. *Let $n \geq 2$ and let $1 \leq i_1 < i_2 < \cdots < i_r = n$ be some integers. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ be some positive integers. Then*

$$I = < x_{i_1}^{\alpha_1}, x_{i_2}^{\alpha_2}, \ldots, x_{i_r}^{\alpha_r} >_{\mathbf{d}} = \sum_{q=1}^{r} I^{(q)},$$

with

$$I^{(q)} = \sum_{\substack{\gamma_1, \ldots, \gamma_q \leq_{\mathbf{d}} \alpha_q, \\ \gamma_1 + \cdots + \gamma_i < \alpha_i, \ for\ i < q \\ \gamma_1 + \cdots + \gamma_i <_d \alpha_q, \ for\ i < q \\ \gamma_1 + \cdots + \gamma_q = \alpha_q}} \prod_{e=1}^{q} \prod_{t=0}^{s} (\mathbf{n}_e^{[d_t]})^{\gamma_{et}},$$

where $\mathbf{n}_e = \{x_{i_{e-1}+1}, \ldots, x_{i_e}\}$, $\mathbf{n}_e^{[d_t]} = \{x_{i_{e-1}+1}^{d_t}, \ldots, x_{i_e}^{d_t}\}$, $i_0 = 0$ and $\gamma_e = \sum_{t=0}^{s} \gamma_{et} d_t$.

Proof. Let $\mathbf{m}_q = \{x_1, \ldots, x_{i_q}\}$ for $1 \leq q \leq r$. Obviously, $\mathbf{n}_q = \mathbf{m}_q \setminus \mathbf{m}_{q-1}$ for $q > 1$ and $\mathbf{m}_1 = \mathbf{n}_1$. Using the simple fact that I is the sum of principal

d-fixed ideals generated by the **d**-generators of I together with [4, Proposition 1.6], we get:

$$I = \sum_{q=1}^{r} \prod_{t=0}^{s} (\mathbf{m}_q^{[d_t]})^{\alpha_{qt}}, \ where \ \alpha_q = \sum_{t=0}^{s} \alpha_{qt} d_t.$$

Denote $S_q = K[x_1, \ldots, x_{i_q}]$ for $1 \leq q \leq r$. In order to obtain the required formula, we use induction on $r \geq 1$, the case $r = 1$ being obvious. Let $r > 1$ and assume that the assertion is true for $r - 1$, i.e

$$I' =< x_{i_1}^{\alpha_1}, \ldots, x_{i_{r-1}}^{\alpha_{r-1}} >_{\mathbf{d}} =$$

$$= \sum_{q=1}^{r-1} \sum_{\substack{\gamma_1, \ldots, \gamma_q \leq_{\mathbf{d}} \alpha_q, \\ \gamma_1 + \cdots + \gamma_i < \alpha_i, \ for \ i < q \\ \gamma_1 + \cdots + \gamma_i <_d \alpha_q, \ for \ i < q \\ \gamma_1 + \cdots + \gamma_q = \alpha_q}} \prod_{e=1}^{q} \prod_{t=0}^{s} (\mathbf{n}_e^{[d_t]})^{\gamma_{et}} \subset S_{r-1}.$$

