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New Exact traveling wave solutions of the (2+1) dimensional Zakharov-Kuznetsov (ZK) equation

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Abstract

The repeated homogeneous balance method is used to construct new exact traveling wave solutions of the (2+1) dimensional Zakharov-Kuznetsov (ZK) equation, in which the homogeneous balance method is applied to solve the Riccati equation and the reduced nonlinear ordinary differential equation, respectively. Many new exact traveling wave solutions are successfully obtained. This method is straightforward and concise, and it can be also applied to other nonlinear evolution equations.

The nonlinear evolution equations have a wide array in application of many fields, which described the motion of the isolated waves, localized in a small part of space, in many fields such as hydrodynamic, plasma physics, nonlinear optics, etc. The investigation of the exact traveling wave solutions of nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena, for example, the wave phenomena observed in fluid dynamics, elastic media, optical fibers, etc. Since the knowledge of closed-form solutions of nonlinear evolution equations NEEs facilitates the testing of numerical solvers, and aids in the stability analysis.

The ZK equation is another alternative version of nonlinear model describing two-dimensional modulation of a kdv soliton [1, 2]. If a magnetic field is

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directed along the x-axis, the ZK equation in renormalized variables [3] takes the form

$$u_t + auu_x + \nabla u_x^2 = 0, \tag{1}$$

where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the isotropic Laplacian. This means that the ZK equation is given by

$$u_t + auu_x + (u_{xx} + u_{yy})_x = 0, \tag{2}$$

and

$$u_t + auu_x + (u_{xx} + u_{yy} + u_{zz})_x = 0, \tag{3}$$

in two-and three-dimensional spaces. The ZK equation governs the behavior of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [1, 2]. The ZK equation, which is more isotropic two-dimensional, was first derived for describing weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma in two dimensions [4]. The ZK equation is not integrable by the inverse scattering transform method. It was found that the solitary-wave solutions of the ZK equation are inelastic.

Motivated by the rich treasure of the ZK equations in the literature of the nonlinear development of ion-acoustic waves in a magnetized plasma, an analytical study will be conducted on the ZK equation (2) and (3). For more details about the solitary-wave solutions and the ZK equations, the reader is advised to read [1-8].

In recent years Wong et al. presented a useful homogeneous balance (HB) method [1-3] for finding exact solutions of given nonlinear partial differential equations. Fan [14]used HB method to search for Backlund transformation and similarity reduction of nonlinear partial differential equations. Also, he showed that there is a close connection among the HB method, Wiess, Tabor, Carnevale(WTC)method and Clarkson, Kruskal(CK)method.

In this paper, we use the HB method to solve the Riccati equation $\phi' = \alpha\phi^2 + \beta$ and the reduced nonlinear ordinary differential equation for the (2+1) ZK equation, respectively. It makes the HB method use more extensively.

For the (2+1) ZK equation [15]

$$u_t + a(u^2)_x + (bu_{xx} + ku_{yy})_x = 0, \tag{4}$$

where a,b and k are constants. Let us consider the traveling wave solutions

$$u(x, y, t) = u(\zeta), \qquad \zeta = x + y - dt, \tag{5}$$

where d is constant.

Substituting (5) into (4), then (1)is reduced to the following nonlinear ordinary differential equation

$$(b+k)u''' + a(u^2)' - du' = 0. \tag{6}$$

We now seek the solutions of Eq.(6) in the form

$$u = \sum_{i=0}^{m} q_i \phi^i, \tag{7}$$

where q_i are constants to be determined later and ϕ satisfy the Riccati equation

$$\phi' = \alpha\phi^2 + \beta, \tag{8}$$

where α, β are constants. It is easy to show that $m = 2$, by balancing u'' with uu'. Therefore we use the ansatz (auxiliary) equation

$$u = q_0 + q_1\phi + q_2\phi^2. \tag{9}$$

Substituting Eq.(8) and (9) into Eq.(6) and equating the coefficients of the same powers of $\phi^i (i = 0, 1, 2, 3, 4, 5)$ to zero yield the system of algebraic equations in q_0, q_1, q_2 and d

