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MAXIMAL COHEN-MACAULAY MODULES OVER HYPERSURFACE RINGS

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Abstract

This paper is a brief survey on various methods to classify maximal Cohen-Macaulay modules over hypersurface rings. The survey focuses on the contributions in this topic of Dorin Popescu together with his collaborators.

1 Introduction

Let k be a field, $Y = \{Y_1, Y_2, \ldots, Y_n\}$ be a set of indeterminates and let $R = k[[Y]]/J$, where J is an ideal of $k[[Y]]$. We are mainly interested in the study of maximal Cohen-Macaulay modules over hypersurfaces, that is $J = (f)$, for some nonzero and non invertible power series f.

A nonzero finitely generated $R-$module M is *maximal Cohen-Macaulay over* R (briefly, MCM(R)-module), if depth M=dim R. (For more details on Cohen-Macaulay rings and modules see [BH], [Y].) These modules preserve many properties from the Artinian case. Note that if dim $R = 0$, then all finitely generated modules over R are maximal Cohen-Macaulay. For instance, if R is an isolated singularity, that is $R_{\mathfrak{p}}$ is regular for any prime ideal $\mathfrak{p} \neq (Y)$, then there exist almost split sequences and the first Brauer-Thrall conjecture holds in the category MCM(R).

For any finitely generated $R-$module M, its n^{th} syzygy, $\Omega^n_R(M)$, that is the n^{th} kernel in a free resolution of M :

$$0 \to \Omega^n_R(M) \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M,$$

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is MCM for $n \geq \dim R$. This implies that every finitely generated $R-$module can be approximated modulo a part of its free resolution by a MCM(R)-module.
If R is a hypersurface ring, then, according to Eisenbud [Ei], the MCM(R)-modules are described in terms of matrix factorizations. A *matrix factorization* of a $f \in k[[Y]],\ Y = \{Y_1, \ldots, Y_n\}$ is a pair of $d \times d$ matrices (φ, ψ) with entries in $k[[Y]]$, such that $\varphi\psi = \psi\varphi = fI_d$. If we write $^-$ for the reduction modulo (f), then a matrix factorization (φ, ψ) yields a periodic complex of R–modules:

$$\ldots \xrightarrow{\bar{\psi}} R^d \xrightarrow{\bar{\varphi}} R^d \xrightarrow{\bar{\psi}} R^d \xrightarrow{\bar{\varphi}} R^d \to \mathrm{Coker}(\bar{\varphi}) \to 0,$$

which is exact since f is regular in $k[[Y]]$. Therefore, we get a free resolution of $M = \mathrm{Coker}(\bar{\varphi})$ over R. Let M be a MCM(R)–module. If its periodic resolution $\mathbb{F}$ over R comes from a matrix factorization (φ, ψ) of f over $k[[Y]]$, then $\mathbb{F}$ is minimal if and only if (φ, ψ) is reduced, that is $\mathrm{Im}\,\varphi, \mathrm{Im}\,\psi \subset \mathfrak{m}k[[Y]]$, where $\mathfrak{m}$ is the maximal ideal of $k[[Y]]$. In this case M has no free summands. By [Ei] there is a bijection between the equivalence classes of reduced matrix factorizations of f over $k[[Y]]$ and the isomorphism classes of MCM(R)–modules without free summands.
The paper is organized as follows. In Section I we are concerned with extensions of Knörrer Periodicity Theorem which are applied in the study of the structure of the MCM modules over the hypersurface of type $(Y_1^t + Y_2^3 + \ldots + Y_n^3)$. In Section 2 we take into consideration the graded MCM modules over the rings $k[Y_1, \ldots, Y_n]/(Y_1^3 + Y_2^3 + \ldots + Y_n^3)$, for $n = 3, 4$. The case $n = 3$ is completely described in the paper [LPP]. The description is mainly based on Atiyah's theory of the vector bundles classification over elliptic curves. A different approach is used for the case $n = 4$ in [BEPP]. In the last section we present a new method which has been applied in the study of the MCM modules over a ring of type $A[[x]]/(x^n)$, where A is a discrete valuation ring.

2 Extensions of Knörrer Theorem

Suppose $f = Y_n^s + h$, where $s \geq 2$ and $0 \neq h \in k[[Y_1, Y_2, \ldots, Y_{n-1}]]$ and let $R = k[[Y_1, Y_2, \ldots, Y_n]]/(f),\ A = k[[Y_1, \ldots, Y_{n-1}]]/(h)$.