Obviously, $I = I'S + < x_n^{\alpha_r} >_{\mathbf{d}} = I'S + \prod_{t=0}^{s} (\mathbf{m}_r^{[d_t]})^{\alpha_{rt}}$. Also, $I'S$ and I' have the same set of minimal generators and none of the minimal generators of $I'S$ is in $I^{(r)}$. But, a minimal generator of $< x_n^{\alpha_r} >_{\mathbf{d}}$ is of the form $w = \prod_{t=0}^{s} \prod_{j=1}^{n} x_j^{\lambda_{tj} d_t}$ with $0 \leq \lambda_{tj}$ and $\sum_{j=1}^{n} \lambda_{tj} = \alpha_{rt}$. Suppose $w \notin I'S$. In order to complete the proof, we shall show that $w \in I^{(r)}$. Let $v_q = \prod_{t=0}^{s} \prod_{j=i_{q-1}+1}^{i_q} x_j^{\lambda_{tj} d_t}$ and let $w_q = \prod_{e=1}^{q} v_e$. Obviously, $w = v_1 \cdots v_r = w_r$. Since $w \notin I'$ it follows that $w_q \notin I^{(q)}$ for any $1 \leq q \leq r-1$. But $w_q \notin I^{(q)}$ implies $(*)$ $\sum_{t=0}^{s} \sum_{j=1}^{i_q} \lambda_{tj} d_t < \alpha_q$, otherwise $w_q \in < x_{i_q}^{\alpha_q} S_q >_{\mathbf{d}} S_{r-1} \subset I'$ and thus $w \in I'$, a contradiction. We choose $\gamma_e = \sum_{t=0}^{s} \sum_{j=i_{e-1}+1}^{i_e} \lambda_{tj} d_t$ for $1 \leq e \leq r$. For $1 \leq q < r$, the inequality $(*)$ implies $\gamma_1 + \cdots + \gamma_q < \alpha_q$. On the other hand, it is obvious that $\gamma_1 + \cdots + \gamma_e \leq_d \alpha_r$ for any $1 \leq e \leq r$ and $\gamma_1 + \cdots + \gamma_r = \alpha_r$. Thus $w \in I^{(r)}$ as required. $\square$

Example 2.3. *Let* $\mathbf{d} : 1|2|4|12$ *and let* $I =< x_2^7, x_3^{10}, x_5^{17} >_{\mathbf{d}} \subset K[x_1, \ldots, x_5]$. *We have* $7 = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 4$, $10 = 1 \cdot 2 + 2 \cdot 4$, $17 = 1 \cdot 1 + 1 \cdot 4 + 1 \cdot 12$. *We have*

$$I^{(1)} =< x_2^7 >_{\mathbf{d}} = (x_1, x_2)(x_1^2, x_2^2)(x_1^4, x_2^4).$$

In order to compute $I^{(2)}$, *we need to find all the pairs* (γ_1, γ_2) *such that* $\gamma_1 < 7$, $\gamma_1 <_{\mathbf{d}} 10$ *and* $\gamma_2 = 10 - \gamma_1$. *We have* 4 *pairs, namely* $(0, 10)$, $(2, 8)$, $(4, 6)$ *and* $(6, 4)$, *thus*

$$I^{(2)} = (x_1^2, x_2^2)(x_1^4, x_2^4)x_3^4 + (x_1^4, x_2^4)x_3^6 + (x_1^2, x_2^2)x_3^8 + (x_3^{10}).$$

In order to compute $I^{(3)}$, we need to find all $(\gamma_1,\gamma_2,\gamma_3)$ such that $\gamma_1 < 7$, $\gamma_1+\gamma_2 < 10$, $\gamma_1 <_{\mathbf{d}} 17$, $\gamma_1+\gamma_2 <_{\mathbf{d}} 17$ and $\gamma_3 = 17-\gamma_1+\gamma_2$. If $\gamma_1 = 0$ then, the pair (γ_2,γ_3) is one of the following pairs: $(0,17)$,$(1,16)$,$(4,13)$ or $(5,12)$. If $\gamma_1 = 1$ then, the pair (γ_2,γ_3) is one of the following pairs: $(0,16)$ and $(4,12)$. If $\gamma_1 = 4$ then, the pair (γ_2,γ_3) is one of the pairs: $(0,13)$ and $(1,12)$. If $\gamma_1 = 5$ then, the pair (γ_2,γ_3) is $(0,12)$. Thus

$$I^{(3)} = (x_1,x_2)(x_1^4,x_2^4)(x_4^{12},x_5^{12}) + (x_1^4,x_2^4)x_3(x_4^{12},x_5^{12})+$$
$$+(x_1^4,x_2^4)(x_4,x_5)(x_4^{12},x_5^{12}) + (x_1,x_2)x_3^4(x_4^{12},x_5^{12})+$$
$$+(x_1,x_2)(x_4^4,x_5^4)(x_4^{12},x_5^{12}) + x_3(x_4^4,x_5^4)(x_4^{12},x_5^{12})+$$
$$+x_3^4(x_4,x_5)(x_4^{12},x_5^{12}) + x_3^5(x_4^{12},x_5^{12}) + (x_4,x_5)(x_4^4,x_5^4)(x_4^{12},x_5^{12}).$$

By Proposition 2.2, we get $I = I^{(1)} + I^{(2)} + I^{(3)}$.