$$\begin{gathered}
2(b+k)q_1\alpha\beta^2 + (2aq_0 - d)q_1\beta = 0, \\
16(b+k)q_2\alpha\beta^2 + 2(aq_1^2 - dq_2 + 2aq_0q_2)\beta = 0, \\
2(3aq_2 + 4\alpha^2(b+k))q_1\beta + (2aq_0 - d)q_1\alpha = 0, \\
4(aq_2 + 10\alpha^2(b+k))q_2\beta + 2(aq_1^2 - dq_2 + 2aq_0q_2)\alpha = 0, \\
6(aq_2 + \alpha^2(b+k))q_1\alpha = 0, \\
4(aq_2 + 6\alpha^2(b+k))q_2\alpha = 0,
\end{gathered} \tag{10}$$

for which, with the aid of "Mathematica", we get the following solution

$$q_0 = \frac{d - 8(b+k)\alpha\beta}{2a}, \qquad q_1 = 0, \qquad q_2 = -\frac{6\alpha^2(b+k)}{a}. \tag{11}$$

For the Riccati Eq.(8), we can solve it by using the HB method as follows (I) Let $\phi = \Sigma_{i=0}^{m} b_i \tanh^i \zeta$. Balancing ϕ' with ϕ^2 leads to

$$\phi = b_0 + b_1 \tanh\zeta. \tag{12}$$

Substituting Eq.(12)into Eq.(8) we obtain the following solution of Eq.(8)

$$\phi = \beta\tanh\zeta = -\frac{1}{\alpha}\tanh\zeta, \qquad \alpha\beta = -1. \tag{13}$$

From Eq.(9), (11) and (13), we get the following traveling wave solutions of (2+1) ZK equation (4)

$$u(x,y,t)=\frac{1}{2a}(d-8(b+k)\alpha\beta-12(b+k)\tanh^2(x+y-dt)). \tag{14}$$

Similarly, let $\phi=\Sigma_{i=0}^{m}b_i\coth^i\zeta$, then we obtain the following traveling wave solutions of (2+1) ZK equation (4)

$$u(x,y,t)=\frac{1}{2a}(d-8(b+k)\alpha\beta-12(b+k)\coth^2(x+y-dt)). \tag{15}$$

(II) From [16], when $\alpha=1$, the Riccati equation 8)has the following solutions

$$\phi=\begin{cases}-\sqrt{-\beta}\tanh(\sqrt{-\beta}\zeta), & \beta<0,\\ -\frac{1}{\zeta}, & \beta=0,\\ \sqrt{\beta}\tan(\sqrt{\beta}\zeta). & \beta>0.\end{cases} \tag{16}$$

From (9),(11) and (16), we have the following traveling wave solutions of (2+1) ZK equation (4).

When $\beta<0$, we have

$$u(x,y,t)=\frac{1}{2a}(d-8(b+k)\beta+12(b+k)\beta\tanh^2(\sqrt{-\beta}(x+y-dt))). \tag{17}$$

When $\beta=0$, we have

$$u(x,y,t)=\frac{d-8(b+k)\beta}{2a}-\frac{6(b+k)}{a(x+y-dt)^2}. \tag{18}$$

When $\beta>0$, we have

$$u(x,y,t)=\frac{1}{2a}(d-8(b+k)\beta-12(b+k)\beta\tan^2(\sqrt{-\beta}(x+y-dt))). \tag{19}$$

(III) We suppose that the Riccati equation (8) has the following solutions of the form

$$\phi=A_0+\sum_{i=1}^{m}(A_if^i+B_if^{i-1}g), \tag{20}$$

with

$$f=\frac{1}{\cosh\zeta+r},\qquad g=\frac{\sinh\zeta}{\cosh\zeta+r},$$

which satisfy

$$f'(\zeta)=-f(\zeta)g(\zeta),\quad g'(\zeta)=1-g^2(\zeta)-rf(\zeta),$$

$$g^2(\zeta) = 1 - 2rf(\zeta) + (r^2-1)f^2(\zeta).$$

Balancing ϕ' with ϕ^2 leads to

$$\phi = A_0 + A_1 F + B_1 g. \tag{21}$$

Substituting Eq.(21) into (8), collecting the coefficient of the same power $f^i g^j$ $(i = 0,1,2; j = 0,1)$ and setting each of the obtained coefficients to zero yield the following set of algebraic equations

