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Conservative Projective Curvature Tensor On Trans-sasakian Manifolds With Respect To Semi-symmetric Metric Connection

C.S.Bagewadi, D.G.Prakasha and Venkatesha

Abstract

We obtain results on the vanishing of divergence of Riemannian and Projective curvature tensors with respect to semi-symmetric metric connection on a trans-Sasakian manifold under the condition $\phi(grad\alpha) = (n-2)grad\beta$.

1 Introduction

In 1924, Friedman and Schouten [8] introduced the notion of semi-symmetric linear connection on a differentiable manifold. Then in 1932, Hayden [10] introduced the idea of metric connection with a torsion on a Riemannain manifold. A systematic study of semi-symmetric metric connection on a Riemannain manifold has been given by Yano [14] in 1970 and later studied by K.S.Amur and S.S.Pujar [1], C.S.Bagewadi[2], U.C.De et al [7], Sharafuddin and Hussain [12] and others.

The authors U.C.De et al [7], C.S.Bagewadi and N.B.Gatti [3] and C.S.Bagewadi and Venkatesha [13] have obtained results on the conservativeness of projective, pseudo projective, conformal, concircular and quasi-conformal curvature tensors on K-contact, Kenmotsu and Trans-sasakian manifolds.

In this paper we study the conservativeness of curvature tensor and projective curvature tensors to trans-Sasakian manifold under the condition $\phi(grad\alpha) =$

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$(n-2)grad\beta$ admitting semi-symmetric metric connection. The paper is organised as follows: After preliminaries in Section 2, we study in Section 3 some basic results on trans-Sasakian manifold under the above condition with respect to semi-symmetric metric connection. In Section 4, we study conservative Riemannian curvature tensor with respect to semi-symmetric metric connection. In the last section, we study conservative projective curvature tensor with respect to this connection and obtain some interesting results.

2 Preliminaries

Let M^n be an almost contact metric manifold [5] with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta.\phi = 0, \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{4}$$

for all $X, Y \in TM$.

An almost contact metric structure (ϕ, ξ, η, g) on M^n is called a trans-Sasakian structure [11] if $(M^n \times R, J, G)$ belongs to the class w_4 [9], where J is the almost complex structure on $M^n \times R$ defined by $J(X, \lambda d/dt) = (\phi X - \lambda\xi, \eta(X)d/dt)$ for all vector fields X on M^n and smooth functions λ on $M^n \times R$ and G is the product metric on $M^n \times R$. This may be expressed by the condition [6]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{5}$$

for some smooth functions functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β).

Let M be a n-dimensional trans-Sasakian manifold. From (5), it is easy to see that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{6}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{7}$$

In an n-dimensional trans-Sasakian manifold, we have

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X), \tag{8}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{9}$$

$$S(X, \xi) = ((n-1)(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (n-2)X\beta - (\phi X)\alpha. \tag{10}$$

Further, in a trans-Sasakian manifold of type(α,β), we have

$$\phi(grad\alpha) = (n-2)grad\beta, \tag{11}$$

Using (11), the equations (8) and (10) reduces to

$$R(\xi, X)\xi = (\alpha^2 - \beta^2)(\eta(X)\xi - X), \tag{12}$$
$$S(X,\xi) = (n-1)(\alpha^2 - \beta^2)\eta(X). \tag{13}$$

In this paper we study trans-Sasakian manifold under the condition (11)

Let (M^n, g) be an n-dimensional Riemannian manifold of class C^∞ with metric tensor g and ∇ be the Levi-Civita connection on M^n. A linear connection $\widetilde{\nabla}$ on (M^n, g) is said to be *semi-symmetric* [14], if the torsion tensor T of the connection $\widetilde{\nabla}$ satisfies

$$T(X,Y) = \pi(Y)X - \pi(X)Y, \tag{14}$$

where π is an 1-form on M^n with the associated vector field ρ, i.e.,$\pi(X) = g(X,\rho)$ for any differentiable vector field X on M^n.

A semi-symmetric connection $\widetilde{\nabla}$ is called *semi-symmetric metric connection* [10], if $\widetilde{\nabla}g = 0$.