Remark 2.4. *For any $1 \le q \le r$ and any nonnegative integers $\gamma_1,\dots,\gamma_q \le_{\mathbf{d}} \alpha_q$ such that $\gamma_1+\cdots+\gamma_i < \alpha_i$, $\gamma_1+\cdots+\gamma_i <_{\mathbf{d}} \alpha_q$ for $1 \le i < q$ and $\gamma_1+\cdots+\gamma_q = \alpha_q$ we denote $I^{(q)}_{\gamma_1,\dots,\gamma_q} = \prod_{e=1}^{q}\prod_{t=0}^{s}(\mathbf{n}_e^{[d_t]})^{\gamma_{et}}$. Proposition 2.2 implies:*

$$I = \sum_{q=1}^{r}\sum_{\gamma_1,\dots,\gamma_q} I^{(q)}_{\gamma_1,\dots,\gamma_q}.$$

Let $\mathbf{m} = (x_1,\dots,x_n) \subset S$ be the irrelevant ideal of S. We have:

$$(I :_S \mathbf{m}) = \bigcap_{j=1}^{n}(I : x_j) = \bigcap_{j=1}^{n}((\sum_{q=1}^{r}\sum_{\gamma_1,\dots,\gamma_q} I^{(q)}_{\gamma_1,\dots,\gamma_q}) : x_j) =$$
$$= \bigcap_{j=1}^{n}(\sum_{q=1}^{r}\sum_{\gamma_1,\dots,\gamma_q}(I^{(q)}_{\gamma_1,\dots,\gamma_q} : x_j)).$$

On the other hand, if $x_j \in \mathbf{n}_p$ for some $1 \le p \le q$, then

$$J^{(q),j}_{\gamma_1,\dots,\gamma_q} := (I^{(q)}_{\gamma_1,\dots,\gamma_q} : x_j) =$$
$$= \prod_{e\neq p}^{q}\prod_{t=0}^{s}(\mathbf{n}_e^{[d_t]})^{\gamma_{et}}\mathbf{n}_{\mathbf{p},\hat{\mathbf{j}}}{}^{[d_t]}(\mathbf{n}_{\mathbf{p}}{}^{[d_t]})^{\gamma_{pt}-1}(\sum_{\gamma_{pt}>0}\prod_{j\neq t}(\mathbf{n}_e^{[d_t]})^{\gamma_{jt}}),$$

where $\mathbf{n}_{\mathbf{p},\hat{\mathbf{j}}}{}^{[d_t]} = (x_{i_{p-1}+1}^{d_t},\dots,x_j^{d_t-1},\dots,x_{i_p}^{d_t})$ and $\mathbf{n}_{\mathbf{p},\hat{\mathbf{j}}}{}^{[d_t]}(\mathbf{n}_{\mathbf{p}}{}^{[d_t]})^{\gamma_{pt}-1} := S$ if $\gamma_{pt} = 0$. Thus

$$(I :_S \mathbf{m}) = \sum_{q^1=1}^{r}\sum_{\gamma_1^1,\dots,\gamma_{q^1}^1}\cdots\sum_{q^n=1}^{r}\sum_{\gamma_1^n,\dots,\gamma_{q^n}^n}\bigcap_{j=1}^{n} J^{(q^j),j}_{\gamma_1^j,\dots,\gamma_{q^j}^j},$$

where, for a given $q = q^j$, we take the second j^{th} sum for $\gamma_1^j, \ldots, \gamma_q^j \leq_{\mathbf{d}} \alpha_q$ such that $\gamma_1^j + \cdots + \gamma_i^j < \alpha_i$, $\gamma_1^j + \cdots + \gamma_i^j <_{\mathbf{d}} \alpha_q$ for $1 \leq i < q^j$ and $\gamma_1^j + \cdots + \gamma_q^j = \alpha_q$.