Question 2.1. *Is it possible to describe the MCM(R)-modules in connection with the MCM modules over A?*

For $s = 2$, an affirmative answer is given by the well–known

Theorem 2.2 ([Kn])**.** *If char $k \neq 2$, then:*

(i) If N is a MCM(R)-module without free summands, then

$$\Omega_R^1(N/Y_nN) \cong N \oplus \Omega_R^1(N).$$

(ii) Every indecomposable MCM(R)-module is a direct summand in a module of type $\Omega_R^1(M)$, for an indecomposable maximal Cohen-Macaulay module M over A.

(ii) follows from (i) and it may be used to describe inductively MCM-modules over f, for special f. Knörrer Periodicity Theorem is one of the main applications of Eisenbud's Matrix Factorization Theorem and the result was completed with the characteristic two case by Pfister and Popescu in [PP2].

This theorem was extended for an arbitrary s in the following

Theorem 2.3 ([Po])**.** *If s is not a multiple of char k, then every MCM(R)-module N without free summands is a direct summand in $\Omega_R^1(N/Y_n^{s-1}N)$. Moreover, $N/Y_n^{s-1}N$ is a deformation of the MCM(A)-module N/Y_nN to $R/(Y_n^{s-1})$.*

We recall that an $\tilde{R} = R/(Y_n^{s-1})-$module L is a *deformation* of an $A-$module M to $\tilde{R}$ if

$$L/Y_nL \cong M \text{ and } \mathrm{Tor}_i^{\tilde{R}}(A, L) = 0, \text{ for } i \geq 1.$$

The case $f = Y_n^s + h$ may be considered as a special type of a general problem: **How can we relate the study of MCM-modules over $k[[Y, Z]]/(h + g)$ with the MCM modules over $k[[Y]]/(h)$ and $k[[Z]]/(g)$?** Here $Y = \{Y_1, \dots, Y_m\}$, $Z = \{Z_1, \dots, Z_n\}$ are indeterminates and $h \in k[[Y]]$, $g \in k[[Z]]$ are two nonzero and non invertible power series. This is called a *Thom-Sebastiani problem* after the name of the authors who have studied for the first time this kind of questions.
The above problem has been studied by Herzog and Popescu in [HP]. They proved that if the ideal Δ_g, generated in $k[[Z]]$ by the partial derivatives $\partial g/\partial Z_i$ is $(Z)-$ primary, then, given an MCM-module N over $S := k[[Y, Z]]/(h + g)$, we have that N/Δ_gN is a deformation to S/Δ_gS of the module N/ZN, which is MCM over $k[[Y]]/(h)$ and N is a direct summand of $\Omega_S^n(N/\Delta_gN) \oplus \Omega_S^{n+1}(N/\Delta_gN)$. In the case we have considered, that is $f = h + Y_n^s$, $h \in k[[Y_1, \dots, Y_{n-1}]]$, this theorem leads to the result obtained by Popescu in [Po]. In particular, if $s = 3$, then any indecomposable MCM(R)-module is a direct summand in the first syzygy module $\Omega_R^1(T)$ of an indecomposable infinitesimal deformation T of a MCM-module over A. Moreover, this module T is liftable to $R/(Y_n^4)$, that is there exists a lifting of T to $R/(Y_n^4)$. Using this procedure one may describe MCM-modules over hypersurfaces f of type $Y_1^t + Y_2^3$

since we may reduce to compute all the deformations of the finitely generated $k[[Y_1]]/(Y_1^t)-$ modules to $k[[Y_1, Y_2]]/(Y_1^t, Y_2^2)$ which are liftable to $R/(Y_2^4)$. This is still difficult. Along this idea we have the following:

Theorem 2.4. *Let $1 \leq i \leq j \leq t$ be two positive integers and $\mathcal{M}_{ij}$ be the class of all MCM(R)-modules N such that N/Y_2N is a direct sum of copies of $P_i = k[[Y_1]]/(Y_1^i)$, $P_j = k[[Y_1]]/(Y_1^j)$. Then the following statements are true.*

(i) If $i + j \neq t$, then $\mathcal{M}_{ij}$ is of finite representation Cohen-Macaulay type and all the indecomposable modules of $\mathcal{M}_{ij}$ are described by some matrix factorizations of size 4.