In an almost contact manifold, semi-symmetric metric connection is defined by identifying the 1-form π of (14) with the contact form η, i.e., by setting [12]

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{15}$$

with ξ as associated vector field. i.e., $g(X,\xi) = \eta(X)$.

The relation between the semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ of M^n has been obtained by K.Yano[14] and it is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{16}$$

where $\eta(Y) = g(Y,\xi)$.

Further, a relation between the curvature tensors R and $\widetilde{R}$ of type $(1,3)$ of the connections ∇ and $\widetilde{\nabla}$ respectively is given by [14],

$$\widetilde{R}(X,Y)Z = R(X,Y)Z - K(Y,Z)X + K(X,Z)Y - g(Y,Z)AX + g(X,Z)AY, \tag{17}$$

where α is a tensor field of type $(0,2)$ defined by

$$\begin{aligned} K(Y,Z) = g(AY,Z) &= (\nabla_Y\eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}\eta(\xi)g(Y,Z) \qquad (18)\\ &= (\widetilde{\nabla}_Y\eta)(Z) - \frac{1}{2}\eta(\xi)g(Y,Z), \end{aligned}$$

for any vector fields X and Y.
From (17), it follows that

$$\widetilde{S}(Y,Z) = S(Y,Z) - (n-2)K(Y,Z) - a.g(Y,Z), \tag{19}$$

where $\widetilde{S}$ denotes the Ricci tensor with respect to $\widetilde{\nabla}$, $a = Tr.K$.
Differentiating (19) covariantly with respect to X, we obtain

$$\begin{aligned}(\widetilde{\nabla}_X \widetilde{S})(Y,Z) = (\nabla_X S)(Y,Z) - (n-2)(\nabla_X \alpha)(Y,Z) - \eta(Y)S(X,Z) - \\ -\eta(Z)S(X,Y) + (n-2)\eta(Y)\alpha(X,Z) + (n-2)\eta(Z)\alpha(Y,X) + \\ +g(X,Y)S(\xi,Z) + g(X,Z)S(Y,\xi) - (n-2)g(X,Z)\alpha(Y,\xi) - \\ -(n-2)g(X,Y)\alpha(Z,\xi).\end{aligned} \tag{20}$$

Now let e_i be an orthogonal basis of the tangent space at each point of the manifold M^n for $i = 1, 2, ..., n$. Putting $Y = Z = e_i$ in (20) and then taking summation over the index i, we get

$$\widetilde{\nabla}_X \widetilde{r} = \nabla_X r - (n-2)(\nabla_X a). \tag{21}$$

We recall the following definition which is used in later section,

Definition 1 *A trans-Sasakian manifold M^n is said to be η-**Einstein**, if its Ricci tensor S is of the form $S(X,Y) = Pg(X,Y) + Q\eta(X)\eta(Y)$, for any vector fields X, Y, where P, Q are functions on M*

3 Some Basic Results

Theorem 1. *For a trans-Sasakian manifold M^n, $n > 1$, under the conditon (11), we have*

$$\begin{aligned}&[(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z)] = \\ &= \beta S(Y,Z) - (n-1)(\alpha^2 - \beta^2)\beta g(Y,Z) - \\ &-(n-1)(\alpha^2 - \beta^2)\alpha g(Y,\phi Z) + \alpha S(Y,\phi Z).\end{aligned} \tag{22}$$

Proof. For a symmetric endomorphism Q of the tangent space at a point of M, we express the Ricci tensor S as

$$S(X,Y) = g(QX,Y). \tag{23}$$

Further, it is known that [6]

$$(L_\xi g)(X,Y) = 2\beta[g(X,Y) - \eta(X)\eta(Y)], \tag{24}$$

for all X and Y, where L is the Lie derivation.
In a trans-Sasakian manifold, from (23) and (24), we have

$$(L_\xi S)(X,Y) = 2\beta S(X,Y) - 2\beta(n-1)(\alpha^2 - \beta^2)\eta(X)\eta(Y). \tag{25}$$