$$\begin{gathered}
\alpha A_1^2 + \alpha(r^2-1)B_1^2 + (r^2-1)B_1 = 0, \\
2\alpha A_1 B_1 + A_1 = 0, \\
2\alpha A_0 A_1 - 2\alpha r B_1^2 - rB_1 = 0, \\
2\alpha A_0 B_1 = 0, \\
\alpha A_0^2 + \alpha B_1^2 + \beta = 0,
\end{gathered} \tag{22}$$

which have solutions

$$A_0 = 0, \quad A_1 = \pm\sqrt{\frac{(r^2-1)}{4\alpha^2}}, \quad B_1 = -\frac{1}{2\alpha}, \tag{23}$$

where $4\alpha\beta = -1$. From Eqs.(20),(23), we have

$$\phi = \frac{-1}{2\alpha}\left(\frac{\sinh\zeta \mp \sqrt{(r^2-1)}}{\cosh\zeta + r}\right). \tag{24}$$

From Eqs.(9), (11) and (24), we obtain

$$u(x,y,t) = \frac{1}{2a}((d - 8(b+k)\alpha\beta - 3(b+k)(\frac{\pm\sqrt{r^2-1} - \sinh(\zeta)}{r + \cosh(\zeta)})^2), \tag{25}$$

where

$$\zeta = x + y - dt.$$

(IV) We take ϕ in the Riccati equation(8) as being of the form

$$\phi = e^{p_1\zeta}\rho(z) + p_4(\zeta), \tag{26}$$

where

$$z = e^{p_2\zeta} + p_3, \tag{27}$$

where p_1, p_2 and p_3 are constants to be determined.

Substituting (26) and (27) into (8), we have

$$p_2 e^{(p_1+p_2)\zeta}\rho' - \alpha e^{2p_1\zeta}\rho^2 + (p_1 - 2\alpha p_4)e^{p_1\zeta}\rho + p_4' - \alpha p_4^2 - \beta = 0. \tag{28}$$

Setting $p_1+p_2=2p_1$, we get $p_1=p_2$. If we assume that $p_4=\frac{p_1}{2\alpha}$ and $\beta=-\frac{p_1^2}{4\alpha}$, then Eq.(28) becomes

$$p_2\rho'-\alpha\rho^2=0. \tag{29}$$

By solving Eq.(29), we have

$$\rho=-\frac{p_1}{\alpha z}=-\frac{p_1}{\alpha e^{p_1\zeta}+p_3}. \tag{30}$$

Substituting (30) and $p_4=\frac{p_1}{2\alpha}$ into (26), we have

$$\phi=-\frac{p_1e^{p_1\zeta}}{\alpha(e^{p_1\zeta}+p_3)}+\frac{p_1}{2\alpha}. \tag{31}$$

If $p_3=1$ in (31), we get

$$\phi=-\frac{p_1}{2\alpha}\tanh(\frac{1}{2}p_1\zeta). \tag{32}$$

If $p_3=-1$ in (31), we get

$$\phi=-\frac{p_1}{2\alpha}\coth(\frac{1}{2}p_1\zeta). \tag{33}$$

From (9), (11) and (31), we obtain the following traveling wave solutions of (2+1) ZK equation (1)

$$u(x,y,t)=\frac{1}{2a}(d-8(b+k)\alpha\beta-3p_1^2(b+k)(\frac{2e^{p_1(x+y-dt)}-1}{e^{p_1(x+y-dt)}+p_3})^2). \tag{34}$$

When $p_3=1$, we have, from (32),

$$u(x,y,t)=\frac{1}{2a}(d-8(b+k)\alpha\beta-3p_1^2(b+k)\tanh^2(\frac{p_1}{2}(x+y-dt)). \tag{35}$$

Clearly, (14) is the special case of (35) with $p_1=2$. When $p_3=-1$, we have from (33),

$$u(x,y,t)=\frac{1}{2a}(d-8(b+k)\alpha\beta-3p_1^2(b+k)\coth^2(\frac{p_1}{2}(x+y-dt)). \tag{36}$$

Clearly, (15) is the special case of (36) with $p_1=2$.