(ii) If $i + j = t$, $t > 5$, $t \neq 3i$, then $\mathcal{M}_{ij}$ is of infinite Cohen-Macaulay representation type.

The first part of this theorem was proved in [EP1] and the second part was proved in [PP1]. An attempt to apply iteratively the method used for $Y_1^t + Y_2^3$ to hypersurfaces of type $Y_1^t + Y_2^3 + \ldots + Y_r^3$, for $r \geq 2$, has the disadvantage that usually we are not able to describe completely all the MCM-modules in the case $r-1$. The optimal generalization of Knörrer theorem for hypersurfaces of type $f = Y_1^t + Y_2^3 + \ldots + Y_r^3$, $r \geq 2$, is given in [OP1], [OP2] as a consequence of a more general result. It is showed that every indecomposable MCM(R)-module N (we denoted $R = k[[Y]]/(f)$) is a direct summand in $\Omega_R^r(L)$, for a certain lifting L of a $k[[Y_1]]/(Y_1^t)-$module to $\tilde{R} := R/(Y_2^2, \ldots, Y_r^2)$, which can be chosen to be indecomposable and *weakly liftable*, that is liftable to $R/(Y_2^2, \ldots, Y_r^2)^2$. As in the case $r = 1$, the correspondence

$$N \to \tilde{R} \otimes_R N,$$

gives "almost" an injection from the isomorphism classes of MCM(R)-modules to the isomorfism classes of liftings of the finitely generated $k[[Y_1]]/(Y_1^t)-$modules to $\tilde{R}$, in the sense that if $\tilde{R} \otimes_R N$ is isomorphic to $\tilde{R} \otimes_R N'$ for some indecomposable MCM(R)-modules N, N', then either $N \cong N'$, or $N \cong \Omega_R^1(N')$. Given a matrix factorization of the module L over $\tilde{R}$, the authors define the matrix factorization of the MCM(R)-module $\Omega_R^r(L)$ by a construction involving a countably iterated mapping cone. They also give conditions for L to be weakly liftable. Thus, using this technique, we should be able to describe the indecomposable weakly liftable deformations of the finitely generated $k[[Y_1]]/(Y_1^t)-$modules to $\tilde{R}$. From a theoretical point of view, the problem is clear, but any practical attempt to describe these deformations is very hard. Even for the case $r = 2$ the computations of weakly liftable deformations of the $k[[Y_1]]/(Y_1^t)-$modules which are direct sums of copies of $k[[Y_1]]/(Y_1^i)$ lead to very hard calculations with matrix equations over $k[[Y_1]]/(Y_1^i)$.

3 Maximal graded Cohen-Macaulay modules over hypersurfaces of type $Y_1^3 + Y_2^3 + \ldots + Y_n^3$

Let $R_n := k[Y_1, \ldots, Y_n]/(f)$, where $f_n = Y_1^3 + Y_2^3 + \ldots + Y_n^3$ and k is an algebraically closed field of characteristic 0. Using the classification of vector bundles over elliptic curves obtained by Atiyah [At], C. Kahn gives a "geometrical" description of the graded MCM-modules over R_3 and also describe the Auslander-Reiten quivers of MCM(R_3) [K]. His method does not give the matrix factorizations of the indecomposable MCM(R_3)-modules. In [LPP], Laza, Pfister and Popescu use Atiyah classification to describe the matrix factorizations of the graded, indecomposable, reflexive modules over R_3. The description depends on two discrete invariants, the rank and the degree of the bundle, and on a continuous invariant, the points of the curve $Z = V(f_3)$. They give canonical normal forms for the matrix factorizations of these modules of rank one and show how one may obtain the modules of rank ≥ 2 using `SINGULAR` [GPS]. Since over the completion $k[[Y_1, Y_2, Y_3]]/(f_3)$ of R_3, every reflexive module is gradable, the authors obtain a description of the MCM-modules over $k[[Y_1, Y_2, Y_3]]/(f_3)$.