Consider

$$\begin{aligned}(\nabla_\xi S)(Y,Z) &= \xi S(Y,Z) - S(\nabla_\xi Y, Z) - S(Y, \nabla_\xi Z)\\ &= \xi S(Y,Z) - S([\xi,Y] + \nabla_Y \xi, Z) - S(Y, [\xi, Z] + \nabla_Z \xi)\\ &= \xi S(Y,Z) - S([\xi,Y],Z) - S(\nabla_Y \xi, Z) - S(Y,[\xi,Z]) - S(\nabla_Z \xi)\\ &= (L_\xi S)(Y,Z) - S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi).\end{aligned}$$

Using (6), (13) and (25) in the above equation, we obtain

$$(\nabla_\xi S)(Y,Z) = 0. \tag{26}$$

We know that

$$(\nabla_Y S)(\xi, Z) = YS(\xi,Z) - S(\nabla_Y \xi, Z) - S(\xi, \nabla_Y \xi).$$

By virtue of (6) and (13), the above equation takes the form

$$(\nabla_Y S)(\xi, Z) = (n-1)(\alpha^2 - \beta^2)Y.\eta(Z) - S(-\alpha\phi Y + \beta(Y - \eta(Y)\xi, Z)))-$$

$$-(n-1)(\alpha^2 - \beta^2)\eta(\nabla_Y Z).$$

Further simplifying by using (7), we get

$$\begin{aligned}(\nabla_Y S)(\xi, Z) = (n-1)(\alpha^2 - \beta^2)\beta g(Y,Z) - \beta S(Y,Z) +\\ +(n-1)(\alpha^2 - \beta^2)\alpha g(Y, \phi Z) - \alpha S(Y, \phi Z).\end{aligned} \tag{27}$$

By putting (26) and (27) in the left hand side of (22), the result follows.

Theorem 2. *For a trans-Sasakian manifold M^n, under the condition (11),* *the following results are true:*

$$\begin{aligned}
&(i) K(Y,Z) = \alpha g(Y,\phi Z) + \left(\beta + \frac{1}{2}\right) g(Y,Z) - (\beta+1)\eta(Y)\eta(Z);\\
&(ii) K(Y,\xi) = K(\xi,Y) = -\frac{1}{2}\eta(Y);\\
&(iii) K(\nabla_Y \xi, Z) = -\alpha^2[g(Y,Z) - \eta(Y)\eta(Z)] - 2\alpha\beta g(\phi Y, Z)\\
&\qquad -\frac{\alpha}{2} g(\phi Y, Z) + \beta\left(\beta + \frac{1}{2}\right)[g(Y,Z) - \eta(Y)\eta(Z)]; \qquad (28)\\
&(iv) K(Y, \nabla_Z \xi) = \alpha^2[g(Y,Z) - \eta(Y)\eta(Z)] + \frac{\alpha}{2} g(\phi Y, Z)\\
&\qquad +\beta\left(\beta + \frac{1}{2}\right)[g(Y,Z) - \eta(Y)\eta(Z)].
\end{aligned}$$

Proof. Using (2) and (18) in (18), we get (28(i)).

By Taking $Z = \xi$ in (28(i)) and then using (2) and (4), we have the result (28(ii)).

Next, by considering $Y = \nabla_Y \xi$ in (28(i)) and then by using (2) and (6), the result (28(iii)) follows.

From the result (28(iii)), the proof of (28(iv)) is obvious.

Theorem 3. *For a trans-Sasakian manifold M^n, under the condition (11), we have*

$$\begin{aligned}
[(\nabla_\xi K)(Y,Z) - (\nabla_Y K)(\xi, Z)] &= \alpha(2\beta+1) g(Y,\phi Z)\\
&-[\alpha^2 - \beta(\beta+1)][g(Y,Z) - \eta(Y)\eta(Z)]. \qquad (29)
\end{aligned}$$

Proof. From (24) and $K(X,Y) = g(AX,Y)$, we have

$$(L_\xi K)(Y,Z) = 2\beta K(Y,Z) + \beta\eta(Y)\eta(Z). \qquad (30)$$

We take

$$\begin{aligned}
&(\nabla_\xi K)(Y,Z) = \xi K(Y,Z) - K(\nabla_\xi Y, Z) - K(Y, \nabla_\xi Z) =\\
&= \xi K(Y,Z) - K([\xi,Y] + \nabla_Y \xi, Z) - K(Y, [\xi, Z] + \nabla_Z \xi) =\\
&= \xi K(Y,Z) - K([\xi,Y], Z) - K(\nabla_Y \xi, Z) - K(Y, [\xi,Z]) - K(\nabla_Z \xi) =\\
&= (L_\xi K)(Y,Z) - K(\nabla_Y \xi, Z) - K(Y, \nabla_Z \xi).
\end{aligned}$$

