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ON THE SPACE OF MARKOV CHAIN'S FINITE PROJECTIONS

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Dedicated to Professor Mirela Ştefănescu on the occasion of her 60th birthday

Abstract

For Markov chains in this paper the connection between their Martin kernel and finite projections was established. As a consequence we have a new interpretation of $h-$ transformation of Markov probability which may serve as a new base for construction of Boundary theory for Markov chains, using only conditioning of Markov probability.

1. Preliminaries. Paper [1] described the finite projection for countable Markov chains. Thus, we deal with a Markov chain $X = (X_n)_{n\geqslant 0}$ with discrete state space $E = \{0, 1, 2, ...\}$, specified by the transition matrix $(p_{ij})_{i,j\in E}$ and initial distribution $\gamma = (\gamma(i))_{i\in E}$. $(\Omega, \mathcal{F}, \mathbf{P})$ denotes the corresponding probability sampling space, where $\mathcal{F} = \sigma(X_0, X_1, ...)$ is the $\sigma-$ algebra generated by random variables (r.v.) $X_0, X_1, ...$. To reproduce from [1] the main result used in our paper we need the following quantities:

$$\begin{aligned}
p_{ij}^{(n)} &= \mathbf{P}(X_n = j/X_0 = i), p_{ij}^{(0)} = \delta(i,j) = \begin{cases} 1, if\ i \neq j, \\ 0, if\ i = j, \end{cases} \\
f_{ij}^{(n)} &= \mathbf{P}(X_n = j, X_m \neq j, m = \overline{1, n-1}/X_0 = i), f_{ij}^{(0)} = 0; \\
f_{ij} &= \sum_{n=1}^{\infty} f_{ij}^{(n)}, m_{ij} = \sum_{n=1}^{\infty} n \cdot f_{ij}^{(n)}, i, j \in E.
\end{aligned}$$

To avoid the common situations we suppose that for Markovian probability all the states from E are $\mathbf{P}-$possible, i.e.,

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$$\sum_{i\in E}\gamma(i)f_{ij} > 0, \forall j \in E. \qquad (1.1)$$

Associated with the state 0 there are hitting times

$T(i) = \inf\{n > T(i-1)|\ X_n = 0\}, i \geqslant 1$, considering $T(0) = \inf\{n \geqslant 1|\ X_n = 0\}$, $\inf\{\emptyset\} = \infty$.

Proposition [1] *If* $0 < \mathbf{P}(T(1) < +\infty) < 1$ *and* $(\mathbf{P}_n)_{n\geqslant 1}$ *is a sequence of probability measures associated with* $(\Omega, \mathcal{F})$ *defined by formula* $\mathbf{P}_n(\cdot) = \mathbf{P}_n(\cdot/T(n) < \infty)$, *then*

a) there exists a unique probability measure $\mathbf{P}^*$ *defined on* $(\Omega, \mathcal{F})$ *such that for any* $A \in \sigma(X_0, X_1, ...X_k), k \geqslant 0$

$$\lim_{n\to\infty}\mathbf{P}_n\{A\} = \mathbf{P}^*\{A\};$$

b) the chain X *associated with the probability space* $(\Omega, \mathcal{F}, \mathbf{P}^*)$ *is Markovian with the state space* $E^* \subseteq E, E^* = \{i \in E\ / f_{i0} > 0\}$ *. It is specified by transition matrix* $\left(p^*_{ij}\right)_{i,j\in E^*}$ *with initial distribution* $(p^*_i)_{i\in E^*}$, *where*

$$\begin{aligned} p^*_{ij} &= p_{ij}f_{j0}^{1-\delta(0,j)}/f_{i0}, i, j \in E^*,\\ p^*_i &= p_i f_{i0}^{1-\delta(0,i)}\left(\sum_{j\neq 0}p_j f_{j0} + p_0\right)^{-1}, i \in E^*; \end{aligned}$$

c) for Markov chain X *associated with* $(\Omega, \mathcal{F})$ *the state* 0 *is recurrent and it is positive recurrent if and only if* $m_{00} < \infty$; *in add* $f^{*(n)}_{i0} = f^{(n)}_{i0}/f_{i0}$, $i \in E^*, n \geqslant 1$.

Definition 1. The Markov probability $\mathbf{P}^*$ determined according to the above proposition will be called *the finite projection of the Markov probability* $\mathbf{P}$ *with respect to the state* 0, *or with respect to the sequence* $T = (T(i))_{i\geqslant 0}$.

Similarly [1] defined the finite projection $\mathbf{P}^{*l}$ with respect to the state $l \in E$, when $0 < f_{ll} < 1$ and $\mathbf{P}(\inf\{k > 0/X_k = l\}) > 0$. Of course, $\mathbf{P}^* = \mathbf{P}^{*0}$.