It is of high interest to classify vector bundles, in particular ACM bundles (i.e. those which corresponds to MCM modules) over the singularities of higher dimension. In the paper [EP2] the matrix factorizations which define the graded MCM modules of rank one over $f_4 = Y_1^3 + Y_2^3 + Y_3^3 + Y_4^3$ are described. There is a finite number of such modules which correspond to 27 lines, 27 pencils of quadrics and 72 nets of twisted cubic curves lying on the surface $Y = V(f_4) \subset \mathbb{P}^3$. From a geometrical point of view, the problem is easy, but the effective description of the matrix factorizations is difficult and `SINGULAR` has been intensively used.
The study of the graded MCM modules over R_4 is continued in [BEPP] for the modules of rank two. The complete description of these modules which are orientable is given using a technique based on the results of Herzog and Kühl [HK] concerning the Bourbaki sequences.
For the rest of this section we shall denote $S = k[Y_1, \ldots, Y_4]$ and $f = f_4$. An exact sequence of R–modules

$$0 \to F \to M \to I \to 0, \tag{3.1}$$

where F is free, M is a MCM(R)–module and I is an ideal of R, is called a *Bourbaki sequence*. If $\mu(M) = \mu(I) + \operatorname{rank} F$, the above sequence is called *tight* ($\mu(E)$ denotes the minimal number of generators of the R–module E). The Bourbaki sequences play an important role in the study of the MCM modules over hypersurface rings (see [HK], [BEPP]). If M is an MCM(R)–module, then M is torsion–free. Then, by [B], there exists a finite free submodule $F \subset M$ such that M/F is isomorphic with an ideal I of R and the canonical map $F \otimes k \to M \otimes k$ is injective, that is the Bourbaki sequence

$$0 \to F \to M \to I \to 0$$

is tight. In the case when $\operatorname{rank} M = 2$, the above sequence becomes

$$0 \to R \to M \to I \to 0 \tag{3.2}$$

and

$$\mu(M) = \mu(I) + 1.$$

If M is orientable, then the ideal I in the above sequence is Gorenstein of codimension 2.
According to Herzog and Kühl [HK], any non–free graded orientable MCM(R)–module of rank two must have 4 or 6 generators. With the help of the Buchsbaum–Eisenbud theorem, it is obtained a general description of the MCM orientable modules of rank two. The modules are given by skew–symmetric matrix factorizations. Let $\varphi = (a_{ij})_{1\le i,j\le 2s}$ be a generic skew–symmetric matrix, that is

$$a_{ii} = 0, a_{ij} = -a_{ji}, \text{ for all } i, j = 1, \ldots, 2s.$$

Then

$$\det(\varphi) = \operatorname{pf}(\varphi)^2,$$

where $\operatorname{pf}(\varphi)$ denotes the Pfaffian of φ (see [Bo1] or [BH]). Set

$$\psi = \frac{1}{\operatorname{pf}(\varphi)} B,$$

where B is the adjoint matrix of φ. Then

$$\varphi\psi = \psi\varphi = \operatorname{pf}(\varphi) \operatorname{Id}_{2s}.$$

Theorem 3.1. ([BEPP, Theorem 6]) *Let φ be a homogeneous skew–symmetric matrix over S of order 4 or 6 such that $\det\varphi = f^2$. Then the cokernel of φ defines a graded MCM R–module M of rank 2. Conversely, each non–free graded orientable MCM R–module M of rank 2 is the cokernel of a map given by a skew–symmetric homogeneous matrix φ over S of order 4 or 6, whose determinant is f^2. φ together with ψ defined above forms the matrix factorization of M.*

The matrix factorizations of the graded, orientable, rank 2, 4–generated MCM modules are parameter families indexed by the points of the surface $Y = V(f)$, that is, two–parameter families and some finite ones in bijection with rank 1 MCM modules described in [EP2]. Here an important fact is that two Gorenstein ideals of codimension 2 define the same MCM module via the associated Bourbaki sequence if and only if they belong to the same even linkage class. The description of graded, rank 2, 6-generated MCM modules is different from what one could expect, since a part of them, given by Gorenstein ideals defined by 5 general points on $Y = V(f)$, forms a 5-parameter family (see [Mig], [IK]). The complete description in the 6–generated case has been done with the help of `SINGULAR`.

Now we pass to the study of non-orientable MCM(R)–modules. The idea is to find a method similar with that used in the case of the orientable modules. The following propositions are slightly generalizations of the corresponding results in [BEPP].