Using (6), (28(ii),(iii)& (iv)) and (30) in above, we obtain

$$(\nabla_\xi K)(Y,Z) = 0. \qquad (31)$$

We know that

$$(\nabla_Y K)(\xi, Z) = YK(\xi, Z) - K(\nabla_Y \xi, Z) - K(\xi, \nabla_Y \xi).$$

Using (6) and (28(ii),(iii)& (iv)) in the above equation and simplifying, we get

$$(\nabla_Y K)(\xi, Z) = \alpha g(\phi Y, Z) + 2\alpha\beta g(\phi Y, Z) + [\alpha^2 - \beta(\beta+1)](g(Y, Z) - \eta(Y)\eta(Z)). \tag{32}$$

By putting (31) and (32) in left hand side of (29), the result follows.

4 Trans-Sasakian Manifold Admitting A Semi-symmetric Metric Connection With $Div\widetilde{R} = 0$

Theorem 4. *Suppose a trans-Sasakian manifold M^n, $(n > 2)$ under the condition (11) admits a semi-symmetric metric connection whose curvature tensor with respect to this connection is conservative. Then the manifold M^n is η-Einstein with respect to Levi-Civita connection; and moreover, the scalar curvature of the manifold is constant if and only if $\beta = -1$.*

Proof. Let us suppose that in a trans-Sasakian manifold M^n under the condition (11) with respect to semi-symmetric metric connection, $Div\widetilde{R} = 0$, where Div denotes the divergence.

Then (17) gives

$$\begin{aligned}
[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)] = [(\nabla_X K)(Y, Z) - (\nabla_Y K)(X, Z)] + \\
+\eta(X)S(Y, Z) - \eta(Y)S(X, Z) + \eta(Z)S(X, Y) - (n-1)\eta(R(X, Y)Z) - \\
-\eta(Y)K(X, Z) - \eta(Z)K(Y, X) - (n-a-1)\eta(Y)g(X, Z) + \\
+\eta(X)K(Y, Z) + \eta(Z)K(X, Y) + (n-a-1)\eta(X)g(Y, Z) + \\
+g(Y, Z)(da(X)) - g(X, Z)(da(Y)).
\end{aligned} \tag{33}$$

Now putting $X = \xi$ in (33), and then using (2), (4),(13) and (28(ii)), we get

$$\begin{aligned}
[(\nabla_\xi S)(Y, Z) - (\nabla_Y S)(\xi, Z)] = [(\nabla_\xi K)(Y, Z) - (\nabla_Y K)(\xi, Z)] + S(Y, Z) + \\
+[S(Y, \xi)\eta(Z) - S(Z, \xi)\eta(Y)] - (n-1)\eta(R(\xi, Y)Z) - \\
-K(\xi, Z)\eta(Y) + (n-a-1)g(\phi Y, \phi Z) + K(Y, Z) + \\
+g(Y, Z)(\nabla_\xi a) - \eta(Z)(\nabla_Y a).
\end{aligned} \tag{34}$$

Using (12) ,(22), (28(i)) and (29) in above, we get

$$(\beta-1)S(Y,Z)+\alpha S(Y,\phi Z)=$$
$$=\left[(\alpha^2-\beta^2)[(n-1)(\beta-1)-1]+2\beta+(n-a+\frac{1}{2})\right]g(Y,Z)-$$
$$-\alpha[(n-1)(\alpha^2-\beta^2)-2(\beta+1)]g(\phi Y,Z)-\eta(Z)(\nabla_Y a)+$$
$$+\left[(n-1)(\alpha^2-\beta^2)+\alpha^2-(\beta+1)^2-(n-a-\frac{3}{2})\right]\eta(Y)\eta(Z) \qquad (35)$$