Proposition 2.5. *Let $n \geq 2$ and let $1 \leq i_1 < i_2 < \cdots < i_r = n$ be some integers. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ be some positive integers. We consider the ideal $I = \sum_{q=1}^r I_q$, where $I_q = < x_{i_q}^{\alpha_q} >_{\mathbf{d}}$. Then, we have: $reg(I) \leq reg(I_r)$* (We will see later in which conditions we have equality).

Proof. From [4, Corollary 3.6] it follows that $(I_q)_{\geq e}$ is stable, if $e \geq reg(I_q)$ so $(I_q)_{\geq e}$ is stable for $e = \max\{reg(I_1), \ldots, reg(I_r)\}$. Since $I_{\geq e} = \sum_{q=1}^r (I_q)_{\geq e}$ and since a sum of stable ideals is still a stable ideal, it follows that $I_{\geq e}$ is stable. Therefore, from [6, Proposition 12], we get $reg(I) \leq e$. On the other hand, if we denote $s_q = \max\{t | \alpha_{qt} > 0\}$ for any $1 \leq q \leq r$, from [4, Theorem 3.1] we get $reg(I_q) = \alpha_{qs_q} d_{s_q} + (i_q - 1)(d_{s_q} - 1)$, thus $\max\{reg(I_1), \ldots, reg(I_r)\} = reg(I_r)$. In conclusion, $reg(I) \leq reg(I_r)$. □

Proposition 2.6. *With the above notations, for any $1 \leq q \leq r$ we have:*

$$(I_q : \mathbf{m}_q) + (I_1 + \cdots + I_q) \subset ((I_1 + \cdots + I_q) : \mathbf{m}_q) \subset$$

$$\subset ((I_1 + \cdots + I_q) : \mathbf{n}_q) = (I_q : \mathbf{n}_q) + (I_1 + \cdots + I_q).$$

Proof. Fix $1 \leq q \leq r$. The first two inclusions are obvious. In order to prove the last equality, it is enough to show that

$$((I_1 + \cdots + I_q) : x_j) \subset (I_q : x_j) + (I_1 + \cdots + I_q),$$

for any $x_j \in \mathbf{n}_q$. Indeed, suppose $u \in ((I_1 + \cdots + I_q) : x_j)$, therefore $x_j \cdot u \in I_1 + \cdots + I_q$. If $x_j \cdot u \notin I_q$ it follows that $x_j \cdot u \in I_e$ for some $e < q$. Thus $u \in I_e$, since x_j does not divide any minimal generator of I_e. □

Let $n \geq 2$ and let $1 \leq i_1 < i_2 < \cdots < i_r = n$ be some integers. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ be some positive integers. We write $\alpha_q = \sum_{t \geq 0} \alpha_{qt} d_t$. Let $s_q = \max\{t | \alpha_{qt} > 0\}$ for any $1 \leq q \leq r$. Notice that $s_1 \leq s_2 \leq \cdots \leq s_r$. Indeed, assume, by contradiction, that there exist $q < q'$ such that $s_q > s_{q'}$. Then, from the $\mathbf{d}$ - decomposition of $\alpha_{q'}$ and α_q, we have

$$\alpha_{q'} = \sum_{t=0}^{s_{q'}} \alpha_{q't} d_t \leq \sum_{t=0}^{s_{q'}} (\frac{d_{t+1}}{d_t} - 1) d_t = d_{s_{q'}+1} - d_0 \leq d_{s_{q'}+1} \leq d_{s_q} \leq \alpha_q,$$

absurd.
Let $1 \leq q_1 < q_2 < \cdots < q_k = r$ be such that:

$$s_1 = \cdots = s_{q_1} < s_{q_1+1} = \cdots = s_{q_2} < \cdots < s_{q_{k-1}+1} = \cdots = s_{q_k}.$$

