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About some linear operators defined by infinite sums

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Abstract

In this paper we study a general class of linear operators defined by infinite sum. In particular, we obtain the convergence and the evaluation for the rate of convergence in therm of the first modulus of smoothness for the Mirakjan-Favard-Szász, Meyer-König and Zeller operators.

1 Introduction

In this section, we recall some notions and results which we will use in this paper.

For a given interval I we shall use the following function sets: $B(I) = \{f | f : I \to \mathbb{R},\ f$ bounded on $I\}$, $C(I) = \{f | f : I \to \mathbb{R},\ f$ continuous on $I\}$ and $C_B(I) = B(I) \cap C(I)$. For any $x \in I$ consider the functions $\psi_x : I \to \mathbb{R}$, given by $\psi_x(t) = t - x$ and $e_i : I \to \mathbb{R}$, $e_i(t) = t^i$ for any $t \in I$, $i \in \{0, 1, 2\}$.

For $f \in C_B(I)$, by the first order modulus of smoothness of f is meant the function $\omega(f; \cdot) : [0, \infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega(f; \delta) = \sup\left\{|f(x') - f(x'')| : x', x'' \in I,\ |x' - x''| \leq \delta\right\}. \tag{1.1}$$

Let $\mathbb{N}$ be the set of positive integer numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$ consider the operators $S_m : C_2([0, \infty)) \to C([0, \infty))$ defined for any function $f \in C_2([0, \infty))$ by

$$(S_m f)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right), \tag{1.2}$$

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$x \in [0,\infty)$, where $C_2([0,\infty)) = \left\{ f \in C([0,\infty)) : \lim\limits_{x\to\infty} \dfrac{f(x)}{1+x^2} \text{ exists and is finite} \right\}$.

The operators $(S_m)_{m\geq 1}$ are named the Mirakjan-Favard-Szász operators and were introduced in 1941 by G. M. Mirakjan in [6]. They were intensively studied by J. Favard in 1941 in [3] and O. Szász in 1950 in [11].

W. Meyer-König and K. Zeller have introduced in [5] a sequence of linear and positive operators.

These operators, $Z_m : C([0,1]) \to C([0,1])$, defined for any function $f \in C([0,1])$ and $m \in \mathbb{N}$ by

$$(Z_m f)(x) = \begin{cases} (1-x)^{m+1} \sum\limits_{k=0}^{\infty} \dbinom{m+k}{k} x^k f\left(\dfrac{k}{m+k}\right) & \text{if } 0 \leq x < 1 \\ f(1) & \text{if } x = 1 \end{cases} \tag{1.3}$$

are nowadays called the Meyer-König and Zeller operators.

For the following see [10].

Let $I, J \subset \mathbb{R}$ be intervals with $I \cap J \neq \emptyset$. For $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$ consider the function $\varphi_{m,k} : J \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in J$ and the linear positive functional $A_{m,k} : E(I) \to \mathbb{R}$.

For $m \in \mathbb{N}$, let the operator $L_m : E(I) \to F(J)$ be defined by

$$(L_m f)(x) = \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(f) \tag{1.4}$$

for any $f \in E(I)$ and $x \in J$, where $E(I)$ and $F(J)$ are subsets of the set of real functions defined on I and J, respectively. These operators are linear and positive on $E(I \cap J)$.

For $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$ define T_i by

$$(T_i L_m)(x) = m^i \left(L_m \psi_x^i\right)(x) = m^i \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}\left(\psi_x^i\right) \tag{1.5}$$

for any $x \in I \cap J$.

In what follows $s \in \mathbb{N}_0$, s is even.

We suppose that the operators $(L_m)_{m\geq 1}$ verify the conditions:

$$A_{m,k}(e_0) = 1 \tag{1.6}$$

for any $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \varphi_{m,k}(x) = 1 \tag{1.7}$$

for any $x \in I \cap J$ and any $m \in \mathbb{N}$, there exist the smallest α_s, $\alpha_{s+2} \in [0, \infty)$ such that

$$\lim_{m\to\infty} \frac{(T_j L_m)(x)}{m^{\alpha_j}} = B_j(x) \in \mathbb{R} \tag{1.8}$$

for any $x \in I \cap J$, $j \in \{s, s+2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{1.9}$$

Remark 1.1 By (1.6) and (1.7) it results that

$$(T_0 L_m)(x) = 1 \tag{1.10}$$

for any $x \in I \cap J$ and $m \in \mathbb{N}$.

For $s = 0$ and $s = 2$ we have the following theorems.