Next, by replacing Z by ϕZ in above and then using (2), we obtain

$$S(Y,Z)=\frac{1}{\alpha}(1-\beta)S(\phi Y,Z)+[(n-1)(\alpha^2-\beta^2)-$$
$$-2(\beta+1)]g(Y,Z)+[2(\beta+1)]\eta(Y)\eta(Z)+$$
$$+\frac{1}{\alpha}\left[(\alpha^2-\beta^2)[(n-1)(\beta-1)-1]+2\beta+(n-a-\frac{1}{2})\right]g(\phi Y,Z). \qquad (36)$$

Interchanging Y with Z in (36), we have

$$S(Y,Z)=\frac{1}{\alpha}[1-\beta]S(Y,\phi Z)+[(n-1)(\alpha^2-\beta^2)-$$
$$-2(\beta+1)]g(Y,Z)+[2(\beta+1)]\eta(Y)\eta(Z)+$$
$$+\frac{1}{\alpha}\left[(\alpha^2-\beta^2)[(n-1)(\beta-1)-1]+2\beta+(n-a-\frac{1}{2})\right]g(Y,\phi Z). \qquad (37)$$

By adding (36) with (37), and then by using the skew-symmetric property of ϕ, one can get

$$S(Y,Z)=[(n-1)(\alpha^2-\beta^2)-2(\beta+1)]g(Y,Z)+2(\beta+1)\eta(Y)\eta(Z). \qquad (38)$$

Hence by the Definition 1, the manifold is η-Einstein.
Differentiating (38) covariantly with respect to X, and then using (7), we have

$$(\nabla_X S)(Y,Z)=2(\beta+1)[\alpha(g(\phi Y,X)\eta(Z)+g(\phi Z,X)\eta(Y))+$$
$$+\beta(g(X,Y)\eta(Z)g(X,Z)\eta(Y))-2\beta\eta(X)\eta(Y)\eta(Z)]. \qquad (39)$$

Taking an orthonormal frame field and contracting (39) over X and Z, we obtain

$$dr(Y)=2(\beta+1)[\alpha\psi+(n-1)\beta]\eta(Y), \qquad (40)$$

where $\psi=Tr.\phi$. From (40), it follows that

$$dr(Y)=0 \quad \text{if and only if} \quad \beta=-1. \qquad (41)$$

Hence the theorem is proved.

5 Trans-Sasakian Manifold Admitting A Semi-symmetric Metric Connection With $Div\widetilde{P} = 0$

Theorem 5. *If a trans-Sasakian manifold M^n $(n > 2)$ under the conditon (11) admits a semi-symmetric metric connection whose projective curvature tensor with respect to this connection is conservative, then the manifold M^n is η-Einstein with respect to Levi-Civita connection. Moreover, the scalar curvature of the manifold is constant if and only if $\beta = 0$.*

Proof. Let us suppose that in a trans-Sasakian manifold M^n $(n > 2)$ under the condition (11) with respect to semi-symmetric metric connection, $Div\widetilde{P} = 0$.
Then by virtue of (17), (20) gives

$$\begin{aligned}\frac{n-2}{n-1}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] -\\ -\frac{1}{n-1}[(\nabla_X K)(Y,Z) - (\nabla_Y K)(X,Z)] =\\ = \frac{n-2}{n-1}[g(Y,Z)K(X,\xi) - g(X,Z)K(Y,\xi)] +\\ +\frac{n}{n-1}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] -\\ -\frac{1}{n-1}[K(X,Z)\eta(Y) + K(Y,X)\eta(Z) -\\ -K(Y,Z)\eta(X) - K(X,Y)\eta(Z)] -\\ -(n-a-1)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] +\\ +S(X,Y)\eta(Z) + (n-1)\eta(R(X,Y)Z) -\\ -(\alpha^2 - \beta^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] +\\ +[g(Y,Z)(\nabla_X a) - g(X,Z)(\nabla_Y a)].\end{aligned} \tag{42}$$