For $1 \leq j \leq k$, we define some positive integers χ_j as follows. If $i_{q_j} - i_{q_j-1} \geq 2$, we put $\chi_j = (d_{s_{q_j}} - 1)(i_{q_j} - i_{q_{j-1}}) + d_{s_{q_j}}(\alpha_{q_j s_{q_j}} - 1)$. Otherwise, suppose that $q = q_j$ and there exists a positive integer $1 \leq l \leq r - q + 1$ such that $s_{q-1} < s_q < \cdots < s_{q+l-1}$ and $i_{q+l-1} = i_{q-1} + l$. Denote $i = i_q$. We define recursively the numbers χ_{i+m-1}, for $1 \leq m \leq l$, starting with $m = l$. Suppose that we have already defined $\chi_{i+m}, \ldots, \chi_{i+l-1}$. If $\alpha_{q+m-2,s_{q+m-2}} > \alpha_{q+m-1,s_{q+m-1}}$, we put $\chi_{q+m-1} := \sum\limits_{t=s_{q+m-2}+1}^{s_{q+m-1}} \alpha_{q+m-1,t} d_t - 1$ and we switch from m to $m-1$. Otherwise, if $\alpha_{q+m-2,s_{q+m-2}} \leq \alpha_{q+m-1,s_{q+m-1}}$ we put

$$\chi_{q+m-1} := (\alpha_{q+m-1,s_{q+m-2}} - \alpha_{q+m-2,s_{q+m-2}} + 1) \cdot d_{s_{q+m-2}} +$$

$$+ \sum_{t=s_{q+m-2}+1}^{s_{q+m-1}} \alpha_{q+m-1,t} d_t - 1$$

and, if $m \geq 2$, we put also $\chi_{q+m-2} := \alpha_{q+m-2,s_{q+m-2}} \cdot d_{s_{q+m-2}} - 1$. We switch from m to $m-2$. We continue this procedure until $m \leq 0$.

With these notations, for the ideal $I =< x_{i_1}^{\alpha_1}, x_{i_2}^{\alpha_2}, \ldots, x_{i_r}^{\alpha_r} >_{\mathbf{d}}$, we have the following theorem:

Theorem 2.7. $\max\{e : \ (Soc(S/I))_e \neq 0\} = \sum_{j=1}^{k} \chi_j$.

Proof. For each integer $1 \leq j \leq k$, we consider the following ideal:

$$J_j = \begin{cases} (x_{i_{q_j}}^{\chi_j}), & if\ i_{q_j} - i_{q_j-1} = 1, \\ (x_{i_{q_{j-1}}+1} \cdots x_{i_{q_j}})^{d_{s_{q_j}}-1} \cdot \sum_{e=q_{j-1}+1}^{q_j} (\mathbf{n}_e^{[d_{s_{q_j}}]})^{\alpha_{e s_e}-1}, & otherwise. \end{cases}$$

Let $J = J_1 \cdot J_2 \cdots J_k$. We claim the following:

(1) $J \subset (I : \mathbf{m})$,

(2) $G(J) \cap G(I) = \emptyset$,

(3) $\max\{e|\ (Soc(S/I))_e \neq 0\} = \max\{e|\ ((J+I)/I)_e \neq 0\}$.

Suppose that we proved $(1), (2)$ and (3). (1) and (2) implies

$$\max\{e|\ ((J+I)/I)_e \neq 0\} = \deg(J) := \max\{\deg(u)|\ u \in G(J)\}.$$

On the other hand, it is obvious that $\deg(J) = \sum_{j=1}^{k} \chi_j$ and thus, by (3), we complete the proof of the theorem.

In order to prove (1), we pick $x_i \in \mathbf{n}_q$ a variable, where $q \in \{1, \ldots, r\}$. Let j be the unique integer with the property that $q \in \{q_{j-1}+1, \ldots, q_j\}$. We want to show that $x_i \cdot J \subset I$. We consider two cases. First, we assume $i_{q_j} - i_{q_{j-1}} \geq 2$. We claim that $x_i J_j \subset I_{q_{j-1}+1} + \cdots + I_{q_j}$. Indeed, for any $e \in \{q_{j-1}+1, \ldots, q_j\}$,

$x_i(x_{i_{q_{j-1}}+1}\cdots x_{i_{q_j}})^{d_{s_{q_j}}-1}(\mathbf{n}_e^{[d_{s_{q_j}}]})^{\alpha_{es_e}-1} \subset I_e$, thus $x_iJ_j \subset I_{q_{j-1}+1}+\cdots+I_{q_j}$, as required. (See the proof of [4, Lema 2.1] for details.)