Proposition 3.2. *Let $B = A/(g)$, where $A = k[x_1, \dots, x_{d+1}]$, $d \geq 2$ and $g \in A$ is a prime homogeneous polynomial of degree e. Let M be a graded non–orientable MCM module over B of* $\operatorname{rank} M = r \geq 2$. *Then M is isomorphic with the second syzygy of a graded ideal I of B with* $\dim B/I = d - 1$ *and* $\operatorname{depth} B/I = d - 2$. *Moreover, $\mu(M) = \mu(I) + r - 1$.*

Proof. Since M is torsion–free, as above, there exists a finite free submodule $F \subset M$ such that M/F is isomorphic with an ideal I of B and the canonical map $F/\mathfrak{m}F \to M/\mathfrak{m}M$ is injective. Thus we get the following exact sequence:

$$0 \to F \to M \to I \to 0. \tag{3.3}$$

Since M is non–orientable, by [HK], codim $I = 1$, that is $\dim R/I = d - 1$ and from (3.3) it follows that $\operatorname{depth} R/I = d - 2$. Also from (3.3) we get $\Omega^2_R(M) \simeq \Omega^2_R(I)$ and so $M \simeq \Omega^2_R(I)$. Applying $\otimes_R R/\mathfrak{m}$ to (3.3), we obtain the exact sequence

$$0 \to k^{r-1} \to M/\mathfrak{m}M \to I/\mathfrak{m}I \to 0,$$

which implies $\mu(M) = \mu(I) + r - 1$.

□

In our special case we get

Corollary 3.3. ([BEPP, Proposition 12]) *Each graded, non–orientable, rank 2, s–generated MCM R–module is the second syzygy $\Omega^2_R(I)$ of an $(s-1)$–generated graded ideal $I \subset R$ with*

$$\operatorname{depth} R/I = 1 \textit{ and } \dim R/I = 2.$$

In the above settings, that is $B = A/(g)$, $A = k[x_1, \ldots, x_{d+1}]$, $d \geq 2$, and $g \in A$ being a prime homogeneous polynomial of degree e, let M be a graded non–orientable MCM module M over B of rank $r \geq 2$ and (φ, ψ) be the matrix factorization which defines M, that is $M \simeq \mathrm{Coker}(\varphi)$. Then

$$0 \to A^{\mu(M)} \xrightarrow{\varphi} A^{\mu(M)} \to 0$$

is a minimal A–resolution of M. Using the Bourbaki sequence (3.3), we get the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A^{r-1} \arrow[r] \arrow[d, "\nu_g"'] & A^{\mu(M)} \arrow[d, "\varphi"] & & \\
 & A^{r-1} \arrow[r, "\sigma"] \arrow[d] & A^{\mu(M)} \arrow[d] & & \\
0 \arrow[r] & B^{r-1} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & I \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

As in [HK], from this diagram, we obtain the minimal free resolution of I over A :

$$0 \to A^{r-1} \xrightarrow{u_3} A^{\mu(M)} \xrightarrow{u_2} A^{\mu(M)-r+1} \xrightarrow{u_1} I \to 0. \tag{3.4}$$

Let $J \subset A$ be a graded ideal such that $g \in J$ and $I = J/(g)$. Then depth $\frac{A}{J} =$ depth $\frac{B}{I} = d - 1$. Let us assume that $g \in \mathfrak{m}J$, and let

$$0 \to A^{s_3} \xrightarrow{d_3} A^{s_2} \xrightarrow{d_2} A^{s_1} \xrightarrow{d_1} J \to 0$$

be the minimal free A–resolution of J. Then the following sequence

$$0 \longrightarrow A^{s_3} \xrightarrow{\binom{d_3}{0}} A^{s_2+1} \xrightarrow{(d_2,v)} A^{s_1} \xrightarrow{\tilde{d}_1} I \longrightarrow 0$$

is exact and forms a minimal free A–resolution of I. Here $v : A \longrightarrow A^{s_1}$ is an A–linear map such that $jd_1v = g\,\mathrm{Id}_A$, where $j : J \longrightarrow A$ is the inclusion. Comparing with (3.4) it results: $s_3 = r-1, s_2 = \mu(M)-1$ and $s_1 = \mu(I) = \mu(M) - r + 1$.