(V) We suppose that the Riccati equation (8) has the following solutions of the form

$$\phi=A_0+\sum_{i=1}^{m}\sinh^{i-1}(A_i\sinh\omega+B_i\cosh\omega),$$

where $d\omega/d\zeta = \sinh\omega$ or $d\omega/d\zeta = \cosh\omega$. It is easy to find that $m = 1$, by balancing ϕ' and ϕ^2. So we choose

$$\phi = A_0 + A_1 \sinh\omega + B_1 \cosh\omega, \tag{37}$$

when $d\omega/d\zeta = \sinh\omega$, we substitute (37) and $d\omega/d\zeta = \sinh\omega$, into (8), and set the coefficient of $\sinh^i\omega\cosh^j\omega (i = 0, 1, 2; j = 0, 1)$ to zero. A set of algebraic equations is obtained as follows

$$\begin{gathered} \alpha A_0^2 + \alpha B_1^2 + \beta = 0, \\ 2\alpha A_0 A_1 = 0, \\ \alpha A_1^2 + \alpha B_1^2 = B_1 \\ 2\alpha A_0 B_1 = 0, \\ 2\alpha A_1 B_1 = A_1, \end{gathered} \tag{38}$$

for which, we have the following solutions

$$A_0 = 0, \quad A_1 = 0, \quad B_1 = \frac{1}{\alpha}, \tag{39}$$

where $c = \frac{-1}{a}$, and

$$A_0 = 0, \quad A_1 = \pm\frac{1}{2\alpha}, \quad B_1 = \frac{1}{2\alpha}, \tag{40}$$

where $\beta = -\frac{1}{4\alpha}$.

To $d\omega/d\zeta = \sinh\omega$, we have

$$\sinh\omega = -csch\zeta, \quad \cosh\omega = -\coth\zeta. \tag{41}$$

From (38)-(41), we obtain

$$\phi = -\frac{coth\zeta}{\alpha}, \tag{42}$$

where $\beta = -\frac{1}{\alpha}$, and

$$\phi = \frac{\coth\zeta \pm csch\zeta}{2\alpha}, \tag{43}$$

where $\beta = -\frac{1}{4\alpha}$.

Clearly, (42)is the special case of (34)with $p_1 = 2$.

From (9),(11),(42) and (43), we get the exact traveling wave solutions of (2+1) ZK equation (4) in the following form

$$u(x, y, t) = \frac{1}{2a}((d - 8(b + k)\alpha\beta - 12(b + k)\coth^2(\zeta)), \tag{44}$$

which is identical with (15).

$$u(x,y,t)=\frac{1}{2a}((d-8(b+k)\alpha\beta-3(b+k)(coth(\zeta)\pm csch(\zeta))^2), \quad (45)$$

where $\zeta = x+y-dt$.

Similarly, when $d\omega/d\zeta = \cosh\omega$, we obtain the following exact traveling wave solutions of (2+1) ZK equation (4) in the following form

$$u(x,y,t)=\frac{1}{2a}((d-8(b+k)\alpha\beta-12(b+k)\cot^2(\zeta)), \quad (46)$$

$$u(x,y,t)=\frac{1}{2a}((d-8(b+k)\alpha\beta-3(b+k)(\cot(\zeta)\pm\csc(\zeta))^2), \quad (47)$$

where $\zeta = x+y-dt$.

In this paper, we exhibited the repeated homogeneous balance method to study the (2+1) dimensional Zakharov-Kuznetsov (ZK) equation. New solitons and periodic solutions were formally derived. These solutions may be helpful to describe waves features in plasma physics. Moreover, the obtained results in this work clearly demonstrate the reliability of the repeated homogeneous balance method.

We now summarize the key steps as follows

Step1: For a given nonlinear evolution equation

$$F(u,u_t,u_x,u_{xt},u_{tt},\ldots)=0, \quad (48)$$

we consider its traveling wave solutions $u(x,y,t)=u(\zeta), \zeta=x+y-dt$ then Eq.(47) is reduced to a nonlinear ordinary differential equation

$$Q(u,u',u'',u''',\ldots)=0, \quad (49)$$

where a prime denotes $\frac{d}{d\zeta}$.

Step2: For a given ansatz equation (for example, the ansatz equation is $\phi' = \alpha\phi^2+\beta$ in this paper), the form of u is decided and the HB method is used on Eq.(49) to find the coefficients of u.

Step3: The HB method is used to solve the ansatz equation.

Step4: Finally, the traveling wave solutions of Eq.(48) are obtained by combining Step2 and Step3.

From the above procedure, it is easy to find that the HB method is more effective and simple than other methods and a lot of solutions can be also applied to other nonlinear evolution equations.

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