Suppose now $i_{q_j}-i_{q_j-1}=1$. Let $j'\leq j$, such that if we denote $q=q_{j'}$, there exists a positive integer $j-j'+1\leq l$ with $s_{q-1}<s_q<\cdots<s_{q+l-1}$, $i_{q+l-1}=i_{q-1}+l$ and $i_{q_{j'+l}}>i_{q+l-1}+1$ when $q+l-1<r$. We prove in fact that $x_i\cdot J_{j'}\cdots J_j\subset I_j$. Note that $i=i_{q+m-1}$, where $m=j-j'+1$. Assume $m\geq 2$. If $\alpha_{q+m-2,s_{q+m-2}}>\alpha_{q+m-1,s_{q+m-2}}$, then

$$x_i\cdot J_{q+m-2}J_{q+m-1}=(x_{i-1}^{\cdots+\alpha_{q+m-2,d_{s_{q+m-2}}}-1}\cdot x_i^{\sum_{t=s_{q+m-2}+1}^{s_{q+m-1}}\alpha_{q+m-1,t}d_t})\subset I_j,$$

because $\alpha_{q+m-2,d_{s_{q+m-2}}}-1\geq\alpha_{q+m-1,d_{s_{q+m-2}}}+d_{s_{q+m-2}}-1$ and therefore

$$x_i\cdot J_{q+m-2}J_{q+m-1}\subset(x_{i-1}^{d_{s_{q+m-2}}-1}\cdot x_{i-1}^{\alpha_{q+m-1,d_{s_{q+m-2}}}}\cdot x_i^{\sum_{t=s_{q+m-2}+1}^{s_{q+m-1}}\alpha_{q+m-1,t}d_t}).$$

Now, the above assertion is obvious. If $m=1$, the same trick works, with the only difference that the first "=" is replaced by "$\subseteq$".

If $m\geq 2$ and $\alpha_{q+m-2,s_{q+m-2}}\leq\alpha_{q+m-1,s_{q+m-2}}$, then $x_i\cdot J_{q+m-2}J_{q+m-1}$ is the ideal generated by the product of the monomial $x_{i-1}^{\alpha_{q+m-2,d_{s_{q+m-2}}}d_{s_{q+m-2}}-1}$ with

$$x_i^{(\alpha_{q+m-1,s_{q+m-2}}-\alpha_{q+m-2,s_{q+m-2}}+1)d_{s_{q+m-2}}+\sum\limits_{t=s_{q+m-2}+1}^{s_{q+m-1}}\alpha_{q+m-1,t}d_t}.$$

By regrouping, we see that $x_i\cdot J_{q+m-2}J_{q+m-1}=$

$$=(x_{i-1}^{d_{s_{q+m-2}-1}}\cdot(x_{i-1}^{(\alpha_{q+m-2,d_{s_{q+m-2}}}-1)d_{s_{q+m-2}}}\cdot$$

$x_i^{(\alpha_{q+m-1,s_{q+m-2}}-\alpha_{q+m-2,s_{q+m-2}}+1)d_{s_{q+m-2}}})\cdot x_i^{\sum\limits_{t=s_{q+m-2}+1}^{s_{q+m-1}}\alpha_{q+m-1,t}d_t})\subset I_j$, as required. If $m=1$ the same trick works, with the only difference that the first "=" is replaced by "$\subseteq$".

In order to prove (2) it is enough to show for any $1\leq j\leq k$ that $G(J_1\cdots J_j)\cap G(I_e)=\emptyset$ for any $e\in\{q_{j-1}+1,\ldots,q_j\}$, because each of the minimal generators of $J_1\cdots J_j$ does not contain variables x_i with $i>i_{q_j}$. We use induction on $1\leq j\leq k$. If $j=1$, then $G(J_1)\cap G(I_1)=\emptyset$ from [4, Lemma 2.1]. Suppose the assertion is true for $j-1$. We must consider two cases.