Proposition 3.4. *Let $I = J/(f)$, J being a graded ideal of A with $g \in J$, such that $\dim R/I = d-1$ and $\operatorname{depth} R/I = d-2$. We suppose that g is not a minimal generator of J. If $\mu(I) = s$ and $\operatorname{rank}(\Omega^2_R(I)) = r$, then $M = \Omega^2_R(I)$ is a graded non–orientable MCM module over R with $\mu(M) = s + r - 1$.*

Proof. Let

$$0 \to A^{s_3} \xrightarrow{d_3} A^{s_2} \xrightarrow{d_2} A^{s_1} \xrightarrow{d_1} J \to 0$$

be the minimal free A–resolution of J.

Since g is not a minimal generator of J, as in [BEPP], we get the following exact sequence

$$B^{s_3+s_1} \xrightarrow{\binom{\bar{h}|\bar{d}_3}{0}} B^{s_2+1} \xrightarrow{(\bar{d}_2,\bar{v})} B^{s_1} \xrightarrow{\bar{d}_1} I \longrightarrow 0,$$

which is part of a minimal free B–resolution of I. Here $v : A \longrightarrow A^{s_1}$ is an A–linear map such that $jd_1v = g\,\mathrm{Id}_A$, where $j : J \longrightarrow A$ is the inclusion, $\tilde{d}_1$ is the composite map $A^{s_1} \xrightarrow{d_1} J \longrightarrow J/(f) = I$ and $h : A^{s_1} \longrightarrow A^{s_2+1}$ is an A–linear map such that $(d_2, v)h = g\,\mathrm{Id}_{S^{s_1}}$. The above sequence shows that $\Omega^2_B(I)$ is the image of the first map. It follows

$$s_3 + s_1 = s_2 + 1.$$

From the exact sequence

$$0 \to \Omega^2_B(I) \to B^{s_2+1} \to B^{s_1} \to I \to 0,$$

we obtain $r - (s_2+1) + s_1 - 1 = 0$. Since $s_1 = s$, it follows $s_2 = r + s - 2$ and $\mu(M) = s_3 + s = r + s - 1$. □

Coming back to our study, let $I = J/(f)$, for some graded ideal $J \subset S$, and such that

$$\dim \frac{R}{I} = 2,\ \operatorname{depth} \frac{R}{I} = 1,\ \mu(I) = s \text{ and } \operatorname{rank} \Omega^2_R(I) = 2.$$

By the above proposition, if $f \in \mathfrak{m}J$, that is f is not a minimal generator of J, then $M = \Omega^2_R(I)$ is a graded non-orientable MCM(R)–module, with

$\mu(M) = s+1$. In [EP3, Theorem 1.3], it is proved that the hypothesis $f \in \mathfrak{m}J$ is fulfilled for $s \geq 4$. By [EP3, Theorem 1.3] and [BEPP], one gets the complete description of the indecomposable, graded, non–orientable MCM modules of rank 2, 4– and 5–generated. There is a finite number of such modules, which corresponds somehow to the rank 1 modules described in [EP2]. These results remind us of the theory of Atiyah and give a small hope that the non-orientable case behaves in the same way for higher ranks.

For the 6–generated case, we have Lemma 18 [BEPP]. We shall give here a different proof.

Lemma 3.5 ([BEPP])**.** *There exist no graded, indecomposable, non–orientable, rank 2, 6–generated MCM modules.*

Proof. Suppose there exist such an MCM module M. Then $M \cong \Omega_R^2\big(J/(f)\big)$ for a certain 5–generated ideal $J = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ of S (see [BEPP, Lemma 11]). As in the proof of [BEPP, Lemma 18], one may suppose that J is generated in degree 2. Let

$$0 \to S(-5) \to S^5(-3) \to S^5(-2) \to S \to \frac{S}{J} \to 0$$

be the minimal free S–resolution of $\frac{S}{J}$. This implies that the Hilbert series of S/J is given by

$$H_{S/J}(z) = \frac{1-5z^2+5z^3-z^5}{(1-z)^4} = \frac{1+3z+z^2}{1-z}.$$

This shows that $\dim \frac{R}{J} = 1$, which is impossible. $\square$

4 Maximal Cohen–Macaulay modules over $K[[x,y]]/(x^n)$

In this section we are concerned with the structure of the MCM modules over the ring $R_n := K[[x,y]]/(x^n),\ n \geq 2$. Let $A := K[[y]],\ S := K[[x,y]]$, and $R_n := A[[x]]/(x^n), n \geq 2$. In order to know the structure of the MCM modules over the ring R_n one should apply a completely different method as we have seen in the previous sections. We shall make a quick summary of the results of [EP4].