Now, taking $X = \xi$ in (42), and then using (2),(4), (13) and (28(ii)), we get

$$\frac{n-2}{n-1}[(\nabla_\xi S)(Y,Z)-(\nabla_Y S)(\xi,Z)]-$$
$$-\frac{1}{n-1}[(\nabla_\xi K)(Y,Z)-(\nabla_Y K)(\xi,Z)]=$$
$$=\frac{n}{n-1}S(Y,Z)+$$
$$+\left[(n-a-1)-(\alpha^2-\beta^2)\frac{n-2}{2(n-1)}+(\nabla_\xi a)\right]$$
$$-\frac{1}{n-1}K(Y,Z)+$$
$$+\left[2n(\alpha^2-\beta^2)+\frac{n}{2(n-1)}-(n-a-1)\right]\eta(Y)\eta(Z). \tag{43}$$

Using (12) ,(22), (28(i)) and (29) in above, we obtain

$$\left[\frac{n-2}{n-1}\beta-\frac{n}{n-1}\right]S(Y,Z)-\frac{n-2}{n-1}\alpha S(\phi Y,Z)=$$
$$=\left[((n-2)\beta-1)(\alpha^2-\beta^2)+(n-a-1)-\frac{1}{2(n-1)}\right]g(Y,Z)-$$
$$-\alpha\left[n-2)(\alpha^2-\beta^2)+\frac{2\beta}{n-1}\right]g(\phi Y,Z)-\eta(Z)\nabla_Y a$$
$$\left[\frac{(\alpha^2-\beta^2)}{n-1}+\frac{(n+2)}{2(n-1)}+2n(\alpha^2-\beta^2)-(n-a-1)\right]\eta(Y)\eta(Z). \tag{44}$$

Next, by replacing Z by ϕZ in above and then using (2), we have

$$\frac{n-2}{n-1}\alpha S(Y,Z)=\left[\frac{n-2}{n-1}\beta-\frac{n}{n-1}\right]S(Y,\phi Z)+$$
$$+\alpha\left[(n-2)(\alpha^2-\beta^2)+\frac{2\beta}{n-1}\right]g(Y,Z)+$$
$$+\left[\frac{1}{2(n-1)}-((n-2)\beta-1)-(n-a-1)\right]g(Y,\phi Z)-\frac{2\beta}{n-1}\alpha\eta(Y)\eta(Z) \tag{45}$$

Interchanging Y with Z in (45), we get

$$\frac{n-2}{n-1}\alpha S(Y,Z) = \left[\frac{n-2}{n-1}\beta - \frac{n}{n-1}\right] S(\phi Y, Z) + \\ +\alpha\left[(n-2)(\alpha^2-\beta^2) + \frac{2\beta}{n-1}\right] g(Y,Z) + \\ +\left[\frac{1}{2(n-1)} - ((n-2)\beta - 1) - (n-a-1)\right] g(\phi Y, Z) - \\ -\frac{2\beta}{n-1}\alpha\eta(Y)\eta(Z). \tag{46}$$

By adding (45) with (46), and then using the skew-symmetric property of ϕ, one can get

$$S(Y,Z) = \left[(n-1)(\alpha^2-\beta^2) + \frac{2\beta}{n-2}\right] g(Y,Z) - \frac{2\beta}{n-2}\eta(Y)\eta(Z) \tag{47}$$

Hence by the Definition (1), the manifold is η-Einstein.
Differentiating (47) covariantly with respect to X, and then using (7), we have

$$(\nabla_X S)(Y,Z) = \frac{2\beta}{n-2}[\alpha(g(\phi Y, X)\eta(Z) + g(\phi Z, X)\eta(Y)) + \\ +\beta(g(X,Y)\eta(Z)g(X,Z)\eta(Y)) - 2\beta\eta(X)\eta(Y)\eta(Z)]. \tag{48}$$

Taking an orthonormal frame field and contracting (48) over X and Z, we obtain

$$dr(Y) = \frac{2\beta}{n-2}[\alpha\psi + (n-1)\beta]\eta(Y), \tag{49}$$

where $\psi = Tr.\phi$. From (49), it follows that

$$dr(Y) = 0 \quad \text{if and only if} \quad \beta = 0. \tag{50}$$

Hence the theorem.

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Department of Mathematics and Computer Science
Kuvempu University, Jnana Sahyadri-577 451,
Shimoga, Karnataka, INDIA.
e-mail:
prof_bagewadi@yahoo.co.in.
prakashadg@gmail.com
vens_2003@rediffmail.com