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ABOUT THE H - MEASURE OF A SET. II.

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Dedicated to Professor Mirela Ştefănescu on the occasion of her 60th birthday

Abstract

In the papers $[B1]$ and $[B2]$ we have established some conditions for the finitude of the Hausdorff h-measure of some set. Now, we shall determine a better minorant for this measure.

1 Definitions

We denote by R^n the euclidean n - dimensional space and by $d(E)$ - the diameter of a set $E \subset R^n$.

Definition 1 *If $r_0 > 0$ is a fixed number, a continuous function $h(r)$, defined on $[0, r_0)$, nondecreasing and such that $\lim_{r\to 0} h(r) = 0$ is called a measure function.*

If $E \subset R^n$ is a bounded set and $\delta \in R_+$, the Hausdorff h-measure of E is defined by:

$$H_h(E) = \lim_{\delta \to 0} \inf \sum_i h(\rho_i),$$

inf being considered over all coverings of E with a countable number of spheres of radius $\rho_i \leq \delta$.

Definition 2 *$f : D(\subset R^n) \to \overline{R}$ is a δ - class Lipschitz function if*

$$|f(x+\alpha) - f(x)| \leq M\, |\alpha|^{\delta}, x \in D, \alpha \in R^n, x+\alpha \in D, M > 0.$$

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Definition 3 *Let $\varphi_1, \varphi_2 > 0$ be functions defined in a neighborhood of $0 \in R^n$. We say that φ_1 and φ_2 are equivalent and we denote by: $\varphi_1 \sim \varphi_2$, for $x \to 0$, if there exist $r > 0$, $Q > 0$, satisfying:*

$$\frac{1}{Q}\varphi_1(x) \leq \varphi_2(x) \leq Q\varphi_1(x), (\forall)x \in R^n, |x| < r.$$

An analogous definition can be given for $x \to \infty$. In this case, $\varphi_1 \sim \varphi_2$ means that the previous inequalities have place in all the space.

If $f : [0,1] \longrightarrow \overline{R}$, the graph of f is the set: $\Gamma = \{(x, f(x)) | x \in [0,1]\}$.

2 Results

Theorem 1 *If $\delta \in [0,1]$, h is a measure function such that*

$$h(t)\tilde{\ }t^2 \tag{1}$$

and $f : [0,1] \to \overline{R}$ is a δ- class Lipschitz function, then: $H_h(\Gamma) < +\infty$.

(see $[B1]$)

Lemma 2 *Consider that $E \subset R^n$ is a closed and bounded set, which has a finite Hausdorff h - measure. Suppose that there exists an additive function $\varphi(U)$,defined on union, U, of n - dimensional intervals of the type*

$$Q = [a_1, b_1) \times ... \times [a_n, b_n), \; a_i, b_i \in R, \; a_i < b_i, i = 1, 2, ..., n, \tag{2}$$

and which satisfies the properties:
(1) $\varphi(U) \geq 0$, for every U;
(2) if $U \supseteq E$ then $\varphi(U) \geq \alpha$, where α is a fixed constant;
(3) there exists a constant $k \neq 0$ such that:

$$\varphi(U) \leq k \cdot h\left[d(U)\right]. \tag{3}$$

Then:
$$H_h(E) \geq \frac{\alpha}{k}.$$

Proof. We denote by **M** the set of all intervals U of the type (2). To determine $H_h(E)$, we consider a covering of E with sets that satisfy Definition 1. From the Heine - Borel - Lebesgue theorem, it results that we can choose a finite number of convex sets $(E_i)_{i \in I}$ (I is finite) such that: $E \subset \cup_{i \in I} E_i$.

Consider $E_i \subset U_i \in$**M,** with:

$$h\left[d(U_i)\right] < (1+\varepsilon)h\left[d(E_i)\right], \; \varepsilon > 0.$$

From (3), we have:

$$h\left[d(U_i)\right] \geq \frac{1}{k}\varphi(U_i).$$

Thus:

$$\sum_{i\in I} h\left[d(E_i)\right] > \frac{1}{1+\varepsilon}\sum_{i\in I} h\left[d(U_i)\right] \geq \frac{1}{k(1+\varepsilon)}\sum_{i\in I}\varphi(U_i) \geq \frac{1}{k(1+\varepsilon)}\varphi(\cup_{i\in I}U_i)$$

because

$$\varphi(\cup_{i\in I}U_i) \leq \sum_{i\in I}\varphi(U_i).$$

But $\cup_{i\in I}U_i \supset E$ and we can apply (3): there exists a constant $\alpha > 0$ such that:

$$\varphi(\cup_{i\in I}U_i) \geq \alpha.$$

Thus

$$\sum_{i\in I} h\left[d(E_i)\right] \geq \frac{1}{k(1+\varepsilon)}\varphi(\cup_{i\in I}U_i) \geq \frac{\alpha}{k(1+\varepsilon)}$$

and $H_h(E) \geq \dfrac{\alpha}{k}$. □

Theorem 3 *In the hypothesis of the previous theorem there exist* α, $k > 0$ *such that:* $H_h(\Gamma) \geq \dfrac{\alpha}{k}$.

Proof. We prove that the conditions of the Lemma 5 are satisfied. $H_h(\Gamma) > 0$, from the Theorem 4. We consider $U = \bigcup_{i=1}^{m} Q_i$,where

$$Q_i = [a_i, b_i) \times [c_i, d_i)\,,\; a_i, b_i \in R,\; a_i < b_i, c_i < d_i, i = 1, 2, ..., m \tag{4}$$

and we define

$$\varphi(U) = \sum_{i=1;c_i d_i>0}^{m} (b_i - a_i) \times \max\{|c_i|\,, |d_i|\} + \sum_{i=1;c_i d_i>0}^{m} (b_i - a_i) \times |d_i - c_i|\,. \tag{5}$$

(i) $\varphi(U) \geq 0$, for every U.

(ii) We denote:

$$\begin{aligned} G_1 &= \left\{(x, y) \in R^2 : 0 \leq x \leq 1, 0 \leq y \leq f(x)\right\} \\ G_2 &= \left\{(x, y) \in R^2 : 0 \leq x \leq 1, f(x) \leq y \leq 0\right\} \end{aligned}$$

and $\alpha-$ the sum of the areas of $G_1$and G_2 :

$$\alpha = \sigma(G_1) + \sigma(G_2).$$

If $U \supseteq \Gamma$, then $\varphi(U) \geq \alpha$.

(iii) From (1), we deduce that there exists a constant $Q > 0$ such that:

$$\frac{1}{Q} d^2(U) \leq h\left[d(U)\right] \leq Q d^2(U).$$

Then

$$\varphi(U) \leq \sum_{i=1}^{m} d(U)^2 = m \cdot d(U)^2 \leq mQh\left[d(U)\right].$$

Using theorem 4, it results:

$$H_h(\Gamma) \geq \frac{1}{mQ}\left[\sigma(G_1) + \sigma(G_2)\right].$$

The proof is complete. □

References

[B1] A. Barbulescu, *La finitude d'une h -mesure Hausdorff d' une ensemble des points dans le plan,* Analele Univ. "Valahia", Târgoviste, 1995/1996, fasc. II, p.93 -99

[B2] A. Barbulescu, *About the h - measure of a set,* Proceedings of "International Conference on Complex Analysis and Related Topics", Brasov, 2001 (to appear)

[B3] A. Barbulescu, *P- modulus and p - capacity (PhD. Thesis),* Iassy, 1997

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