Theorem 1.1 *Let $f : I \to \mathbb{R}$ be a function, $f \in E(I)$. If $x \in I \cap J$ and f is continuous at x, then*

$$\lim_{m\to\infty} (L_m f)(x) = f(x). \tag{1.11}$$

Assume that f is continuous on I and there exists an interval $K \subset I \cap J$ such that $m(0) \in \mathbb{N}$ and $k_2(K) \in \mathbb{R}$ exist, so that for $m \geq m(0)$ and $x \in K$ we have

$$\frac{(T_2 L_m)(x)}{m^{\alpha_2}} \leq k_2(K). \tag{1.12}$$

Then the convergence given in (1.11) is uniform on K and

$$|(L_m f)(x) - f(x)| \leq (1 + k_2(K))\omega\left(f; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right) \tag{1.13}$$

for any $x \in K$ and $m \geq m(0)$.

Theorem 1.2 *Let $f : I \to \mathbb{R}$ be a function, $f \in E(I)$. If $x \in I \cap J$ and f is a two times differentiable function at x with $f^{(2)}$ continuous at x, then*

$$\lim_{m\to\infty} m^{2-\alpha_2}\left[(L_m f)(x) - f(x) - \frac{1}{m}(T_1 L_m)(x) f^{(1)}(x)\right] \tag{1.14}$$
$$= \frac{1}{2} B_2(x) f^{(2)}(x).$$

Assume that f is a two times differentiable function on I with $f^{(2)}$ continuous on I and an interval $K \subset I \cap J$ exists such that $m(2) \in \mathbb{N}$ and $k_j(K)$ exist, so that for any $m \geq m(2)$ and $x \in K$ we have

$$\frac{(T_j L_m)(x)}{m^{\alpha_j}} \leq k_j(K), \tag{1.15}$$

where $j \in \{2, 4\}$. Then the convergence given in (1.14) is uniform on K.

It is known that (see [10])

$$(T_0S_m)(x) = 1, \tag{1.16}$$

$$(T_1S_m)(x) = 0, \tag{1.17}$$

$$\lim_{m\to\infty} \frac{(T_2S_m)(x)}{m} = x \tag{1.18}$$

for any $x \in [0, \infty)$,

$$(T_0S_m)(x) = 1 = k_0, \tag{1.19}$$

$$\frac{(T_2S_m)(x)}{m} \leq b = k_2, \tag{1.20}$$

$$\frac{(T_4S_m)(x)}{m^2} \leq 3b^2 + b = k_4 \tag{1.21}$$

for any $m \in \mathbb{N}$ and $x \in K = [0, b]$, where $b > 0$, and

$$(T_0Z_m)(x) = 1, \tag{1.22}$$

$$(T_1Z_m)(x) = 0, \tag{1.23}$$

$$\lim_{m\to\infty} \frac{(T_2Z_m)(x)}{m} = x(1-x)^2 \tag{1.24}$$

for any $x \in [0, 1]$,

$$(T_0Z_m)(x) = 1 = k_0, \tag{1.25}$$

$$\frac{(T_2Z_m)(x)}{m} \leq 2 = k_2 \tag{1.26}$$

for any $m \in \mathbb{N}$ and $x \in [0, 1]$.

2 Preliminaries

In this section we construct a general class of linear positive operators. Let I, J be intervals with $I \cap J \neq \emptyset$.

For $m \in \mathbb{N}$ let $b_m : J \to \mathbb{R}$ be a indefinitely differentiable function such that

$$b_m(x) > 0 \tag{2.1}$$

for any $x \in J$ and for any compact $K \subset J$ there exists $M(K)$ such that

$$\left| b_m^{(k)}(x) \right| \leq M(K) \tag{2.2}$$

for any $x \in K$ and $k \in \mathbb{N}_0$.

Then, it is known that

$$b_m(x) = \sum_{k=0}^{\infty} \frac{1}{k!} b_m^{(k)}(0) x^k \tag{2.3}$$

for any $x \in K$ and $m \in \mathbb{N}$.
For $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$ consider the linear positive functionals $A_{m,k} : E(I) \to \mathbb{R}$.

Definition 2.1 For $m \in \mathbb{N}$ define the operator $L_m : E(I) \to F(J)$ by

$$(L_m f)(x) = \frac{1}{b_m(x)} \sum_{k=0}^{\infty} \frac{1}{k!} b_m^{(k)}(0) x^k A_{m,k}(f) \tag{2.4}$$

for any $f \in E(I)$ and $x \in J$.

Remark 2.1 The sets $E(I)$, $F(J)$ are subsets of the set of real functions defined on I and J, respectively such that the series from (2.4) is convergent.

Definition 2.2 For $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$ define T_i by

$$(T_i L_m)(x) = m^i \left(L_m \psi_x^i \right)(x) = m^i \frac{1}{b_m(x)} \sum_{k=0}^{\infty} \frac{1}{k!} b_m^{(k)}(0) x^k A_{m,k}\left(\psi_x^i \right), \tag{2.5}$$

where $x \in I \cap J$.