2. Connection between Martin kernel and finite projection for Markov chains.

Let $\mathcal{M}$ be the set of all Markovian probabilities defined on the measurable space $(\Omega, \mathcal{F})$, including null measure defined on this space. We endow $\mathcal{M}$ with the $F-$topology determined by convergence of the finite dimensional distributions (f.d.d.) according to the

Definition 2. We say that the sequence $(\mathbf{P}_n)_{n\geqslant 1}$ of Markovian probabilities converges to the measure $\widetilde{\mathbf{P}}\in \mathcal{M}$ in the $F-$topology if all f.d.d. which correspond to the probabilities $(\mathbf{P}_n)_{n\geqslant 1}$ converge to the f.d.d corresponding to the $\widetilde{\mathbf{P}}$ when $n\to\infty$. More exactly $\mathbf{P}_n \to_{n\to\infty} \widetilde{\mathbf{P}}$ in the $F-$topology if and only if

$$\lim_{n\to\infty}\mathbf{P}_n(X_0=i_0,X_1=i_1,....,X_k=i_k)=\widetilde{\mathbf{P}}\ (X_0=i_0,X_1=i_1,....,X_k=i_k),$$

$\forall\ i_0,....,i_k\in E$.

Our aim is to study the properties of $\mathcal{M}^*$ as a set of all finite projections $\mathbf{P}^*$ for Markovian probabilities $\mathbf{P}\in\mathcal{M}$. Of course $\mathcal{M}^*\subseteq\mathcal{M}$ and $\mathcal{M}$ may be considered a topological subspace of $\mathcal{M}$.

First of all let's recall some notions related to the boundary theory for Markov chains [2,3]. The function $h: E\longmapsto\mathbb{R}$ is called $\mathbf{P}-$*excessive* if $h(i)\geqslant \sum_{j\in E} p_{ij}h(j), \forall i\in E, \mathbf{P}$ being Markovian probability.

For $\mathbf{P}-$excessive function h, such that $h(i)\geqslant 0, \forall i\in E$, we have the $h-$transformation of Markovian probability $\mathbf{P}$ as the probability $\mathbf{P}^h$ determined by one-step transition probabilities

$$p^h_{ij}=\begin{cases} p_{ij}h(j)/h(i), \text{if } h(i)\neq 0, i\in E,\\ 0, if\ h(i)=0 \text{ and } p_{ij}h(j)=0, i,j\in E.\end{cases}$$

If $h\equiv 0$ then we consider as a corresponding $h-$transformation the null measure on the selection's measurable space $(\Omega,\mathcal{F})$.

In the boundary theory for transient Markov chains, it was constructed the compactification E^+ of state set E called Martin compactification. This construction is based on the Martin kernel

$$K(i,j)=\frac{G(i,j)}{\sum_{l\in E}\gamma(l)G(l,j)},$$

where $G(i,j)$ is the Green function:

$$G(i,j)=\sum_{n=1}^{\infty}p^{(n)}_{ij}.$$

In our paper we consider only the case of Markov chains with a finite Green function G, i.e., for transient Markov chains [2]. We say that Martin kernel $K(i,j)$ is correctly defined if $\sum_{j\in E}\gamma(j)G(j,l)>0$, $\forall l\in E$. In this case the initial distribution γ is called a standard distribution [2]. In fact the condition (1.1) assures us that the initial distribution γ is a standard distribution.

The topology of compact E^+ is called M^+-topology and the sequence of states $(l_n)_{n\geq 1} \subseteq E$ is convergent in this topology if the following hypotheses are verified:

H1. The function $K(i, l_n)$ converges for every $i \in E$.

H2.$l_n \longrightarrow \infty$ or does exist n_0 such that $l_n = c, c - const$ for $\forall\ n \geqslant n_0$.

Because for every fixed $l \in E$ the kernel $K(\cdot, l)$ is the finite excessive function for Markov probability $\mathbf{P}$, we may consider the $K(\cdot, l)-$transformation of probability $\mathbf{P}$.

From equality $G(i,j) = f_{ij}^{1-\delta(i,j)} G(j,i)$ it results that $p_{ij}^{*l} = p_{ij}^{K(\cdot,l)} f_{il}^{1-\delta(i,l)}$, i.e., for $i \neq l$ the finite projections of one-step transition probabilities from the state i into the state j with respect to the state $l \in E$ coincides with one-step transition probabilities from the state i into the state j which corresponds to the $K(\cdot, l)-$transformations of Markovian probability $\mathbf{P}$.

The following results emphasize the connection between the Markov's chains finite projections and the corresponding Martin boundary. In fact this means that the construction of boundary theory for Markov chains is possible using conditioning of Markov probabilities only.