First, suppose $i_{q_j}-i_{q_{j-1}}\geq 2$. It follows $J_j=(x_{i_{q_{j-1}}+1}\cdots x_{i_{q_j}})^{d_{s_{q_j}}-1}\cdot\sum_{e=q_{j-1}+1}^{q_j}(\mathbf{n}_e^{[d_{s_{q_j}}]})^{\alpha_{es_e}-1}$. Since $s_{q_{j-1}}<s_{q_j}$, it follows that $J_1\cdots J_{j-1}\cdot J_j\subset$

$(x_1,\ldots,x_{i_{q_{j-1}}})^{d_{s_{q_j}}-1}J_j$, and it is easy to note that none of the minimal generator of the ideal from left is included in some I_e with $q_{j-1}+1 \leq e \leq q_j$.

Suppose now $i_{q_j} - i_{q_{j-1}} = 1$. Let $j' \leq j$, such that if we denote $q = q_{j'}$, there exists an positive integer $j - j' + 1 \leq l$ with $s_{q-1} < s_q < \cdots < s_{q+l-1}$, $i_{q+l-1} = i_{q-1} + l$ and $i_{q_{j'+l}} > i_{q+l-1} + 1$ when $q + l - 1 < r$. We prove in fact that $x_i \cdot J_{j'} \cdots J_j \subset I_j$. Note that $i = i_{q+m-1}$, where $m = j - j' + 1$. Assume $m \geq 2$. If $\alpha_{q+m-2,s_{q+m-2}} > \alpha_{q+m-1,s_{q+m-2}}$, then

$$J_1\cdots J_j = (J_1\cdots J_{j-2})\cdot (x_{i-1}^{\cdots+\alpha_{q+m-2,d_{s_{q+m-2}}}-1}\cdot x_i^{\sum_{t=s_{q+m-2}+1}^{s_{q+m-1}}\alpha_{q+m-1,t}d_t-1}) \subset$$

$$(x_1,\ldots,x_{i_{q_{j-2}}})^{d_{s_{q_{j-1}}}-1}(x_{i-1}^{\cdots+\alpha_{q+m-2,d_{s_{q+m-2}}}-1}\cdot x_i^{\sum_{t=s_{q+m-2}+1}^{s_{q+m-1}}\alpha_{q+m-1,t}d_t-1}),$$

and it is easy to see that none of the minimal generators of the last ideals is in I_j. The subcase $\alpha_{q+m-2,s_{q+m-2}} \leq \alpha_{q+m-1,s_{q+m-2}}$ is similar. Also, the case $m = 1$.

In order to prove (3) it is enough to show the "$\leq$" inequality, since obviously $(J+I)/I \subset Soc(S/I)$. Let $u = x_1^{\beta_1}\cdots x_n^{\beta_n} \in (I : \mathbf{m})$ be a monomial such that $u \notin I$. We claim that $\deg(u) \leq \sum\limits_{j=0}^{k} \chi_j$. More precisely, we claim the following:
(a) $\sum_{i=i_{q_{j-1}}+1}^{i_{q_j}} \beta_i \leq \chi_j$, for all $1 \leq j \leq r$ such that $i_{q_j} - i_{q_{j-1}} \geq 2$.
(b) For each j with the property that there exists an positive integer $1 \leq l \leq r - q + 1$ (where $q = q_j$) such that $s_{q-1} < s_q < \cdots < s_{q+l-1}$, $i_{q_j} - i_{q_{j-1}} \geq 2$ and $i_{q+l-1} = i_{q-1} + l$, we have $\sum_{i=i_{q_{j-1}+1}}^{i_{q_{j-1+l}}} \beta_i \leq \sum_{m=1}^{l} \chi_{j+m-1}$.

Obviously, (a) and (b) implies (3).
In order to prove (a), assume that $\sum_{i=i_{q_{j-1}}+1}^{i_{q_j}} \beta_i > \chi_j$, therefore

$$\sum_{i=i_{q_{j-1}}+1}^{i_{q_j}} \beta_i \geq (d_{s_{q_j}} - 1)(i_{q_j} - i_{q_{j-1}} - 1) + \alpha_{q_j s_{q_j}} d_{s_{q_j}}.$$