3 Main results

In this section we study the operators that we introduced in the previous section.

Proposition 3.1 *The operators L_m, $m \in \mathbb{N}$ are linear and positive on $E(I \cap J)$.*

Proof. The proof follows immediately.

In the following we suppose that the operators $(L_m)_{m \geq 1}$ verify the conditions:

$$A_{m,k}(e_0) = 1 \tag{3.1}$$

for any $m \in \mathbb{N}$, $k \in \mathbb{N}_0$,

$$A_{m,0}(e_1) = 0 \tag{3.2}$$

for any $m \in \mathbb{N}$,

$$b_m^{(k)}(0) A_{m,k}(e_1) = k b_m^{(k-1)}(0) \tag{3.3}$$

for any $m, k \in \mathbb{N}$,

$$\lim_{m\to\infty} m^{2-\alpha_2}\left[(L_m e_2)(x) - x^2\right] = B_2(x) \in \mathbb{R} \tag{3.4}$$

for any $x \in I \cap J$, where α_2 is the smallest and

$$2 < \alpha_2. \tag{3.5}$$

Lemma 3.1 *We have*

$$(L_m e_0)(x) = 1, \tag{3.6}$$

$$(L_m e_1)(x) = 0 \tag{3.7}$$

for any $x \in J$ and any $m \in \mathbb{N}$,

$$(T_0 L_m)(x) = 1, \tag{3.8}$$

$$(T_1 L_m)(x) = 0, \tag{3.9}$$

$$(T_2 L_m)(x) = m^2\left[(L_m e_2)(x) - x^2\right] \tag{3.10}$$

for any $x \in I \cap J$ and $m \in \mathbb{N}$.

Proof. By (3.1) we have

$$(L_m e_0)(x) = \frac{1}{b_m(x)} \sum_{k=0}^{\infty} \frac{1}{k!} b_m^{(k)}(0) x^k$$

and from (2.3), (3.6) results. By (3.2) and (3.3) we have

$$(L_m e_1)(x) = \frac{1}{b_m(x)} \sum_{k=0}^{\infty} \frac{1}{k!} b_m^{(k)}(0) x^k A_{m,k}(e_1) = \frac{1}{b_m(x)} \sum_{k=1}^{\infty} \frac{1}{k!}(0) A_{m,k}(e_1) x^k$$
$$= x\frac{1}{b_m(x)} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} b_m^{(k-1)}(0) x^{k-1}$$

and (3.7) results. From (3.6) and (3.7) we obtain (3.8) – (3.10).

In the following we suppose that the function $B_2 : I \cap J \to \mathbb{R}$ is bounded on any compact interval K, $K \subset I \cap J$.

Lemma 3.2 *We have*

$$\lim_{m\to\infty} \frac{(T_2 L_m)(x)}{m^{\alpha_2}} = B_2(x) \tag{3.11}$$

for any $x \in I \cap J$ and if $K \subset I \cap J$, K is a compact interval, $m(2) \in \mathbb{N}$ and $k_2(K) \in \mathbb{R}$ exist, so that for any $m \geq m(0)$ and $x \in K$

$$\frac{(T_2 L_m)(x)}{m^{\alpha_2}} \leq k_2(K). \tag{3.12}$$

Proof. From (3.4) and (3.10), (3.11) results. Because the function B_2 is bounded on any compact K, $K \subset I \cap J$, it results the inequality from (3.12).

Theorem 3.1 *Let $f : I \to \mathbb{R}$ be a function, $f \in E(I)$. If $x \in I \cap J$ and f is continuous at x, then*

$$\lim_{m\to\infty} (L_m f)(x) = f(x). \tag{3.13}$$

If f is continuous on $I \cap J$, then the convergence given in (3.13) is uniform on any compact $K \subset I \cap J$ and $m(0) \in \mathbb{N}$ and $k_2(K) \in \mathbb{R}$ exist, so that for any $m \geq m(0)$ and $x \in K$ we have

$$|(L_m f)(x) - f(x)| \leq (1 + k_2(K))\omega\left(f; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right). \tag{3.14}$$

Proof. It results from Theorem 1.1, Lemma 3.1 and Lemma 3.2.