Theorem. *If* $(l_n)_{n\geq 1}$ *is a sequence of states* $l_n \in E$ *such that* $\liminf_{n\to\infty} l_n = \infty$, *then the following assertions are equivalent:*

a) There exists $\lim_{n\to\infty} l_n = l^+ \in E^+$ *(in the sense of* M^+-*topology);*

b) For every $i \in E$ *exists* $\lim_{n\to\infty} K(i, l_n) = K(i, l^+)$ *and* $K(i, l^+) < +\infty$.

c) The sequence of the finite projections $(\mathbf{P}^{*l_n})_{n\geqslant 1}$ *converges to the* $K(\cdot, l^+)-$ *transformation of the Markovian probability* $\mathbf{P}$ *with initial distribution* $\gamma^{K(\cdot,l^+)}$ *given by formula*

$$\gamma^{K(\cdot,l^+)}(i) = \mathbf{P}^{K(\cdot,l^+)}(X_0 = i) = \gamma(i) K(i, l^+), \forall\ i \in E.$$

Proof. The equivalence of assertions a) and b) is a consequence of the definition of M^+-topology, the property $K(i, l^+) < +\infty$ being a consequence of the inequality

$$K(i,j) \leqslant 1 / \sum_{l \in E} \gamma(l) f_{lj}.$$

To prove the equivalence of b) and c) we observe that from [1] the finite projection of Markov probability with respect to the state $l \in E$ is such that

$$\mathbf{P}^{*l}(X_0 = i_0, X_1 = i_1, ..., X_k = i_k) = \gamma^{*l}(i_0) p^{*l}_{i_0 i_1} ... p^{*l}_{i_{k-1} i_k} =$$

$$= \mathbf{P}(X_0 = i_0, X_1 = i_1, ..., X_k = i_k) \frac{f_{i_k l}^{1-\delta(i_k,l)} \sum_{j \in E} \gamma(j) f_{jl}^{1-\delta(j,l)}}{\prod_{m=1}^{k-1}} \cdot f_{i_m l}^{-\delta(i_m,l)}.$$

This means that

$$\begin{aligned} \mathbf{P}^{*l}(X_0 &= i_0, X_1 = i_1, ..., X_k = i_k) = \\ &= \mathbf{P}(X_0 = i_0, X_1 = i_1, ..., X_k = i_k) K(i_k, l) \prod_{m=1}^{k-1} f_{i_m l}^{-\delta(i_m,l)}. \qquad (2.1) \end{aligned}$$

Supposing that in M^+-topology there exists $\lim_{n\to\infty} l_n = l^+ \in E^+ \setminus E$, from (2.1) and hypotheses H1-H2, we deduce that

$$\lim_{n\to\infty} \mathbf{P}^{*l_n}(X_0 = i_0, X_1 = i_1, ..., X_k = i_k) =$$

$$= \mathbf{P}(X_0 = i_0, X_1 = i_1, ..., X_k = i_k) K(i_k, l^+) \lim_{n\to\infty} \prod_{m=1}^{k-1} f_{i_m l}^{-\delta(i_m,l_n)}, .(2.2)$$

where $K(i_k, l^+) = \lim_{n\to\infty} K(i_k, l_n)$. But

$$\lim_{n\to\infty} \prod_{m=1}^{k-1} f_{i_m l}^{-\delta(i_m,l_n)} = 1, \qquad (2.3)$$

since $\liminf_{n\to\infty} l_n = \infty$.

As a limit of excessive functions, the function $K(\cdot, l^+)$ is excessive too. So, we may define the $K(\cdot, l^+)-$transformation of the Markov probability $\mathbf{P}$. From (2.2) and (2.3)it results that the sequence of Markov probabilities $\mathbf{P}^{*l_n}$ converges to the $K(\cdot, l^+)-$transformation of the Markov probability $\mathbf{P}$ with the initial distribution $\gamma^{K(\cdot,l^+)}(i) = \mathbf{P}^{K(\cdot,l^+)}(X_0 = i) = \gamma(i) K(i, l^+), \forall\ i \in E$. Consequently b) implies c).

The inverse implication is a consequence of the equality (2.1) and the fact that the initial distribution γ is a standard distribution. Actually, from (1.1) we have that for every $i \in E$ the states $i_0, i_1, ..., i_k \in E$ are such that $i_k = i$ and $\mathbf{P}(X_0 = i_0, X_1 = i_1, ..., X_k = i) > 0$. Then from equality $\liminf_{n\to\infty} l_n = \infty$ we deduce the validity of (2.3). Thus, from convergence of $(\mathbf{P}^{*l_n})_{n\geqslant 1}$ and (2.1)it results the existence of $\lim_{n\to\infty} K(i, l_n) < \infty, \quad \forall\ i \in E.$■

Remark. If the initial distribution γ is such that $\gamma(i) > 0$, $\forall\ i \in E$, then the connection between the Martin kernel and the finite projection is given by the equality

$$K(i,j) = \mathbf{P}^{*j}(X_0 = i)/\gamma(i) = \mathbf{P}^{*j}(X_0 = i)/\mathbf{P}(X_0 = i).$$

This shows that we may interpret the $h-$transformation of the Markov probability $\mathbf{P}$ as a limit of its finite projections with respect to the different states $l \in E, i.e.$, in fact the $h-$transformation represents the given Markov probability conditioned by events that corresponding chain X will reach the subset of states from boundary $\partial E = E^{+}\backslash E$.

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