It follows that we can write $u_j = x_i^{d_{s_{q_j}}-1} \cdot w$, with

$$w \in (x_{i_{q_{j-1}}+1}^{d_{s_{q_j}}},\ldots,x_{i_{q_j}}^{d_{s_{q_j}}})^{\alpha_{q_j s_{q_j}}},$$

for some $i \in \{x_{i_{q_{j-1}}+1},\ldots,x_{i_{q_j}}\}$, and thus $u_j \in I_{q_j}$, a contradiction.
Consider now the case (b) and assume that

$$\sum_{i=i_{q_{j-1}+1}}^{i_{q_{j-1+l}}} \beta_i > \sum_{m=1}^{l} \chi_{j+m-1}.$$

Using similar arguments as in the case (a), we get $u_j \in I_{q_j}$, a contradiction. □

Corollary 2.8. *With the previous notations, $reg(I) = \sum_{j=1}^{k} \chi_k + 1$.*

Proof. Since I is an artinian ideal, $reg(I) = \max\{e : \ Soc(S/I)_e \neq 0\} + 1$ so the required result follows immediately from the previous theorem. □

Remark 2.9. *We have already seen that $reg(I) \leq reg(I_r)$. Now, we are able to say when we have equality, and this is only in the case when $k = 1$, i.e. $s_1 = s_2 = \cdots = s_r$. Indeed, if $k = 1$, by [4, 3.1], $reg(I_r) = (d_{s_r} - 1)(n-1) + d_{s_r}(\alpha_{rs_r} - 1) + 1 = \chi_1 + 1$. Conversely, if $k > 1$ then $\chi_1 + \cdots + \chi_k < reg(I_r)$, because $\chi_j < (d_{s_r} - 1)(i_{q_j} - i_{q_j - 1}) + d_{s_r}(\alpha_{rs_r} - 1)$ for any $j < k$.*

Example 2.10.

1. *Let $\mathbf{d} : 1|2|6|12$ and $I = < x_2^7, x_3^{10}, x_5^{17} >_{\mathbf{d}} \subset K[x_1, \ldots, x_5]$. We have $k = 2$, $\chi_1 = 15$ and $\chi_2 = 22$. Therefore, $reg(I) = 27$. An element of maximal degree in $Soc(S/I)$ is $x_1^5 x_2^5 x_3^5 x_4^{11} x_5^{11}$.*

2. *Let $\mathbf{d} : 1|4|12$ and $I = < x_1^2, x_2^7, x_3^{16} >_{\mathbf{d}} \subset K[x_1, x_2, x_3]$. We have $k = 3$. Since $2 = 2 \cdot 1$, $7 = 3 \cdot 1 + 1 \cdot 4$ and $16 = 1 \cdot 4 + 1 \cdot 12$, we get $\chi_1 = 1$, $\chi_2 = 3$ and $\chi_3 = 19$. Therefore, $reg(I) = 23$. An element of maximal degree in $Soc(S/I)$ is $x_1 x_2^3 x_3^{19}$.*

References

[1] A. Aramova, J. Herzog, *p-Borel principal ideals*, Illinois J.Math., **41** (1997),103-121.

[2] D. Bayer, M. Stillman, *A criterion for detecting m-regularity*, Invent. Math **87** (1987), 1-11.

[3] G. Caviglia, E. Sbarra, *Characteristic-free bounds for the Castelnuovo Mumford regularity*, Compos. Math., **141**(2005), 1365-1373.

[4] M. Cimpoeaş, *A generalisation of Pardue's formula*, Bull. Math. Soc. Sci. Math. Roumanie, **49**(97), no. 4, 2006, 315-334.

[5] M. Cimpoeaş, *A stable property of Borel type ideals*, to appear in Communications in Algebra.

[6] D. Eisenbud, A. Reeves, B.Totaro, *Initial ideals, Veronese subrings and rates of algebras*, Adv.Math., **109** (1994), 168-187.

[7] J. Herzog, D. Popescu, M. Vladoiu, *On the Ext-Modules of ideals of Borel type*, Contemporary Math., **331** (2003), 171-186.

[8] J. Herzog, D. Popescu, *On the regularity of p-Borel ideals*, Proceed.of AMS, **129** (2001), no.9, 2563-2570.

[9] I. Anwar, S. Ahmad, *Regularity of ideals of Borel type is linearly bounded*, Preprint, 2006.

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