Theorem 3.2 *Let $f : I \to \mathbb{R}$ be a function, $f \in E(I)$. If the smallest $\alpha_4 \in [0, \infty)$ exists, such that*

$$\lim_{m\to\infty} \frac{(T_4 L_m)(x)}{m^{\alpha_4}} \in \mathbb{R} \tag{3.15}$$

for any $x \in I \cap J$ and

$$\alpha_4 < \alpha_2 + 2, \tag{3.16}$$

then for $x \in I \cap J$ and f a two times differentiable function at x with $f^{(2)}$ continuous at x, we have

$$\lim_{m\to\infty} m^{2-\alpha_2} \left[(L_m f)(x) - f(x)\right] = \frac{1}{2} B_2(x) f^{(2)}(x). \tag{3.17}$$

Assume that f is a two times differentiable function on I with $f^{(2)}$ continuous on I and for $K \subset I \cap J$, K is a compact interval, $m(2) \in \mathbb{N}$ and $k_4(K) \in \mathbb{R}$ exist, so that for any $m \geq m(2)$ and $x \in k_4(K)$ we have

$$\frac{(T_4 L_m)(x)}{m^{\alpha_4}} \leq k_4(K). \tag{3.18}$$

Then the convergence given in (3.17) is uniform on K.

Proof. It results from Theorem 1.2, Lemma 3.1 and Lemma 3.2.

Now we discuss some particular cases.

Example 3.1 We consider $I = J = [0, \infty)$, $E(I) = C_2([0, \infty))$, $F(J) = C([0, \infty))$, $b_m(x) = e^{mx}$ for any $x \in [0, \infty)$ and $m \in \mathbb{N}$, $A_{m,k}(f) = f\left(\frac{k}{m}\right)$ for any $f \in C_2([0, \infty))$, $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then $b_m^{(k)}(0) = m^k$ for any $x \in [0, \infty)$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and we obtain the Mirakjan-Favard-Szász operators.

Theorem 3.3 *Let $f : [0, \infty) \to \mathbb{R}$ be a function, $f \in C_2([0, \infty))$. If f is a s times differentiable at $x \in [0, \infty)$ with $f^{(s)}$ continuous at x, then*

$$\lim_{m\to\infty} (S_m f)(x) = f(x) \tag{3.19}$$

if $s = 0$ and

$$\lim_{m\to\infty} m\left[(S_m f)(x) - f(x)\right] = \frac{1}{2}\, x f^{(2)}(x) \tag{3.20}$$

if $s = 2$.

If f is a s times differentiable function on $[0, \infty)$ with $f^{(s)}$ continuous on $[0, \infty)$, then the convergence given in (3.19) *and* (3.20) *are uniform on every compact $[0, b] \subset [0, \infty)$, where $b > 0$.*

Moreover

$$|(S_m f)(x) - f(x)| \le (1 + b)\omega\left(f; \frac{1}{\sqrt{m}}\right) \tag{3.21}$$

for any $f \in C([0, \infty))$, $m \in \mathbb{N}$ and $x \in [0, b]$.

Proof. We have $\alpha_0 = 0$, $\alpha_2 = 1$, $\alpha_4 = 2$, $k_2 = b$, $k_4 = 3b^2 + b$ (see [10]) and we apply Theorem 3.1 and Theorem 3.2.

Example 3.2 Let $I = J = [0, 1]$, $E(I) = F(J) = C([0, 1])$, $b_m(x) = (1 - x)^{-m-1}$ for $x \in [0, 1)$ and $m \in \mathbb{N}$, $A_{m,k}(f) = f\left(\frac{k}{m+k}\right)$ for any $f \in C([0, 1])$, $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then $b_m^{(k)}(x) = k!\binom{m+k}{k}(1 - x)^{-m-k-1}$ for any $x \in [0, 1)$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and we obtain the Meyer-König and Zeller operators.

Theorem 3.4 *Let $f : [0, 1] \to \mathbb{R}$ be a function, $f \in C([0, 1])$. If f is a s times differentiable at $x \in [0, 1]$ with $f^{(s)}$ continuous at x, then*

$$\lim_{m\to\infty} (Z_m f)(x) = f(x) \tag{3.22}$$

if $s = 0$ and

$$\lim_{m\to\infty} m\left[(Z_m f)(x) - f(x)\right] = \frac{1}{2}\, x(1 - x)^2 f^{(2)}(x) \tag{3.23}$$

if $s = 2$.

If f is continuous on $[0, 1]$, then the convergence given in (3.22) *is uniform on $[0, 1]$ and*

$$|(Z_m f)(x) - f(x)| \le 3\omega\left(f; \frac{1}{\sqrt{m}}\right) \tag{3.24}$$

for any $x \in [0, 1]$ and $m \in \mathbb{N}$.

Proof. We have $\alpha_0 = 0$, $\alpha_2 = 1$, $k_2 = 2$, $\lim_{m\to\infty} \frac{(T_2 Z_m)(x)}{m} = x(1 - x)^2$ (see [10]), and we apply Theorem 3.1 and Theorem 3.2.

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