Any MCM R_n–module M is free over A of finite rank. Giving a MCM R_n–module M is equivalent with giving the action of x on the free A–module M, that is with giving an endomorphism $u \in \mathrm{End}_A(M)$ such that $u^n = 0$, which can be represented by its matrix T in some basis of M over A. Obviously, $T^n = 0$.

Proposition 4.1 ([En]). *Let*

$$\mathcal{T} = \{T \mid T \text{ is an } m \times m - \text{matrix over } A,\ T^n = 0,\ m \geq 1\}$$

and

$$\mathcal{M} = \{M \mid M \text{ is an MCM } R_n - \text{module}\}.$$

Then the map $\phi : \mathcal{T} \to \mathcal{M}$ defined by $\phi(T)$ to be the MCM R_n–module associated to the matrix factorization

$$(x \operatorname{Id}_m - T, x^{n-1} \operatorname{Id}_m + x^{n-2}T + \ldots + T^{n-1})$$

is surjective.

We briefly recall the proof:

Proof. Let us consider an MCM R_n–module M whose minimal free R_n–resolution is

$$\ldots \xrightarrow{\bar{\psi}} R_n^q \xrightarrow{\bar{\varphi}} R_n^q \xrightarrow{\bar{\psi}} R_n^q \xrightarrow{\bar{\varphi}} R_n^q \longrightarrow M \longrightarrow 0,$$

where (φ, ψ) is a matrix factorization of x^n over $K[[x, y]]$ which defines M. Let $m := \operatorname{rank}_A M$ and let T be the nilpotent $m \times m$–matrix with entries in A which gives the action of x on the finite free A–module M, and let N be the MCM R_n–module given by the periodic resolution

$$\ldots \xrightarrow{\bar{\mu}} R_n^m \xrightarrow{\bar{\nu}} R_n^m \xrightarrow{\bar{\mu}} R_n^m \xrightarrow{\bar{\nu}} R_n^m \longrightarrow N \longrightarrow 0,$$

where

$$\nu = x \operatorname{Id}_m - T,\ \mu = x^{n-1} \operatorname{Id}_m + x^{n-2}T + \ldots + T^{n-1}.$$

Then N is an A–free module of rank m and the action of x on N is given by T. This means that the R_n–modules M and N are isomorphic, hence the module M has the matrix factorization (ν, μ). $\square$

Remark 4.2. *The matrix factorization (ν, μ) from above can be not reduced, as we can see in the following example and, by consequence, the R_n–free resolution provided by this factorization could be not minimal.*

Example 4.3. *Let us consider the MCM R_3–module given by the matrix factorization $\left(\varphi := \begin{pmatrix} x & -y \\ 0 & x^2 \end{pmatrix}, \psi := \begin{pmatrix} x^2 & y \\ 0 & x \end{pmatrix}\right)$. Then, as an A–module, M has rank 3 and the action of x on M is given by the matrix $T := \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, that is the matrix factorization (ν, μ) is given by*

$$\nu = \begin{pmatrix} x & -y & 0 \\ 0 & x & -1 \\ 0 & 0 & x \end{pmatrix}, \mu = \begin{pmatrix} x^2 & xy & y \\ 0 & x^2 & x \\ 0 & 0 & x^2 \end{pmatrix}.$$

In order to find the matrix factorizations of the MCM R_n–modules, we need to study the structure of the nilpotent matrices over A. The general structure of these matrices, equivalently of nilpotent endomorphisms of finite free A–modules, is given in a more general setting in [En].
In order to find the matrix factorizations of the MCM R_n–modules, we need to study the structure of the nilpotent matrices over A. The general structure of these matrices, equivalently of nilpotent endomorphisms of finite free A–modules, is given in a more general setting in [En]. Namely, let E be a finite free module of rank m over a principal ideal domain A. Let $u \in \mathrm{End}_A(E)$ such that $u^n = 0$ and $u^{n-1} \neq 0,\ n \geq 2$. For $0 \leq k \leq n$, we denote $E_k := \ker(u^k)$. Then, for any $0 \leq k \leq n-1, E_{k+1}/E_k$ is a non–zero free module over A, and the morphism

$$\bar{u}_k : \frac{E_{k+1}}{E_k} \to \frac{E_k}{E_{k-1}},\ \bar{u}_k(x + E_k) = u(x) + E_{k-1},\ x \in E_{k+1},$$

induced by u, is injective, $\forall\ 1 \leq k \leq n-1$. In the above notations we have:

Theorem 4.4 ([En])**.** *There exists a basis B of E such that the matrix of u in the basis B has the form:*

$$M_B(u) = \left(\begin{array}{c|c|c|c|c|c} 0 & \Lambda_1 & \Delta_{11} & \Delta_{12} & \dots & \Delta_{1,n-2} \\ \hline 0 & 0 & \Gamma_1\Lambda_2 & \Delta_{22} & \dots & \Delta_{2,n-2} \\ \hline 0 & 0 & 0 & \Gamma_2\Lambda_3 & \dots & \Delta_{3,n-2} \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \Gamma_{n-2}\Lambda_{n-1} \\ \hline 0 & 0 & 0 & 0 & \dots & 0 \end{array}\right),$$

where $\Lambda_1 = \left(\frac{\mathrm{diag}(a_{11}, \dots, a_{1r_2})}{0}\right)$ is of size $r_1 \times r_2$ and has the last $r_1 - r_2$ rows 0, $\Lambda_k = \mathrm{diag}(a_{k1}, \dots, a_{kr_{k+1}}),\ k \geq 2$, Γ_k is left invertible and of size $r_{k+1} \times r_{k+2}$, for any k, and Δ_{ij} is of size $r_i \times r_{j+2}$, for any i, j. Moreover, for $k \geq 1$, the elements $a_{k1} \mid a_{k2} \mid \dots \mid a_{kr_{k+1}}$ are the invariants of the A–free modules $\bar{u}_k(\frac{E_{k+1}}{E_k}) \subset \frac{E_k}{E_{k-1}}$.

As an immediate consequence we get the known form of the indecomposable MCM modules over $R_2 = k[[x, y]]/(x^2)$ (see [BGS, Proposition 4.1], [Y, Example 6.5]).

Proposition 4.5. *Let M be an indecomposable MCM–module over $K[[x, y]]/(x^2)$. Then M has a matrix factorization of the following form:*

$$((x), (x)),\ \text{or}\ \left(\begin{pmatrix} x & y^t \\ 0 & x \end{pmatrix}, \begin{pmatrix} x & -y^t \\ 0 & x \end{pmatrix}\right),$$

for some positive integer t.

From Theorem 4.4, one can see that the structure of the nilpotent endomorphisms which have the nilpotency index $n \geq 3$ is more complicate. As a consequence, the structure of the MCM modules over the ring $R_n,\ n \geq 3$, is more complicated.
Let M be an MCM R_n–module, $n \geq 2$. For $1 \leq i \leq n$ we denote $M_i := 0 :_M x^i$. Then the action of x on M yields the filtration

$$(0) \subset M_1 \subset M_2 \ldots \subset M_{n-1} \subset M_n = M,$$

where all the factors $\frac{M_i}{M_{i-1}}, i \geq 1$, are free A–modules. The composition map

$$M_{i+1} \xrightarrow{x} M_i \longrightarrow \frac{M_i}{M_{i-1}}$$

induces an injective map

$$\frac{M_{i+1}}{M_i} \to \frac{M_i}{M_{i-1}},\ m + M_i \mapsto xm + M_{i-1}.$$

We set $M_0 := 0$ and $r_i := \mathrm{rank}_A(\frac{M_i}{M_{i-1}}),\ 1 \leq i \leq n$. Then

$$r_1 \geq r_2 \geq \ldots \geq r_n \text{ and } r_1 + r_2 + \ldots + r_n = \mathrm{rank}_A(M).$$

The sequence $(r_1, r_2, \ldots, r_n)$ is an invariant of the module M. We shall call this sequence the *canonical sequence associated to* M. The main result in [EP4] states that, for an arbitrary decreasing sequence of positive integers $(r_1, r_2, \ldots, r_n)$, $n \geq 3$, there exist infinitely many non-isomorphic indecomposable MCM R_n–modules with the canonical associated sequence $(r_1, r_2, \ldots, r_n)$. The construction procedure is inductive. The most laborious step is to find infinite families of indecomposable non-isomorphic MCM R_3–modules for arbitrary decreasing sequences. Moreover, in [EP4, Theorem 1.12] are explicitly given infinite families of indecomposable MCM R_n–modules of arbitrary rank $r \geq 1$.

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